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ON THE BEHAVIOR NEAR 0 AND NEAR ∞ OF FUNCTIONALS ON $W_0^{1,p}(\Omega)$ INVOLVING NONLINEAR OSCILLATING TERMS

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Abstract. The behavior near 0 and near ∞ of energy functionals on $W_0^{1,p}(\Omega)$ associated to boundary value problems for quasilinear elliptic equations is studied. As a consequence, some results concerning the existence of infinitely many solutions for the Dirichlet problem are established.

1. **Introduction**

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with $C^{1,1}$ boundary $\partial\Omega$.

Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. In recent years, some authors have investigated the problem of finding infinitely many solutions for the problem

$$\begin{cases} -\Delta_p u = f(x,u) \ \text{ in } \ \Omega \\ B(u) = 0 \end{cases} \tag{P}$$

in the case in which the nonlinearity $f(x,\cdot)$ has an oscillatory behavior near 0 or near ∞. Here, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-laplacian operator, with $p > 1$, and B is a given boundary operator. The reader is referred, for instance, to [1] [2], [6] and [11] for problem (P) with Neumann boundary condition, that is with $Bu = \frac{\partial u}{\partial \nu}_{|\partial\Omega}$, and to [3], [4], [7], [9] and [10] for problem (P) with Dirichlet boundary condition, that is with $Bu = u_{|\partial\Omega}$ (see also reference of [9], [10] for a wider overview on the subject). In all of these papers, the existence of infinitely many solutions is obtained by showing that the energy functional associated to problem (P)

$$\Psi(u) = \frac{1}{p}\int_\Omega |\nabla u(x)|^p dx - \int_\Omega \left(\int_0^{u(x)} f(x,t)dt \right) dx, \tag{1.1}$$

defined on $W^{1,p}(\Omega)$ or $W_0^{1,p}(\Omega)$ according to whether the Neumann or the Dirichlet boundary condition is considered, possesses infinitely many critical points. Therefore,

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in particular, solutions are always understood in weak sense. In practice, the previous circumstance is realized by showing that if $f(x,\cdot)$ has a suitable oscillatory behavior near ∞ or near 0 then there exists a sequence of local minima $\{u_n\}$ for the functional Ψ which is unbounded with respect to some norm or satisfying $\lim_{n\to+\infty} \Psi(u_n) = 0$ and $\Psi(u_n) < 0$ for all $n \in \mathbb{N}$. In order to find the local minima u_n, a direct variational method is used in [1] where it is proved that the for a certain sequence on spheres $\{S_n\}$ of $W^{1,p}(\Omega)$, the infimum of Ψ on each S_n is strictly greater of the global minimum of Ψ on the closed ball B_n having S_n as boundary; truncation methods are, instead, used in [2], [3], [7], [9], [10] where it is proved that the global minima of certain truncations Ψ_n of Ψ are, actually, local minima of this latter; finally in [4], [6], [11], taking advantage of the compact embedding of $W^{1,p}(\Omega)$ in $C^0(\Omega)$ when $p > N$, a variational result of [12] on the multiplicity of critical points is applied. Once a sequence of local minima is found out, the successive step, as said before, is to show that this sequence contains infinitely many pairwise distinct elements. It is quite simple to realize this when the Neumann problem is considered. Indeed, due to the fact that the constant functions belong to $W^{1,p}(\Omega)$, it is suffice to require that there exists a sequence $\{\xi_n\}$ of real numbers such that

$$a)\qquad \lim_{n\to+\infty} \xi_n \to \begin{cases} 0, \\ \pm\infty \end{cases} ;$$

$$b)\qquad \limsup_{n\to+\infty} \frac{\int_\Omega \int_0^{\xi_n} f(x,t)dt}{|\xi_n|^p} = +\infty$$

(see, for instance, [2], [6], [11] where, however, a slight weaker condition is required). The above question becomes more delicate when the Dirichlet problem is considered, especially if $f(x,\cdot)$ is sign-changing. In this case, the validity of $a)$ and $b)$ is no more sufficient to realize that the sequence of local minima u_n contains infinitely many pairwise distinct elements. To achieve this goal, more sophisticated conditions must be imposed. In particular, in [3] it is showed that under the following assumptions:

there exist a nonempty open set D in Ω, a positive number $\sigma > 0$ and a sequence $\{\xi_n\}$ in $]0,+\infty[$ such that $\lim_{n\to+\infty} \xi_n = 0$ and

$$a_1)\qquad \limsup_{n\to+\infty} \frac{\operatorname*{ess\,inf}_{x\in D} \int_0^{\xi_n} f(x,t)dt}{\xi_n^p} = +\infty;$$

$$b_1)\qquad \operatorname*{ess\,inf}_{x\in D} \inf_{\xi\in[0,\xi_n]} \int_0^{\xi} f(x,t)dt \geq -\sigma \operatorname*{ess\,inf}_{x\in D} \int_0^{\xi_n} f(x,t)dt \qquad \text{for all } n \in \mathbb{N},$$

the sequence of local minima $\{u_n\}$ can be chosen having the following property:

$$\Psi(u_n) < 0 \text{ for all } n \in \mathbb{N} \text{ and } \lim_{n\to+\infty} \Psi(u_n) = 0.$$

Note that, as showed in [3], assumptions $a_1)$ and $b_1)$, when f is independent of x, are weaker than the following ones assumed in [10]

$$a_2) \quad \limsup_{\xi\to 0^+} \frac{\int_0^\xi f(t)dt}{\xi^p} = +\infty \ ;$$

$$b_2) \quad \liminf_{\xi\to 0^+} \frac{\int_0^\xi f(t)dt}{\xi^p} = 0,$$

where similar conclusions are obtained. Finally, in [9] it is showed that the sequence of local minima $\{u_n\}$ satisfies the following property $\lim_{n\to+\infty} \dfrac{u_n(x)}{\mathrm{dist}(x,\partial\Omega)} = +\infty$ uniformly in Ω assuming that $a_2)$, $b_2)$ hold with $\xi \to 0^+$ replaced by $\xi \to +\infty$. It is worth of noticing the fact that in [9] the authors obtained, besides $\{u_n\}$, a sequence of saddle points $\{v_n\}$ having the same property of $\{u_n\}$. Moreover, note that this property proves that u_n and v_n turn out to be positive in Ω rather than simply nonnegative (as in [3] and [10]).

The aim of this paper is to give a new contribution on this topic. In particular, we will establish the existence of a sequence of pairwise distinct local minima for the functional Ψ keeping assumption $a_1)$ but replacing assumption $b_1)$ with the following one

$\tilde{b}_1)$ there exists a positive number $\sigma > 0$ and a nonempty open set D in Ω such that

$$\int_0^{\xi_n} F_+(\xi)d\xi \geq -\sigma \int_0^{\xi_n} F_-(\xi)d\xi$$

for all $n \in \mathbb{N}$,

where $F : \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$F(\xi) \leq \operatorname*{ess\,inf}_{x\in D} \int_0^\xi f(x,t)dt \quad \text{and}$$

$$F_+(t) = \max\{F(t), 0\}, \quad F_-(\xi) = \min\{F(\xi), 0\},$$

for all $\xi \in \mathbb{R}$.

To motivate our main result, we promptly exhibit an example of function f (for simplicity independent of x) which satisfy $a_1)$ and $\tilde{b}_1)$ but not $b_1)$:

let α, β, η be three positive numbers such that $1 < \beta < \alpha < p$ and $\beta > \alpha - \eta$ and let $g \in C^1(]0, \infty[)$ be a bounded nonnegative function such that

$$\int_0^1 \frac{g(t)}{t^\eta} < +\infty \tag{1.2}$$

and satisfying

$$\lim_{n\to+\infty} g(t_n) > 0 \quad \text{and} \quad g(s_n) = 0 \quad \text{for every} \quad n \in \mathbb{N}, \tag{1.3}$$

where $\{t_n\}$ and $\{s_n\}$ are two decreasing sequences in $]0, +\infty[$ such that

$$\sup_{n\in\mathbb{N}} \frac{t_n}{t_{n+1}} < +\infty \quad \text{and} \quad \lim_{n\to+\infty} t_n = \lim_{n\to+\infty} s_n = 0.$$

Put

$$f(t) = t^{\alpha-1} - t^{\beta-1}(\beta^{-1}tg'(t) + g(t)) \;\text{ for }\; t > 0 \;\text{ and }\; f(t) = 0 \;\text{ otherwise.}$$

Let us to show that f is the functions we are looking for. At first, note that

$$F(\xi) = \int_0^\xi f(t)dt = \frac{1}{\alpha}\xi^\alpha - \frac{1}{\beta}\xi^\beta g(\xi) \quad \text{for every} \quad \xi \geq 0.$$

Now, fix any sequence $\{\xi_n\}$ in $]0, +\infty[$ such that $\lim_{n\to+\infty} \xi_n = 0$ and $\xi_n < t_1$ for every $n \in \mathbb{N}$ and denote by k_n the smallest integer such that $t_{k_n} \leq \xi_n$. It follows $\lim_{n\to+\infty} t_{k_n} = 0$ and, for every $n \in \mathbb{N}$,

$$\inf_{\xi\in[0,\xi_n]} F(\xi) \leq F(t_{k_n}) = \frac{1}{\alpha}t_{k_n}^\alpha - \frac{1}{\beta}t_{k_n}^\beta g(t_{n_k})$$

$$F(\xi_n) \leq \sup_{\xi\in[0,\xi_n]} F(\xi) \leq \frac{1}{\alpha}t_{k_n-1}^\alpha.$$

Hence, for every $\sigma > 0$ and $n \in \mathbb{N}$, one has

$$\inf_{\xi\in[0,\xi_n]} F(\xi) + \sigma F(\xi_n) \leq \frac{1}{\alpha}t_{k_n}^\alpha - \frac{1}{\beta}t_{k_n}^\beta g(t_{n_k}) + \sigma\frac{1}{\alpha}t_{k_n-1}^\alpha =$$

$$\frac{1}{\alpha}t_{k_n}^\alpha\left(1 + \sigma\frac{t_{n_k-1}^\alpha}{t_{k_n}^\alpha} - \frac{\alpha}{\beta}t_{k_n}^{\beta-\alpha}g(t_{k_n})\right).$$

Consequently, in view of (1.3), one has

$$\inf_{\xi\in[0,\xi_n]} F(\xi) + \sigma F(\xi_n) < 0 \quad \text{for every} \quad n \in \mathbb{N}, \quad \text{with } n \text{ large enough.}$$

This means that condition $b_1)$ does not hold.

Now, note that for every $\xi \in]0, 1[$ with

$$\xi < \left(\frac{\beta}{2\alpha}\int_0^1 \frac{g(\tau)}{\tau^\eta}d\tau\right)^{\frac{1}{\beta+\eta-\alpha}}$$

one has:

$$\int_{F_\xi^+} F(\tau)d\tau \geq \int_0^\xi F(\tau)d\tau = \frac{\xi^\alpha}{\alpha} - \frac{1}{\beta}\int_0^\xi \tau^\beta d\tau \geq \frac{\xi^\alpha}{\alpha} - \frac{\xi^{\beta+\eta}}{\beta}\int_0^1 \frac{g(\tau)}{\tau^\eta}d\tau \quad \text{and}$$

$$\int_{F_\xi^-} F(\tau)d\tau \geq -\frac{1}{\beta}\int_0^\xi \tau^\beta d\tau \geq -\frac{\xi^{\beta+\eta}}{\beta}\int_0^1 \frac{g(\tau)}{\tau^\eta}d\tau.$$

Therefore,

$$\int_{F_\xi^+} F(\tau)d\tau + \int_{F_\xi^-} F(\tau)d\tau \geq \frac{\xi^\alpha}{\alpha} - 2\frac{\xi^{\beta+\eta}}{\beta}\int_0^1 \frac{g(\tau)}{\tau^\eta}d\tau > 0.$$

Thus, in view of (1.2), condition $\tilde{b}_1)$ holds with $\sigma = 1$. Finally, note that, thanks to the properties of the sequence $\{s_n\}$, condition $a_1)$ holds as well.

To exhibit a concrete example of function g satisfying (1.2) and (1.3), an easy calculation shows that it is enough to take

$$g(t) = e^{-t^{-\rho}\cos^2(t^{-1})}\sin^2(t^{-1}) \qquad \text{with} \qquad \rho > 2(\eta - 1) \qquad \text{if} \quad \eta \geq 1$$

$$g(t) = \sin^2(t^{-1}) \qquad\qquad \text{if} \quad \eta < 1.$$

2. **The results**

In what follows, the following notations will be used:

- for every $\xi \in \mathbb{R}$ and every nonempty open set D in Ω, the set $\{u \in W_0^{1,p}(\Omega) : \ 0 \leq u(x) \leq \xi \ \text{ a.e. in } D\}$ is denoted by $X_{\xi,D}$;
- given a nonempty set A in $\mathbb{R}$ and a positive number ε, the symbol $(A)_\varepsilon$ denotes the (closed) ε-dilatation of A, that is the set $\{t \in \mathbb{R} : \ \inf_{\tau \in A} |t - \tau| \leq \varepsilon\}$. Moreover, we will use the notation $int(A)$ to denote the interior of A;
- given a positive real number τ and a function $h : [0,\tau] \to \mathbb{R}$, the symbols h_τ^+ and h_τ^- denote, respectively, the sets $\{t \in [0,\tau] : \ h(t) > 0\}$ and $\{t \in [0,\tau] : \ h(t) \leq 0\}$;
- for any Lebesgue measurable set A in $\mathbb{R}^N$, the symbol $|A|$ denotes its Lebesgue measure.

Moreover, we equip the space $W_0^{1,p}(\Omega)$ with its standard norm

$$\|\cdot\| = \left(\int_\Omega |\nabla(\cdot)|^p dx\right)^{\frac{1}{p}}$$

and for every continuous function $F : \mathbb{R} \to \mathbb{R}$ we put

$$\Psi_F(u) = \frac{1}{p}\|u\|^p - \int_\Omega F(u(x))dx$$

for all $u \in W_0^{1,p}(\Omega)$ such that $F(u(\cdot)) \in L^1(\Omega)$. Our main result is Theorem 2.1 below. It allow us to determinate an upper estimate of the number $\inf_{X_\xi} \Psi_F$ in terms of constants which depend, besides Ω, only on the ratio of the areas of the regions delimited by the x-axis and the graphs of F_+ and F_- in $[0,\xi]$.

Theorem 2.1. *Let $F : \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that there exist two positive numbers σ, ξ such that*

i) $F(\xi) > 0$*;*

ii) *the set* $\Lambda_F \overset{def}{=} \left\{ \tau \in [0,\xi] : \int_{F_\tau^+} F(t)dt > -\sigma \int_{F_\tau^-} F(t)dt \right\}$ *is nonempty and*

$\sup \Lambda_F = \xi$.

Then, for every nonempty open set D in Ω, there exist two positive constants C_1, C_2 depending only on D and σ such that

$$\inf_{X_{\xi,D}} \Psi_F \leq C_1 \xi^p - C_2 F(\xi) + |D| \sup_{t \in [0, \inf \Lambda_F]} |F(t)| \tag{2.1}$$

Proof. We consider the case in which $\inf_{[0,\xi]} F < 0$ as, otherwise, the proof is similar and simpler. Moreover, note that, without loss of generality, the number σ can be chosen in $]0,1[$. Choose $x_0 \in D$ and fix $r, R > 0$, with $R > r$, such that $\overline{B}(x_0, R) \subset D$. From $i)$ one has $int(F_\xi^+) \neq \emptyset$. Now, let $\varepsilon \in]0,1[$. Note that the connected components of the compact set $(F_\xi^-)_\varepsilon$ are intervals of length at least 2ε. Therefore, $(F_\xi^-)_\varepsilon$ is union of a finite number of pairwise disjoint compact intervals I_l with $l = 1, .., m$, namely

$$(F_\xi^-)_\varepsilon = \cup_{l=1}^m I_l.$$

As a consequence, one has

$$int(F_\xi^+) \supset]0, \xi[\setminus \cup_{l=1}^m I_l = \cup_{\alpha=0}^L]a_{2\alpha+1}, a_{2\alpha}[= int(F_\xi^+) \setminus (F_\xi^-)_\varepsilon \tag{2.2}$$

where a_α is a finite decreasing sequence of positive real numbers. This sequence, of course, depends on ε. However, for brevity reasons, we do not explicitly indicate this dependence. Now, observe that a_{2L+1} is nonincreasing with respect to ε and so we can consider the following limit

$$\lim_{\varepsilon \to 0^+} a_{2L+1} \overset{def}{=} \rho.$$

Let us to show that,

$$\rho \leq \inf \Lambda_F. \tag{2.3}$$

Indeed, arguing by contradiction, assume that

$$\rho > \inf \Lambda_F.$$

Then, it should exist $\tau \in \Lambda_F$ such that $\tau \le \rho$. In particular $F(t) \le 0$ for all $t \in [0, \tau]$ and, consequently, $\int_{F_\tau^+} F(t)dt = 0$ which is absurd if $\tau \in \Lambda_F$. Thus, if we fix any $\eta > 0$, we can choose ε small enough in order that

$$a_{2L+1} < \inf \Lambda_F + \eta. \tag{2.4}$$

Moreover, since

$$\lim_{\varepsilon \to 0^+} \int_{(F_\xi^-)_\varepsilon \cap F_\xi^+} F(t)dt = 0,$$

we can also assume, choosing ε smaller if necessary, that

$$\frac{\int_{int(F_\xi^+) \setminus (F_\xi^-)_\varepsilon}}{\int_{(F_\xi^-)_\varepsilon \cap F_\xi^+} F(t)dt} > 1, \tag{2.5}$$

and, thanks to condition $i)$,

$$\xi \notin (F_\xi^-)_\varepsilon. \tag{2.6}$$

In particular, from (2.6) we have $\xi = a_0$. Moreover, from (2.2) and (2.5) one has

$$\frac{\sum_{\alpha=0}^{L} \int_{a_{2\alpha+1}}^{a_{2\alpha}} F(t)dt}{\int_{(F_\xi^-)_\varepsilon \cap F_\xi^+} F(t)dt} > 1. \tag{2.7}$$

Let us put

$$\eta_1 = \frac{R-r}{a_0} \quad \text{and} \quad \eta_2 = \frac{\eta_1}{4} \cdot \min\left\{ \left(\frac{r}{R}\right)^{N-1}, \frac{r^{N-1}\sigma}{R^{N-1} - \sigma r^{N-1}} \right\}. \tag{2.8}$$

Define the following function

$$u_\xi(x) = \begin{cases} a_0 & \text{if } \; x \in B(x_0, r) \\[2ex] \dfrac{a_\alpha - a_{\alpha+1}}{R_{\alpha+1} - R_\alpha}(R_{\alpha+1} - |x - x_0|) + a_{\alpha+1} & \text{if } \; x \in B(x_0, R_{\alpha+1}) \setminus \overline{B}(x_0, R_\alpha) \\ & \text{and } \; \alpha = 0, .., 2L \\[2ex] \dfrac{a_{2L+1}}{R - R_{2L+1}}(R - |x - x_0|) & \text{if } \; a_{2L+1} \neq 0 \; \text{ and} \\ & x \in \overline{B}(x_0, R) \setminus \overline{B}(x_0, R_{2L+1}) \\[2ex] 0 \qquad \text{otherwise} & \end{cases}$$

where

$$\begin{aligned} &R_{2\alpha+1} = R_{2\alpha} + \eta_1(a_{2\alpha} - a_{2\alpha+1}) \;\text{ for all }\; \alpha = 0, .., L \;\text{ and} \\ &R_{2\alpha} = R_{2\alpha-1} + \eta_2(a_{2\alpha-1} - a_{2\alpha}) \;\text{ for all }\; \alpha = 1, .., L \end{aligned} \tag{2.9}$$

with $R_0 = r$. Note that u_ξ is a Lipschitz function in Ω with $u_{\xi|\partial\Omega} = 0$, hence $u_\xi \in W_0^{1,p}(\Omega)$. Moreover, if we put $\mu = 0$ if $a_{2L+1} = 0$ and

$$\mu = \omega_N \left(\frac{a_{2L+1}}{R - R_{2L+1}} \right)^p (R^N - R_{2L+1}^N)$$

if $a_{2L+1} \neq 0$, taking in mind that $\eta_2 \leq \eta_1$ and in view of (2.8), one has

$$\begin{aligned} \int_\Omega |\nabla u_\xi|^p dx &= \sum_{\alpha=0}^{2L} \omega_N \left(\frac{a_\alpha - a_{\alpha+1}}{R_{\alpha+1} - R_\alpha} \right)^p (R_{\alpha+1}^N - R_\alpha^N) + \mu \\ &= \frac{\omega_N}{\eta_2^{p-1}} \sum_{\alpha=0}^{2L} (a_\alpha - a_{\alpha+1}) \sum_{i=0}^{N-1} R_{\alpha+1}^{N-1-i} R_\alpha^i + \mu \\ &\leq N\omega_N R^{N-1} \left[\frac{4}{\min\left\{ \left(\frac{r}{R}\right)^{N-1}, \frac{r^{N-1}\sigma}{R^{N-1} - \sigma r^{N-1}} \right\}} \right]^{p-1} \frac{a_0^{p-1}}{(R-r)^{p-1}} (a_0 - a_{2L+1}) + \mu. \end{aligned} \tag{2.10}$$

Since from (2.9) it follows $R_{2L+1} \leq R_0 + \eta_1(a_0 - a_{2L+1})$, the following estimate holds

$$\mu \leq N\omega_N R^{N-1} \frac{a_{2L+1}}{(R-r)^{p-1}} a_0^{p-1}. \tag{2.11}$$

Therefore, from (2.10) and (2.11), there exists a constant C depending only on Ω and ρ such that (recall that $a_0 = \xi$)

$$\int_\Omega |\nabla u_\xi|^p dx \leq C\xi^p \tag{2.12}$$

Moreover, since $0 \leq u_\xi(x) \leq \xi$ for all $x \in D$, we finally infer $u_\xi \in X_{\xi,D}$.

Put, for simplicity,

$$D_\alpha = B(x_0, R_{\alpha+1}) \setminus \overline{B}(x_0, R_\alpha)$$

for all $\alpha = 0, .., 2L$, as well as

$$\gamma_\alpha(t) = \frac{a_\alpha - a_{\alpha+1}}{R_{\alpha+1} - R_\alpha}(R_{\alpha+1} - t) + a_{\alpha+1}$$

for all $\alpha = 0, .., 2L$ and $t \in \mathbb{R}$. Finally, put $\delta = 0$ if $a_{2L+1} = 0$ and

$$\delta = \int_{B(x_0,R)\setminus B(x_0,R_{2L+1})} F\left(\frac{a_{2L+1}}{R - R_{2L+1}} (R - |x - x_0|) \right) dx$$

if $a_{2L+1} > 0$.

An upper estimate for $|\delta|$ in the case $a_{2L+1} > 0$ can be obtained noticing that, by (2.4), one gets

$$\begin{aligned}|\delta| \leq \quad & \omega_N \int_0^{a_{2L+1}} \left(R - \frac{R - R_{2L+1}}{a_{2L+1}}\right)^{N-1} \frac{R - R_{2L+1}}{a_{2L+1}} |F(t)| dt \\ \leq \; & \frac{\omega_N R^N}{a_{2L+1}} \int_0^{a_{2L+1}} |F(t)| dt \leq |D| \sup_{t \in [0, \inf \Lambda_F + \eta]} |F(t)|. \end{aligned} \tag{2.13}$$

Now, observe that (2.2) and (2.6) imply

$$\bigcup_{\alpha=1}^{L}]a_{2\alpha}, a_{2\alpha-1}[= \left((F_\xi^-)_\varepsilon \cap F_\xi^+\right) \cup F_\xi^- \tag{2.14}$$

and

$$\{a_0\} \cup ((F_\xi^-)_\varepsilon \cap F_\xi^+) \cup \bigcup_{\alpha=0}^{L}]a_{2\alpha+1}, a_{2\alpha}[= F_\xi^+. \tag{2.15}$$

Thus, if we put

$$I_1 = \sum_{\alpha=0}^{L} \int_{a_{2\alpha+1}}^{a_{2\alpha}} F(t) dt, \quad I_2 = \int_{(F_\xi^-)_\varepsilon \cap F_\xi^+} F(t) dt$$

and

$$I_+ = \int_{F_\xi^+} F(t) dt, \quad I_- = \int_{F_\xi^-} F(t) dt$$

we have

$$I_+ = I_1 + I_2.$$

Moreover, by assumption $ii)$, we also have

$$I_- > -\frac{1}{\sigma} I_+.$$

With this in mind and in view of (2.12), (2.14), (2.15), we get

$$\Psi_F(u_\xi) \leq$$
$$\frac{C}{p}\xi^p - \int_\Omega F(u_\xi) = \frac{C}{p}\xi^p - \sum_{\alpha=0}^{2L}\int_{D_\alpha} F(\gamma_\alpha(|x-x_0|))dx - \int_{B(x_0,r)} F(\xi)dx - \delta$$
$$= \frac{C}{p}\xi^p - \omega_N r^N F(\xi) - \delta -$$
$$\omega_N \sum_{\alpha=0}^{2L}\int_{a_{\alpha+1}}^{a_\alpha}\left[R_{\alpha+1} - \frac{R_{\alpha+1}-R_\alpha}{a_\alpha - a_{\alpha+1}}(t-a_{\alpha+1})\right]^{N-1}\frac{R_{\alpha+1}-R_\alpha}{a_\alpha - a_{\alpha+1}}F(t)dt =$$
$$\frac{C}{p}\xi^p - \omega_N r^N F(\xi) - \delta - \omega_N \sum_{\alpha=0}^{L}\int_{a_{2\alpha+1}}^{a_{2\alpha}}[R_{2\alpha+1} - \eta_1(t-a_{2\alpha+1})]^{N-1}\eta_1 F(t)dt +$$
$$\sum_{\alpha=1}^{L}\int_{a_{2\alpha}}^{a_{2\alpha-1}}[R_{2\alpha} - \eta_2(t-a_{2\alpha})]^{N-1}\eta_2 F(t)dt \leq$$
$$\frac{C}{p}\xi^p - \omega_N r^N F(\xi) + |\delta| - \omega_N\left(\eta_1 r^{N-1}I_1 + \eta_2 r^{N-1}I_2 + \eta_2 R^{N-1}I_-\right) \leq$$
$$\frac{C}{p}\xi^p - \omega_N r^N F(\xi) + |\delta| - \omega_N\left(\eta_1 r^{N-1}I_1 + \eta_2 r^{N-1}I_2 - \eta_2\frac{R^{N-1}}{\sigma}(I_1+I_2)\right) =$$
$$\frac{C}{p}\xi^p - \omega_N r^N F(\xi) + |\delta| -$$
$$\omega_N\left(\left(\eta_1 r^{N-1} - \eta_2\frac{R^{N-1}}{\sigma}\right)I_1 - \eta_2\left(r^{N-1} - \frac{R^{N-1}}{\sigma}\right)I_2\right)$$

At this point, thanks to the choice of η_2 (see (2.8)) and inequality (2.7), one can easily check that

$$\left(\eta_1 r^{N-1} - \eta_2\frac{R^{N-1}}{\sigma}\right)I_1 - \eta_2\left(r^{N-1} - \frac{R^{N-1}}{\sigma}\right)I_2 > 0.$$

Therefore, by the previous inequalities and (2.13), it follows

$$\Psi_F(u_\xi) \leq \frac{C}{p}\xi^p - \frac{\omega_N r^N}{2}F(\xi) + |D|\sup_{t\in[0,\inf\Lambda_F+\eta]}|F(t)|$$

Then, since $u_\xi \in X_\xi$, by the arbitrariness of η conclusion follows. $\square$

Theorem 2.1 can now be applied to study the behavior of Ψ_F near 0 and near ∞. To this end, we have the following two Corollaries

Corollary 2.2. *Let $F:\mathbb{R}\to\mathbb{R}$ be a continuous function with $F(0)=0$. Assume that there exists a sequence $\{\xi_n\}$ in $]0,+\infty[$ with $\lim\limits_{n\to+\infty}\xi_n=0$ and a positive number σ such that*

$$j)\quad \limsup_{n\to+\infty}\frac{F(\xi_n)}{\xi_n^p}=+\infty;$$

$$jj)\quad \int_{F^+_{\xi_n}}F(t)dt\geq-\sigma\int_{F^-_{\xi_n}}F(t)dt\quad \textit{for all } n\in\mathbb{N}.$$

Then, for every nonempty open set D in Ω, up to subsequence of $\{\xi_n\}$, one has

$$\inf\nolimits_{X_{\xi_n,D}}\Psi_F<0 \textit{ for all } n\in\mathbb{N}\quad \textit{and}\quad \lim_{n\to+\infty}\inf\nolimits_{X_{\xi_n,D}}\Psi_F=0.$$

Proof. By assumption $j)$, up to a subsequence, we can suppose $F(\xi_n)>0$ for all $n\in\mathbb{N}$. From this, replacing σ with $\frac{\sigma}{2}$ if necessary, by assumption $jj)$ we have that, for all $n\in\mathbb{N}$, the sets

$$\Lambda^n_F=\left\{\tau\in[0,\xi_n]:\quad \int_{F^+_\tau}F(t)dt>-\sigma\int_{F^-_\tau}F(t)dt\right\}$$

are nonempty with $\sup\Lambda^n_F=\xi_n$. This fact jointly to $\lim_{n\to+\infty}\xi_n=0$ imply $\inf\Lambda^n_F=0$ for all $n\in\mathbb{N}$. Therefore, thanks to Theorem 2.1, there exist two positive constants C_1,C_2 depending only on σ and D such that

$$\inf_{X_{\xi_n,D}}\Psi_F\leq C_1\xi_n^p-C_2F(\xi_n)$$

for all $n\in\mathbb{N}$. Then, in view of $j)$, conclusion follows. □

Corollary 2.3. *Let $F:\mathbb{R}\to\mathbb{R}$ be a continuous function with $F(0)=0$. Assume that there exists a sequence $\{\xi_n\}$ in $]0,+\infty[$ with $\lim\limits_{n\to+\infty}\xi_n=+\infty$ and a positive number σ such that*

$$k)\quad \limsup_{n\to+\infty}\frac{F(\xi_n)}{\xi_n^p}=+\infty;$$

$$kk)\quad \int_{F^+_{\xi_n}}F(t)dt\geq-\sigma\int_{F^-_{\xi_n}}F(t)dt\quad \textit{for all } n\in\mathbb{N}.$$

Then, for every nonempty open set D in Ω, up to subsequence of $\{\xi_n\}$, one has

$$\lim_{n\to+\infty}\inf\nolimits_{X_{\xi_n,D}}\Psi_F=-\infty.$$

Proof. As in Corollary 2.2 we are able to apply Theorem 2.1. In this case, there exist positive constants C_1,C_2 depending only on σ and D such that, up to a subsequence of $\{\xi_n\}$, one has

$$\inf_{X_{\xi_n,D}}\Psi_F\leq C_1\xi_n^p-C_2F(\xi_n)+|D|\sup_{t\in[0,\inf\Lambda^n_F]}|F(t)|.$$

for all $n\in\mathbb{N}$. Since the sequence $\{\inf\Lambda^n_F\}$ is bounded, conclusion follows by assumption $k)$. □

We are now in position to state and prove two results concerning the existence of infinitely many weak solutions for some elliptic boundary value problems. The first one is a variant of Theorem 2.1 of [2].

Theorem 2.4. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that $f(x, 0) = 0$ for a.e. $x \in \Omega$ and let D be a nonempty open bounded set in Ω. Let $F : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $F(0) = 0$ and*

$$F(\xi) \leq \operatorname*{ess\,inf}_{x \in D} \int_0^{\xi} f(x,t)dt \quad for \quad every \quad \xi \in \mathbb{R}.$$

Assume that there exist a nonempty open set D in Ω, two positive numbers σ, s and two sequences $\{\xi_n\}, \{t_n\}$ in $]0, +\infty[$ with $\lim_{n \to +\infty} \xi_n = \lim_{n \to +\infty} t_n = 0$ such that

- *a)* $\sup_{t \in [0,s]} |f(\cdot, s)| \in L^m(\Omega)$ *where* $m \geq 1$ *with* $m > \frac{Np}{Np-N+p}$ *if* $p \leq N$;
- *b)* $f(x, t_n) \leq 0$ *for all* $n \in \mathbb{N}$ *and for almost all* $x \in \Omega$;
- *c)* $\limsup_{n \to +\infty} \dfrac{F(\xi_n)}{\xi_n^p} = +\infty$;
- *d)* $\displaystyle\int_{F^+_{\xi_n}} F(t)dt \geq -\sigma \int_{F^-_{\xi_n}} F(t)dt \quad$ *for all* $n \in \mathbb{N}$;

Then, the following Dirichlet problem

$$\begin{cases} -\Delta_p u = f(x,u) & \text{in} \quad \Omega \\ u_{|\Omega} = 0 \end{cases} \qquad (P_f)$$

admits a sequence of pairwise distinct nonnegative weak solutions $\{u_n\}$ in $W_0^{1,p}(\Omega) \cap C^1(\overline{\Omega})$ satisfying

$$\lim_{n \to +\infty} \|u_n\| = \lim_{n \to +\infty} \max_{x \in \overline{\Omega}} |u_n(x)| = 0.$$

Proof. Up to subsequences, we can suppose $t_n, \xi_n \in]0, s]$ and $\xi_n \leq t_n$ for all $n \in \mathbb{N}$. After that, fix $n \in \mathbb{N}$ and put

$$\Psi_n(u) = \frac{1}{p}\|u\|^p - \int_{\Omega} \left(\int_0^{u(x)} f_n(x,t)dt \right) dx$$

for all $u \in W_0^{1,p}(\Omega)$, where

$$f_n(x,t) = \begin{cases} f(x,t) & \text{if} \quad (x,t) \in \Omega \times [0, t_n] \\ f(x,t_n) & \text{if} \quad (x,t) \in \Omega \times]t_n, +\infty[\\ 0 & \text{if} \quad (x,t) \in \Omega \times]-\infty, 0[\end{cases} .$$

By standard results, condition $a)$ implies that functional Ψ_n is sequentially weakly lower semicontinuous and Gateâux differentiable in $W_0^{1,p}(\Omega)$. Moreover, one has

$$\lim_{\|u\| \to +\infty} \Psi_n(u) = +\infty.$$

Therefore, Ψ_n achieves its global minimum at a point $u_n \in W_0^{1,p}(\Omega)$. Thus, u_n is a weak solution of the problem

$$\begin{cases} -\Delta_p u = f_n(x,u) & \text{in } \Omega \\ u_{|\partial\Omega} = 0 \end{cases}.$$

By regularity results ([5]), one has $u \in C^1(\overline{\Omega})$. Moreover, by the Maximum Principle we easily infer that $0 \leq u_n(x) \leq t_n$ (see for instance Lemma 4.1 of [8]). Consequently, u_n is actually a weak solution of problem (P_f). Finally, note that

$$\Psi_n(u_n) = \inf_{W_0^{1,p}(\Omega)} \Psi_n \leq \inf_{X_{\xi_n,D}} \Psi_n \leq \inf_{X_{\xi_n,D}} \Psi_F \tag{2.16}$$

where

$$\Psi_F(u) = \frac{1}{p}\|u\|^p - \int_\Omega F(u(x))dx, \qquad u \in W_0^{1,p}(\Omega).$$

At this point, assumptions $c)$, $d)$ allows us to apply Corollary 2.2 and this yields, thanks to (2.16),

$$\Psi_n(u_n) < 0 \qquad \text{for all } n \in \mathbb{N} \qquad \text{and} \qquad \lim_{n\to+\infty} \Psi_n(u_n) = 0.$$

Then, conclusion easily follows. □

Our second application is a result analogous to Theorem 2.4 which states the existence of a norm-unbounded sequence of pairwise distinct weak solutions for problem (P_f). The proof is practically the same of Theorem 2.4 and so it is omitted.

Theorem 2.5. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that $f(x,0) = 0$ for a.e. $x \in \Omega$ and let D be a nonempty open bounded set in Ω. Let $F : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $F(0) = 0$ and*

$$F(\xi) \leq \operatorname*{ess\,inf}_{x\in D} \int_0^\xi f(x,t)dt \quad \text{for every} \quad \xi \in \mathbb{R}.$$

Assume that there exist a nonempty open set D in Ω, a positive number σ and two sequences $\{\xi_n\}, \{t_n\}$ in $]0,+\infty[$ with $\lim_{n\to+\infty} \xi_n = \lim_{n\to+\infty} t_n = +\infty$ such that

- *a)* $\sup_{t\in[0,s]} |f(\cdot,s)| \in L^m(\Omega)$ *for every $s > 0$, where $m \geq 1$ with $m > \frac{Np}{Np-N+p}$ if $p \leq N$;*
- *b)* $f(x,t_n) \leq 0$ *for all $n \in \mathbb{N}$ and for almost all $x \in \Omega$;*
- *c)* $\limsup_{n\to+\infty} \frac{F(\xi_n)}{\xi_n^p} = +\infty$*;*
- *d)* $\int_{F_{\xi_n}^+} F(t)dt \geq -\sigma \int_{F_{\xi_n}^-} F(t)dt$ *for all $n \in \mathbb{N}$;*

Then, the following Dirichlet problem

$$\begin{cases} -\Delta_p u = f(x,u) & \text{in} \quad \Omega \\ u_{|\Omega} = 0 \end{cases} \qquad (P_f)$$

admits a sequence of pairwise distinct nonnegative weak solutions $\{u_n\}$ *in* $W_0^{1,p}(\Omega) \cap C^1(\overline{\Omega})$ *satisfying*

$$\lim_{n\to+\infty} \|u_n\| = +\infty.$$

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