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A REMARK ON PERTURBED ELLIPTIC NEUMANN PROBLEMS

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Abstract. The aim of this paper is to establish the existence of infinitely many solutions for perturbed eigenvalue elliptic Neumann problems involving the p–Laplacian. To be precise, we show that an appropriate oscillating behaviour of the nonlinear term, even under small perturbations, ensures again the existence of infinitely many solutions.

1. **Introduction**

Very recently in [6], presenting a version of the infinitely many critical points theorem of B. Ricceri (see [12, Theorem 2.5]), the existence of an unbounded sequence of weak solutions for a Sturm-Liouville problem, having discontinuous nonlinearities, has been established. In a such approach, an appropriate oscillating behavior of the nonlinear term either at infinity or at zero is required. This type of methodology has been used then in several works in order to obtain existence results for different kinds of problems (see, for instance, [2, 3, 5, 7, 8, 9, 10, 11]).

In particular, in [3, Theorem 3], by following this approach, it was proved the existence of infinitely many solutions for the following elliptic Neumann problem

$$\begin{cases} -\Delta_p u + q(x)|u|^{p-2}u = \lambda h(x,u) & \text{in } \Omega \\ \partial u/\partial\nu = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

where $\Omega \subset \mathbf{R}^N$ be a bounded open set with smooth boundary $\partial\Omega$, ν is the outer unit normal to $\partial\Omega$, $\Delta_p = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p–Laplacian, $p > N$, $q \in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_\Omega q > 0$, $h : \Omega \times \mathbf{R} \to \mathbf{R}$ is a continuous function and λ is a positive real parameter. For reader's convenience, Theorem 3 of [3] is here recalled.

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Theorem 1.1. *Let $h : \Omega \times \mathbf{R} \to \mathbf{R}$ be an L^1–Carathéodory function. Put $H(x, \xi) := \int_0^\xi h(x, t)\, dt$ for all $(x, \xi) \in \Omega \times \mathbf{R}$ and assume that*

$$\liminf_{\xi \to +\infty} \frac{\int_\Omega \max_{|t| \le \xi} H(x, t) dx}{\xi^p} < \frac{1}{c^p \|q\|_1} \limsup_{\xi \to +\infty} \frac{\int_\Omega H(x, \xi) dx}{\xi^p}, \tag{1.1}$$

where

$$c = \sup_{u \in W^{1,p}(\Omega) \setminus \{0\}} \frac{\sup_{x \in \overline{\Omega}} |u(x)|}{\left(\int_\Omega |\nabla u(x)|^p dx + \int_\Omega q(x)|u(x)|^p dx \right)^{\frac{1}{p}}}, \tag{1.2}$$

and $\|q\|_1 := \int_\Omega q(x) dx$.

Then, for each

$$\lambda \in \left] \frac{\|q\|_1}{p \limsup_{\xi \to +\infty} \frac{\int_\Omega H(x, \xi) dx}{\xi^p}}, \frac{1}{p c^p \liminf_{\xi \to +\infty} \frac{\int_\Omega \max_{|t| \le \xi} H(x, t) dx}{\xi^p}} \right[,$$

problem (P_λ) possesses an unbounded sequence of weak solutions in $W^{1,p}(\Omega)$.

In recent years, multiplicity results for Neumann problems have widely been investigated (see [13] and [14]) as well as the existence of three solutions for perturbed Neumann problems has been obtained (see [15] and [4]). The aim of this note is to point out, as a consequence of Theorem 1.1, existence results of infinitely many solutions for perturbed Neumann problems. To be precise, we prove the existence of infinitely many weak solutions for the following perturbed Neumann problem

$$\begin{cases} -\Delta_p u + q(x)|u|^{p-2}u = \lambda f(x, u) + \mu g(x, u) & \text{in } \Omega \\ \\ \partial u / \partial \nu = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{$N^{f,g}_{\lambda,\mu}$}$$

where $f, g : \Omega \times \mathbf{R} \to R$ are two L^1–Carathéodory functions and λ, μ are real parameters.

Precisely, requiring that the nonlinear term f has a suitable oscillating behavior at infinity, in Theorem 2.1, we establish the existence of a precise interval Λ such that for every $\lambda \in \Lambda$ and every L^1–Carathéodory function g which satisfies a certain growth at infinity, choosing μ sufficiently small, the perturbed problem $(N^{f,g}_{\lambda,\mu})$ admits an unbounded sequence of weak solutions in $W^{1,p}(\Omega)$.

As an example, we present here a special case of our main result.

Theorem 1.2. *Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous nonnegative function. Put $F(\xi) := \int_0^\xi f(t)\,dt$ for all $\xi \in \mathbf{R}$, and assume that*

$$\liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^2} = 0, \quad \text{and} \quad \limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^2} = +\infty.$$

Then, for every nonnegative continuous function $g : \Omega \times \mathbf{R} \to \mathbf{R}$ such that

$$g'_\infty)\ \ G^\star_\infty := \lim_{\xi\to+\infty} \frac{\int_0^1 \left(\int_0^\xi g(x,t)dt\right) dx}{\xi^2} < +\infty,$$

and for every $\mu \in \left[0, \frac{1}{4G^\star_\infty}\right[$, the following problem

$$\begin{cases} -u'' + u = f(u) + \mu g(x,u) & \text{in} \quad]0,1[\\ u'(0) = u'(1) = 0, \end{cases} \tag{$N^{f,g}_\mu$}$$

admits a sequence of pairwise distinct positive classical solutions.

An analogous result (see Theorem 2.6) can be obtained if we replace the oscillating behavior condition at infinity, by a similar one at zero. In this setting, a sequence of pairwise distinct non-zero solutions which converges to zero is achieved.

The note is arranged as follows. In Section 2 we show our abstract results, while in Section 3 a concrete example of application is given.

2. Main results

We recall here some basic definitions and notations. A function $h : \Omega \times \mathbf{R} \to \mathbf{R}$ is called an L^1-Carathéodory function if $x \mapsto h(x,t)$ is measurable for all $t \in \mathbf{R}$, $t \mapsto h(x,t)$ is continuous for almost every $x \in \Omega$ and for all $M > 0$ one has $\sup_{|t|\le M} |h(x,t)| \in L^1(\Omega)$. Clearly, if h is continuous in $\overline{\Omega} \times \mathbf{R}$, then it is L^1-Carathéodory.

Let $W^{1,p}(\Omega)$ be the usual Sobolev space endowed with the norm

$$\|u\| = \Big(\int_\Omega |\nabla u(x)|^p dx + \int_\Omega q(x)|u(x)|^p dx\Big)^{\frac{1}{p}},$$

that is equivalent to the usual one.

A *weak solution* of the problem $(N^{f,g}_{\lambda,\mu})$ is any $u \in W^{1,p}(\Omega)$, such that

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla v(x)dx + \int_\Omega q(x)|u(x)|^{p-2}u(x)v(x)dx +$$

$$-\lambda \int_\Omega f(x,u(x))v(x)dx - \mu \int_\Omega g(x,u(x))v(x)dx = 0, \quad \forall v \in W^{1,p}(\Omega).$$

Set $F(x,\xi) := \displaystyle\int_0^\xi f(x,t)dt$ for every $(x,\xi) \in \Omega \times \mathbf{R}$, and

$$\lambda_1 := \frac{\|q\|_1}{p \limsup\limits_{\xi\to+\infty} \dfrac{\int_\Omega F(x,\xi)dx}{\xi^p}}, \quad \lambda_2 := \frac{1}{pc^p \liminf\limits_{\xi\to+\infty} \dfrac{\int_\Omega \max_{|t|\le\xi} F(x,t)dx}{\xi^p}}. \tag{2.1}$$

Our result reads as follows

Theorem 2.1. *Assume that*

$$\liminf_{\xi\to+\infty} \frac{\int_\Omega \max_{|t|\le\xi} F(x,t)dx}{\xi^p} < \frac{1}{c^p\|q\|_1} \limsup_{\xi\to+\infty} \frac{\int_\Omega F(x,\xi)dx}{\xi^p}, \tag{2.2}$$

Then, for each $\lambda \in]\lambda_1, \lambda_2[$, for every L^1-Carathédory function $g : \Omega\times\mathbf{R} \to \mathbf{R}$ whose potential $G(x,\xi) := \displaystyle\int_0^\xi g(x,t)\ dt$, $\forall\ (x,\xi) \in \Omega \times \mathbf{R}$, is a nonnegative function satisfying

$$g_\infty)\ \ G_\infty := \lim_{\xi\to+\infty} \frac{\int_\Omega \max_{|t|\le\xi} G(x,t)dx}{\xi^p} < +\infty,$$

and for every $\mu \in [0, \mu_{g,\lambda}[$, where

$$\mu_{g,\lambda} := \frac{1}{pc^pG_\infty}\left(1 - \lambda pc^p \liminf_{\xi\to+\infty} \frac{\int_\Omega \max_{|t|\le\xi} F(x,t)dx}{\xi^p}\right),$$

the problem $(N^{f,g}_{\lambda,\mu})$ admits a sequence of weak solutions which is unbounded in $W^{1,p}(\Omega)$.

Proof. Our aim is to apply Theorem 1.1. To this end, fix $\overline{\lambda} \in]\lambda_1, \lambda_2[$ and let g be a function satisfies hypothesis g_∞). In the non-perturbed case, i.e. $\mu = 0$, the thesis is trivial. Owing to $\overline{\lambda} < \lambda_2$, one has

$$\mu_{g,\overline{\lambda}} := \frac{1}{pc^pG_\infty}\left(1 - \overline{\lambda} pc^p \liminf_{\xi\to+\infty} \frac{\int_\Omega \max_{|t|\le\xi} F(x,t)dx}{\xi^p}\right) > 0.$$

Take $0 < \overline{\mu} < \mu_{g,\overline{\lambda}}$ and put

$$\eta_1 := \lambda_1 = \frac{\|q\|_1}{p \limsup\limits_{\xi\to+\infty} \dfrac{\int_\Omega F(x,\xi)dx}{\xi^p}}, \quad \eta_2 := \frac{1}{pc^p \liminf\limits_{\xi\to+\infty} \dfrac{\int_\Omega \max_{|t|\le\xi} F(x,t)dx}{\xi^p} + pc^p \dfrac{\overline{\mu}}{\overline{\lambda}} G_\infty}.$$

If $G_\infty = 0$ clearly one has $\eta_1 = \lambda_1$, $\eta_2 = \lambda_2$ and

$$\overline{\lambda} \in \Lambda^\star :=]\eta_1, \eta_2[.$$

If $G_\infty \neq 0$, from $\overline{\mu} < \mu_{g,\overline{\lambda}}$, it follows that

$$\overline{\lambda} pc^p \liminf_{\xi \to +\infty} \frac{\displaystyle\int_\Omega \max_{|t| \leq \xi} F(x,t)dx}{\xi^p} + pc^p G_\infty \overline{\mu} < 1,$$

that means

$$\overline{\lambda} < \frac{1}{pc^p \displaystyle\liminf_{\xi \to +\infty} \frac{\displaystyle\int_\Omega \max_{|t| \leq \xi} F(x,t)dx}{\xi^p} + pc^p \frac{\overline{\mu}}{\overline{\lambda}} G_\infty} = \eta_2.$$

On the other hand, by our hypothesis, $\overline{\lambda} > \eta_1$.
Hence, one has

$$\overline{\lambda} \in \Lambda^\star :=]\eta_1, \eta_2[.$$

Now, put

$$H(x,\xi) := F(x,\xi) + \frac{\overline{\mu}}{\overline{\lambda}} G(x,\xi),$$

for every $x \in \Omega$ and $\xi \in \mathbf{R}$.
Then

$$\frac{\displaystyle\int_\Omega \max_{|t| \leq \xi} H(x,t)dx}{\xi^p} \leq \frac{\displaystyle\int_\Omega \max_{|t| \leq \xi} F(x,t)dx}{\xi^p} + \frac{\overline{\mu}}{\overline{\lambda}} \frac{\displaystyle\int_\Omega \max_{|t| \leq \xi} G(x,t)dx}{\xi^p},$$

and taking into account hypothesis g_∞), it follows that

$$\liminf_{\xi \to +\infty} \frac{\displaystyle\int_\Omega \max_{|t| \leq \xi} H(x,t)dx}{\xi^p} \leq \liminf_{\xi \to +\infty} \frac{\displaystyle\int_\Omega \max_{|t| \leq \xi} F(x,t)dx}{\xi^p} + \frac{\overline{\mu}}{\overline{\lambda}} G_\infty. \tag{2.3}$$

Moreover, taking into account that the potential G is a nonnegative function, we obtain

$$\limsup_{\xi \to +\infty} \frac{\displaystyle\int_\Omega H(x,\xi)dx}{\xi^p} \geq \limsup_{\xi \to +\infty} \frac{\displaystyle\int_\Omega F(x,\xi)dx}{\xi^p}. \tag{2.4}$$

Conditions (2.3) and (2.4) yield

$$\overline{\lambda} \in \Lambda^\star \subseteq \left] \frac{\|q\|_1}{p \displaystyle\limsup_{\xi \to +\infty} \frac{\displaystyle\int_\Omega H(x,\xi)dx}{\xi^p}}, \frac{1}{pc^p \displaystyle\liminf_{\xi \to +\infty} \frac{\displaystyle\int_\Omega \max_{|t| \leq \xi} H(x,t)dx}{\xi^p}} \right[. \tag{2.5}$$

So the conclusion follows at once from Theorem 1.1 observing that, from (2.5), condition (1.1) is clearly verified. The proof is complete. $\square$

Remark 2.2. If Ω is convex, an explicit upper bound for the constant c is

$$c \leq 2^{\frac{p-1}{p}} \max\left\{ \frac{1}{\|q\|_1^{\frac{1}{p}}}, \frac{d}{N^{\frac{1}{p}}}\left(\frac{p-1}{p-N}\operatorname{meas}(\Omega)\right)^{\frac{p-1}{p}} \frac{\|q\|_\infty}{\|q\|_1} \right\},$$

where "$\operatorname{meas}(\Omega)$" denotes the Lebesgue measure of the set Ω, $d := \operatorname{diam}(\Omega)$ and $\|q\|_\infty := \max_{x\in\overline{\Omega}} |u(x)|$. See, for instance, [1, Remark 1].

Remark 2.3. If

$$\liminf_{\xi\to+\infty} \frac{\displaystyle\int_\Omega \max_{|t|\leq\xi} F(x,t)dx}{\xi^p} = 0, \quad \text{and} \quad \limsup_{\xi\to+\infty} \frac{\displaystyle\int_\Omega F(x,\xi)dx}{\xi^p} = +\infty,$$

clearly, hypothesis (2.2) is verified and Theorem 2.1 guarantees the existence of infinitely many weak solutions for problem $(N^{f,g}_{\lambda,\mu})$, for every pair $(\lambda,\mu) \in D$, where

$$D :=]0,+\infty[\times \left[0, \frac{1}{pc^pG_\infty}\right[.$$

Moreover, under the assumption $G_\infty = 0$, the main result ensures the existence of infinitely many weak solutions for the problem $(N^{f,g}_{\lambda,\mu})$, for every $\mu > 0$.

Remark 2.4. Assuming that, in Theorem 2.1, the L^1–Carathéodory function f is nonnegative, hypothesis (2.2) can be written as follows

$$\liminf_{\xi\to+\infty} \frac{\displaystyle\int_\Omega F(x,\xi)dx}{\xi^p} < \frac{1}{c^p\|q\|_1} \limsup_{\xi\to+\infty} \frac{\displaystyle\int_\Omega F(x,\xi)dx}{\xi^p},$$

as well as

$$\mu_{g,\lambda} := \frac{1}{pc^pG_\infty}\left(1 - \lambda pc^p \liminf_{\xi\to+\infty} \frac{\displaystyle\int_\Omega F(x,\xi)dx}{\xi^p}\right).$$

Moreover if in addition, we consider the autonomous case, condition (2.2) assumes the following form

$$\liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^p} < \frac{1}{c^p\|q\|_1} \limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^p}.$$

Further, in this setting, one has

$$\lambda_1 := \frac{\|q\|_1}{p \operatorname{meas}(\Omega) \limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^p}}, \quad \lambda_2 := \frac{1}{pc^p \operatorname{meas}(\Omega) \liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^p}},$$

and

$$\mu_{g,\lambda} := \frac{1}{pc^p G_\infty}\left(1 - \lambda p\,\mathrm{meas}(\Omega)c^p \liminf_{\xi\to+\infty}\frac{F(\xi)}{\xi^p}\right).$$

Remark 2.5. We point out that Theorem 1.2 in Introduction is a particular case of Theorem 2.1 taking into account Remarks 2.3 and 2.4.

Replacing the conditions at infinity of the potential by a similar at zero, the same result holds and, in addition, the sequence of pairwise distinct solutions uniformly converges to zero. Precisely, set

$$\lambda_1^* := \frac{\|q\|_1}{p\limsup\limits_{\xi\to0^+}\dfrac{\int_\Omega F(x,\xi)dx}{\xi^p}}, \quad \lambda_2^* := \frac{1}{pc^p\liminf\limits_{\xi\to0^+}\dfrac{\int_\Omega \max\limits_{|t|\le\xi}F(x,t)dx}{\xi^p}}. \tag{2.6}$$

From Theorem 4 of [3], arguing as in the proof of Theorem 2.1, we obtain the following result.

Theorem 2.6. *Assume that*

$$\liminf_{\xi\to0^+}\frac{\int_\Omega \max\limits_{|t|\le\xi}F(x,t)dx}{\xi^p} < \frac{1}{c^p\|q\|_1}\limsup_{\xi\to0^+}\frac{\int_\Omega F(x,\xi)dx}{\xi^p}. \tag{2.7}$$

Then, for each $\lambda \in]\lambda_1^*, \lambda_2^*[$*, for every* L^1-*Carathédory function* $g : \Omega\times\mathbf{R}\to\mathbf{R}$ *whose potential* $G(x,\xi) := \int_0^\xi g(x,t)\,dt$, $\forall\,(x,\xi)\in\Omega\times\mathbf{R}$, *is a nonnegative function satisfying*

$$g_0)\ \ G_0 := \lim_{\xi\to0^+}\frac{\int_\Omega \max\limits_{|t|\le\xi}G(x,t)dx}{\xi^p} < +\infty,$$

and for every $\mu\in[0,\mu_{g,\lambda}[$*, where*

$$\mu_{g,\lambda} := \frac{1}{pc^pG_0}\left(1 - \lambda pc^p\liminf_{\xi\to0^+}\frac{\int_\Omega \max\limits_{|t|\le\xi}F(x,t)dx}{\xi^p}\right),$$

the problem $(N^{f,g}_{\lambda,\mu})$ *possesses a sequence of non-zero weak solutions which strongly converges to* 0 *in* $W^{1,p}(\Omega)$.

3. **Application**

Let $g : \Omega \times \mathbf{R} \to \mathbf{R}$ be a continuous and nonnegative function such that

$$\lim_{\xi \to +\infty} \frac{\displaystyle\int_0^1 \left(\int_0^\xi g(x,t)dt \right) dx}{\xi^2} < +\infty.$$

Consider the following Neumann problem

$$\begin{cases} -u'' + u = \lambda f(u) + \mu g(x,u) & \text{in} \quad]0,1[\\ u'(0) = u'(1) = 0, \end{cases} \tag{$N^{f,g}_{\lambda,\mu}$}$$

where $f : \mathbf{R} \to \mathbf{R}$ is defined as follows

$$f(t) := \begin{cases} t\cos^2(\ln(t)) & \text{if } t > 0 \\ \\ 0 & \text{if } t \leq 0. \end{cases}$$

A direct computation ensures that

$$\liminf_{\xi \to +\infty} \frac{F(\xi)}{\xi^2} = \frac{2-\sqrt{2}}{8},$$

and

$$\limsup_{\xi \to +\infty} \frac{F(\xi)}{\xi^2} = \frac{2+\sqrt{2}}{8}.$$

Moreover,

$$\liminf_{\xi \to +\infty} \frac{F(\xi)}{\xi^2} < \frac{1}{2} \limsup_{\xi \to +\infty} \frac{F(\xi)}{\xi^2}.$$

From Theorem 2.1, for each $\lambda \in \Lambda := \left] \frac{4}{2+\sqrt{2}}, \frac{2}{(2-\sqrt{2})} \right[$, and for every

$$0 \leq \mu < \frac{1}{4} \left(1 - \lambda \frac{(2-\sqrt{2})}{2} \right) \left(\lim_{\xi \to +\infty} \frac{\displaystyle\int_0^1 \left(\int_0^\xi g(x,t)dt \right) dx}{\xi^2} \right)^{-1},$$

problem $(N^{f,g}_{\lambda,\mu})$ possesses a sequence of pairwise distinct classical solutions. For instance, for every $(\lambda,\mu) \in \Lambda \times [0,+\infty[$, the problem

$$\begin{cases} -u'' + u = \lambda f(u) + \mu \dfrac{\sqrt{|u|}}{1+\sqrt{x}} & \text{in} \quad]0,1[\\ u'(0) = u'(1) = 0, \end{cases} \tag{$N^{f,g}_{\lambda,\mu}$}$$

possesses a sequence of pairwise distinct classical solutions.

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