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INFINITELY MANY SOLUTIONS FOR A CLASS OF ELLIPTIC VARIATIONAL-HEMIVARIATIONAL INEQUALITY PROBLEMS

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Abstract. The aim of the present paper is to give some results on the existence of infinitely many solutions for a class of nonlinear elliptic variational-hemivariational inequalities. The approach is based on a result of infinitely many critical points.

1. **Introduction**

In mechanics and physics there is a variety of variational inequality formulations which arise when the material laws or the boundary conditions are derived by a convex, generally not everywhere differentiable and finite superpotential ([12]).The variational inequalities have a precise physical meaning: they express the principle of virtual work (or power) in its inequality form. Moreover, there exists a variety of nonmonotone laws which manifests the need for the derivation of variational formulations for nonconvex and not everywhere differentiable and finite energy functions (nonconvex superpotentials). Such variational formulations have been called by P.D. Panagiotopoulos ([10], [11]) hemivariational inequalities and describe large families of important problems in physics and engineering. It should also be noted that the hemivariational inequalities are closely connected to the notion of the generalized gradient of Clarke, which in the case of lack of convexity plays the same role as the subdifferential in the case of convexity (at least for static mechanical problems). Roughly speaking, variational-hemivariational inequalities may be regarded as hemivariational inequalities subject to variational constraints. Consequently, a further term, namely the subdifferential of some proper, convex, and lower semicontinuous function, appears inside the equation.

Several authors have been interested in the study of variational-hemivariational inequalities, for example, S. A. Marano and D. Motreanu, in the very nice paper

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[8], studied the existence of infinitely many solutions for a class of elliptic variational hemivariational inequality with $p-$Laplacian.

Let Ω be a non-empty, bounded, open subset of the Euclidian space $\mathbb{R}^N$, $N \geq 3$, with a boundary of class C^1, let $p \in]N, +\infty[$, and let $q \in L^\infty(\Omega)$ satisfy $\operatorname{ess\,inf}_{x\in\Omega} q(x) > 0$. Given a closed convex subset K of $W^{1,p}(\Omega)$ containing the constant functions, they consider the following variational-hemivariational inequality problem

Find $u \in K$ fulfilling

$$-\int_\Omega \left[|\nabla u(x)|^{p-2}\nabla u(x)\nabla(v(x)-u(x)) + q(x)|u(x)|^{p-2}u(x)(v(x)-u(x))\right] dx$$

$$\leq \int_\Omega [\alpha(x)F^\circ(u(x);(v(x)-u(x))) + \beta(x)G^\circ(u(x);(v(x)-u(x)))]dx, \quad \forall v \in W^{1,p}(\Omega)$$

where $F(\xi) = \displaystyle\int_0^\xi f(t)dt$, $G(\xi) = \displaystyle\int_0^\xi g(t)dt$ for all $\xi \in \mathbb{R}$, with $f, g : \mathbb{R} \to \mathbb{R}$ locally essentially bounded, α, $\beta \in L^1(\Omega)$ such that $min\{\alpha(x), \beta(x)\} \geq 0$ a.e. in Ω.

In the study of this problem, they apply a result obtained by the same authors ([8, Theorem 1.1]), on the existence of infinitely many critical points.

The main purpose of the present paper is to establish the existence of infinitely many solutions for an elliptic variational-hemivariational inequality with $p-$Laplacian type: *Find $u \in K$ fulfilling*

$$-\int_\Omega \left[|\nabla u(x)|^{p-2}\nabla u(x)\nabla(v(x)-u(x)) + q(x)|u(x)|^{p-2}u(x)(v(x)-u(x))\right] dx$$

$$\leq \lambda \int_\Omega F^\circ(x, u(x);(v(x)-u(x)))dx$$

for all $v \in K$, with λ positive real parameter.

The approach is based on a result of infinitely many critical points due to G. Bonanno and G. Molica Bisci [4] which is a more precise version of [8, Theorem 1.1].

It is worth noticing that our results allow us to consider also the case when the sign of the nonlinear term is constant, see for instance Theorem 3.2 and Example 3.3, in which the nonlinear term h is nonpositive. We observe that this case cannot be investigated by applying [8, Theorem 2.1], (see Remark 3.5).

2. **Preliminaries**

Let $(X, \|\cdot\|)$ be a real Banach space. We denote by X^* the dual space of X, while $\langle\cdot,\cdot\rangle$ stands for the duality pairing between X^* and X. A function $h : X \to \mathbb{R}$ is called locally Lipschitz continuous when to every $x \in X$ there correspond a neighborhood V_x of x and a constant $L_x \geq 0$ such that

$$|h(z) - h(w)| \leq L_x \|z - w\| \quad \forall\, z, w \in V_x\,.$$

If $x, z \in X$, we write $h^\circ(x; z)$ for the generalized directional derivative of h at the point x along the direction z, i.e.,

$$h^\circ(x; z) := \limsup_{w \to x,\, t \to 0^+} \frac{h(w + tz) - h(w)}{t}\,.$$

For locally Lipschitz h_1, $h_2 : X \to \mathbb{R}$, we have

$$(h_1 + h_2)^\circ(x, z) \leq h_1^\circ(x, z) + h_2^\circ(x, z), \qquad \forall x, z \in X. \tag{2.1}$$

The generalized gradient of the function h in x, denoted by $\partial h(x)$, is the set

$$\partial h(x) := \{x^* \in X^* : \langle x^*, z\rangle \leq h^\circ(x; z)\; \forall\, z \in X\}\,.$$

We say that $x \in X$ is a (generalized) critical point of h when

$$h^\circ(x; z) \geq 0 \quad \forall\, z \in X,$$

that clearly signifies $0 \in \partial h(x)$.

When a non-smooth functional, $g : X \to]-\infty, +\infty]$, is expressed as a sum of a locally Lipschitz function, $h : X \to \mathbb{R}$, and a convex, proper, and lower semicontinuous function, $j : X \to]-\infty, +\infty]$, that is $g := h + j$, a (generalized) critical point of g is every $u \in X$ such that

$$h^\circ(u; v - u) + j(v) - j(u) \geq 0,$$

for all $v \in X$ (see [9, Chapter 3]).

Here and in the sequel X is a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ is a sequentially weakly lower semicontinuous functional, $\Upsilon : X \to \mathbb{R}$ is a sequentially weakly upper semicontinuous functional, λ is a positive real parameter, $j : X \to]-\infty, +\infty]$ is a convex, proper and lower semicontinuous functional and $D\,(j)$ is the effective domain of j.
Write

$$\Psi := \Upsilon - j \quad \text{and} \quad I_\lambda := \Phi - \lambda\Psi = (\Phi - \lambda\Upsilon) + \lambda j.$$

We also assume that Φ is coercive and

$$D(j) \cap \Phi^{-1}(]-\infty, r[) \neq \emptyset \tag{2.2}$$

for all $r > \inf_X \Phi$. Moreover, from (2.2) and provided $r > \inf_X \Phi$, we can define

$$\varphi(r) = \inf_{u \in \Phi^{-1}(]-\infty, r[)} \frac{\left(\displaystyle\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)\right) - \Psi(u)}{r - \Phi(u)}$$

and

$$\gamma := \liminf_{r \to +\infty} \varphi(r), \qquad \delta := \liminf_{r \to (\inf_X \Phi)^+} \varphi(r).$$

Assuming also that Φ and Υ are locally Lipschitz functionals, in [4] the authors obtained the following result, which is a more precise version of [8, Theorem 1.1].

Theorem 2.1. *Under the above assumptions on X, Φ and Ψ, one has*

(a) *For every $r > \inf_X \Phi$ and every $\lambda \in]0, \frac{1}{\varphi(r)}[$, the restriction of the functional $I_\lambda = \Phi - \lambda\Psi$ to $\Phi^{-1}(]-\infty, r[)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.*

(b) *If $\gamma < +\infty$ then, for each $\lambda \in]0, \frac{1}{\gamma}[$, the following alternative holds: either*

  (b_1) *I_λ possesses a global minimum,*

  or

  (b_2) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that $\lim_{n \to +\infty} \Phi(u_n) = +\infty$.*

(c) *If $\delta < +\infty$ then, for each $\lambda \in]0, \frac{1}{\delta}[$, the following alternative holds: either*

  (c_1) *there is a global minimum of Φ which is a local minimum of I_λ,*

  or

  (c_2) *there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_λ, with $\lim_{n \to +\infty} \Phi(u_n) = \inf_X \Phi$, which weakly converges to a global minimum of Φ.*

3. **Existence Results**

In this section, we present an applications of Theorem 2.1 to a Neumann-type problem for a variational-hemivariational inequality involving the p-Laplacian.

Let Ω be a non-empty, bounded, open subset of the Euclidian space $\mathbb{R}^N$, $N \geq 3$, with a boundary of class C^1, let $p \in]N, +\infty[$, and let $q \in L^\infty(\Omega)$ satisfy $\operatorname{ess\,inf}_{x\in\Omega} q(x) > 0$. On the space $W^{1,p}(\Omega)$, we consider the norm

$$\|u\| := \left(\int_\Omega (|\nabla u(x)|^p + q(x)|u(x)|^p) dx \right)^{\frac{1}{p}},$$

which is equivalent to the usual one.

Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be locally essentially bounded. Put

$$F(x, \xi) = \int_0^\xi f(x, t) dt.$$

The function F is locally Lipschitz. So, it makes sense to consider its generalized directional derivative F°.

Given a closed convex subset K of $W^{1,p}(\Omega)$ containing the constant functions, denote by (P) the following variational-hemivariational inequality problem:

Find $u \in K$ fulfilling

$$-\int_\Omega \left[|\nabla u(x)|^{p-2}\nabla u(x)\nabla(v(x)-u(x)) + q(x)|u(x)|^{p-2}u(x)(v(x)-u(x))\right] dx$$
$$\leq \lambda \int_\Omega F^\circ(x, u(x); (v(x)-u(x)))dx$$

for all $v \in K$, with λ positive real parameter.

Put

$$c = \sup_{u \in W^{1,p}(\Omega)\setminus\{0\}} \frac{\sup_{x\in\Omega}|u(x)|}{\left(\int_\Omega |\nabla u(x)|^p dx + \int_\Omega q(x)|u(x)|^p dx\right)^{\frac{1}{p}}}. \tag{3.1}$$

From (3.1), we infer at once that

$$c^p \|q\|_1 \geq 1. \tag{3.2}$$

Let

$$A = \liminf_{\xi\to+\infty} \frac{\int_\Omega \max_{|t|\leq\xi} -F(x,t)dx}{\xi^p}, \quad B = \limsup_{\xi\to+\infty} \frac{\int_\Omega -F(x,\xi)dx}{\xi^p},$$

and

$$\lambda_1 = \frac{\|q\|_1}{pB}, \quad \lambda_2 = \frac{1}{pc^p A}. \tag{3.3}$$

Our main result is the following.

Theorem 3.1. *Assume that*

$$\liminf_{\xi\to+\infty} \frac{\int_\Omega \max_{|t|\leq\xi} (-F(x,t))\, dx}{\xi^p} < \frac{1}{c^p\|q\|_1} \limsup_{\xi\to+\infty} \frac{\int_\Omega (-F(x,\xi))\, dx}{\xi^p}. \tag{3.4}$$

Then, for each $\lambda \in]\lambda_1, \lambda_2[$, where λ_1, λ_2 are given in (3.3), problem (P) possesses an unbounded sequence of solutions.

Proof. Our aim is to apply part (b) of Theorem 2.1. Take as X the Sobolev space $W^{1,p}(\Omega)$ endowed with the norm

$$\|u\| = \Big(\int_\Omega |\nabla u(x)|^p dx + \int_\Omega q(x)|u(x)|^p dx\Big)^{\frac{1}{p}}.$$

For each $u \in X$, put

$$\Phi(u) := \frac{1}{p}\|u\|^p, \qquad \Upsilon(u) := \int_\Omega -F(x,u(x))dx.$$

and

$$j(u) = \begin{cases} 0, & \text{if } u \in K, \\ +\infty, & \text{otherwise.} \end{cases}$$

Since, $\Psi := \Upsilon - j$,

$$I_\lambda := \frac{1}{p}\|u\|^p - \lambda\left(\int_\Omega -F(x,u(x))dx - j(u)\right) = \left(\frac{1}{p}\|u\|^p - \lambda\int_\Omega -F(x,u(x))dx\right) + \lambda j.$$

Pick $\lambda \in]\lambda_1, \lambda_2[$. Let $\{\rho_n\}$ be a real sequence such that $\lim_{n\to\infty} \rho_n = +\infty$ and

$$\lim_{n\to\infty} \frac{\int_\Omega \max_{|t|\le\rho_n}(-F(x,t))dx}{\rho_n^p} = A.$$

Put $r_n = \frac{1}{p}\left(\frac{\rho_n}{c}\right)^p$ for all $n \in \mathbb{N}$. Taking into account $\|v\|^p < pr_n$ and $\|v\|_\infty \le c\|v\|$, one has $|v(x)| \le \rho_n$, for every $x \in \Omega$. Therefore,

$$\varphi(r_n) = \inf_{\|u\|^p < pr_n} \frac{\sup_{\|v\|^p<pr_n}\left(\int_\Omega -F(x,v(x))dx - j(v)\right) - \left(\int_\Omega -F(x,u(x))dx - j(u)\right)}{r_n - \frac{\|u\|^p}{p}}$$

$$\le \frac{\sup_{\|v\|^p<pr_n}\left(\int_\Omega -F(x,v(x))dx - j(v)\right)}{r_n} \le \frac{\sup_{\|v\|^p<pr_n}\int_\Omega -F(x,v(x))dx}{r_n}$$

$$\le \frac{\int_\Omega \max_{|t|\le\rho_n} -F(x,t)dx}{r_n}$$

Hence,

$$\varphi(r_n) \le pc^p \frac{\int_\Omega \max_{|t|\le\rho_n}(-F(x,t))dx}{\rho_n^p} \qquad \forall n \in \mathbb{N}.$$

Then,

$$\gamma \le \liminf_{n\to+\infty} \varphi(r_n) \le pc^p A < +\infty.$$

Now, we claim that the functional $\Phi - \lambda\Psi$ is unbounded from below.

Let $\{d_n\}$be a real sequence such that $\lim_{n\to\infty} d_n = +\infty$ and

$$\lim_{n\to\infty} \int_\Omega \frac{-F(x,d_n)dx}{d_n^p} = B. \tag{3.5}$$

For each $n \in \mathbb{N}$, put $w_n(x) = d_n$, for all $x \in \Omega$. Clearly $w_n \in W^{1,p}(\Omega)$ for each $n \in \mathbb{N}$. Hence,

$$\|w_n\|^p = d_n^p\|q\|_1$$

and

$$\Phi(w_n) - \lambda\Psi(w_n) = \frac{\|w_n\|^p}{p} - \lambda \int_\Omega -F(x, w_n(x))dx + \lambda j(w_n)$$
$$= \frac{d_n^p \|q\|_1}{p} - \lambda \int_\Omega -F(x, d_n)dx.$$

Now, if $B < +\infty$, let $\epsilon \in \left]0, B - \frac{\|q\|_1}{p\lambda}\right[$. From (3.5) there exists ν_ϵ such that

$$\int_\Omega -F(x, d_n)dx > (B - \epsilon)d_n^p, \qquad \forall n > \nu_\epsilon.$$

Therefore,

$$\Phi(w_n) - \lambda\Psi(w_n) = \frac{d_n^p \|q\|_1}{p} - \lambda \int_\Omega -F(x, d_n)dx < \frac{d_n^p \|q\|_1}{p} - \lambda d_n^p (B - \epsilon)$$
$$= d_n^p \left(\frac{\|q\|_1}{p} - \lambda(B - \epsilon) \right).$$

From the choice of ϵ, one has

$$\lim_{n \to +\infty} [\Phi(w_n) - \lambda\Psi(w_n)] = -\infty.$$

If $B = +\infty$, fix $M > \frac{\|q\|_1}{p\lambda}$. From (3.5) there exists ν_M such that

$$\int_\Omega -F(x, d_n)dx > M d_n^p, \quad \forall n > \nu_M.$$

Moreover,

$$\Phi(w_n) - \lambda\Psi(w_n) = \frac{d_n^p \|q\|_1}{p} - \lambda \int_\Omega -F(x, d_n)dx < \frac{d_n^p \|q\|_1}{p} - \lambda M d_n^p = d_n^p \left(\frac{\|q\|_1}{p} - \lambda M \right).$$

Taking into account the choice of M, also in this case, one has

$$\lim_{n \to +\infty} [\Phi(w_n) - \lambda\Psi(w_n)] = -\infty.$$

From part (b) of Theorem the functional $\Phi - \lambda\Psi$ admits a sequence of critical points $u_n \subseteq W^{1,p}(\Omega)$ such that $\lim_{n\to\infty} \Phi(u_n) = +\infty$, that means for each point u_n

$$(\Phi - \lambda\Upsilon)^\circ(u_n, v - u_n) + j(v) - j(u_n) \geq 0 \qquad \forall v \in X. \quad (H)$$

Since Φ is bounded on bounded sets and taking into account that $\lim_{n\to+\infty} \Phi(u_n) = +\infty$, then $\{u_n\}$ has to be unbounded. Moreover, from (H) we obtain $u_n \in K, \ \forall n \in \mathbb{N}$, so

$$(\Phi - \lambda\Upsilon)^\circ(u_n, v - u_n) \geq 0 \qquad \forall v \in K.$$

From (2.1) and the regularity of Φ, it follows

$$\Phi^{'}(u_n, v - u_n) + \lambda[-\Upsilon(u_n, v - u_n)]^\circ \geq 0 \qquad \forall v \in K.$$

Therefore,

$$\int_\Omega \left[|\nabla u_n(x)|^{p-2}\nabla u_n(x)\nabla(v(x)-u_n(x)) + q(x)|u_n(x)|^{p-2}u_n(x)(v(x)-u_n(x))\right]dx +$$

$$+\lambda\left[\int_\Omega F(x,u_n(x);(v(x)-u_n(x)))dx\right]^\circ \geq 0 \qquad \forall v\in K.$$

From an inequality concerning the integral functionals ([7]), we have

$$[-\Upsilon(u_n, v-u_n)]^\circ = \left[\int_\Omega F(x,u_n(x);(v(x)-u_n(x)))dx\right]^\circ \leq$$

$$\leq \int_\Omega F^\circ(x,u_n(x);(v(x)-u_n(x)))dx.$$

Then,

$$\int_\Omega \left[|\nabla u_n(x)|^{p-2}\nabla u_n(x)\nabla(v(x)-u_n(x)) + q(x)|u_n(x)|^{p-2}u_n(x)(v(x)-u_n(x))\right]dx +$$

$$+\lambda\int_\Omega F^\circ(x,u_n(x);(v(x)-u_n(x)))dx \geq 0;$$

that is

$$-\int_\Omega \left[|\nabla u_n(x)|^{p-2}\nabla u_n(x)\nabla(v(x)-u_n(x)) + q(x)|u_n(x)|^{p-2}u_n(x)(v(x)-u_n(x))\right]dx$$

$$\leq \lambda\int_\Omega F^\circ(x,u_n(x);(v(x)-u_n(x)))dx.$$

□

Given $\alpha \in L^1(\Omega)$, such that $\alpha(x)\geq 0$ a.e. in Ω, let $h:\mathbb{R}\to\mathbb{R}$ be a locally essentially bounded, such that $h(x)\leq 0$ a.e. in $\mathbb{R}$. Consider the following problem:

(P_H) *Find* $u\in K$ *fulfilling*

$$-\int_\Omega \left[|\nabla u(x)|^{p-2}\nabla u(x)\nabla(v(x)-u(x)) + q(x)|u(x)|^{p-2}u(x)(v(x)-u(x))\right]dx$$

$$\leq \lambda\int_\Omega \alpha(x)H^\circ(u(x);(v(x)-u(x)))dx$$

for all $v\in K$*, with* λ *positive real parameter.*

An immediate consequence of Theorem 3.1 is the following

Theorem 3.2. *Assume that*

$$\liminf_{\xi\to+\infty}\frac{(-H(\xi))}{\xi^p} < \frac{1}{c^p\|q\|_1}\limsup_{\xi\to+\infty}\frac{(-H(\xi))}{\xi^p}.$$

Then for every $\lambda \in \left]\dfrac{\|q\|_1}{p\|\alpha\|_1\limsup\limits_{\xi\to+\infty}\dfrac{(-H(\xi))}{\xi^p}}, \dfrac{1}{pc^p\|\alpha\|_1\liminf\limits_{\xi\to+\infty}\dfrac{(-H(\xi))}{\xi^p}}\right[$*, problem* (P_H) *possesses an unbounded sequence of solutions.*

Example 3.3. Put

$$a_n := \frac{2n!(n+2)!-1}{4(n+1)!}, \quad b_n := \frac{2n!(n+2)!+1}{4(n+1)!}$$

for every $n \in \mathbb{N}$, and define the non-positive (and discontinuous) function $h : \mathbb{R} \to \mathbb{R}$ as follows

$$h(\xi) := \begin{cases} -2(n+1)![n^{p-1}(n+1)!^p - (n-1)^{p-1}n!^p] & \text{if } \xi \in \bigcup\limits_{n\geq 0}]a_n, b_n[\\ 0 & \text{otherwise.} \end{cases}$$

Direct computations ensure that

$$\limsup_{\xi\to+\infty} \frac{-H(\xi)}{\xi^p} = +\infty \quad \text{and} \quad \liminf_{\xi\to+\infty} \frac{-H(\xi)}{\xi^p} = 0.$$

Owing to Theorem 3.2 for each $\lambda > 0$ the problem (P_H) possesses a sequence of solutions.

Remark 3.4. We explicitly observe that we cannot apply [8, Theorem 2.1] to the problem of Example 3.3, since hypotheses (3.6), (3.7), recalled below, do not hold, namely, supposed that there exist two sequences $\{\xi_n\} \subseteq \mathbb{R}$, $\{r_n\} \subseteq \mathbb{R}^+$ such that $\lim_{n\to+\infty} r_n = +\infty$,

$$H(\xi_n) = \inf_{|\xi|\leq c(pr_n)^{1/p}} H(\xi), \quad \forall n \in \mathbb{N}, \tag{3.6}$$

$$\frac{1}{p}\|q\|_1|\xi_n|^p < r_n \quad \forall n \in \mathbb{N}, \tag{3.7}$$

and taking into account that H is nonincreasing, we obtain that $c^p\|q\|_1 < 1$, which contradicts (3.2).

Remark 3.5. When f is an L^1-Carathéodory while $K = W^{1,p}(\Omega)$ the above inequality takes the form

$$-\int_\Omega \left[|\nabla u(x)|^{p-2}\nabla u(x)\nabla(w(x)) + q(x)|u(x)|^{p-2}u(x)(w(x))\right] dx$$

$$= \lambda \int_\Omega f(x, u(x))w(x)dx, \qquad \forall w \in W^{1,p}(\Omega).$$

Therefore, in such a case, a function $u \in W^{1,p}(\Omega)$ solves (P) if and only if it is a weak solution to the Neumann problem

$$\begin{cases} \Delta_p u - q(x)|u|^{p-2}u = \lambda f(x,u) & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{in } \partial\Omega, \end{cases}$$

with ν being the outer unit normal to $\partial\Omega$.

We observe that this problem has been addressed recently in [2], by applying directly Theorem 2.1 to smooth functionals.

The results can be applied to study the above problem with discontinuous nonlinear term (see, for instance [1], [3]).

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