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ASYMPTOTIC BEHAVIOR OF INTERMEDIATE POINTS IN CERTAIN MEAN VALUE THEOREMS. II

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Dedicated to Professor Grigore Ştefan Sălăgean on his 60^{th} *birthday*

Abstract. The paper deals with the asymptotic behavior of the intermediate points in the mean value theorems for integrals as the involved interval shrinks to zero.

1. **Introduction**

Especially in the last two decades a great deal of work has been done in connection with the asymptotic behavior of intermediate points in certain mean value theorems (see, for instance, [1], [2], [3], [5], [9], [12], [13], [14]). The investigations in this direction started with the paper by Azpeitia [3], dealing with the asymptotic behavior of the intermediate point in the Lagrange-Taylor mean value theorem. A significant step forward was realized by Abel [1], who obtained a complete asymptotic expansion of the intermediate point in the Lagrange-Taylor mean value theorem when the length of the involved interval approaches zero. Later, following Abel's method of proof, similar complete asymptotic expansions have been obtained by several authors for other mean value theorems (Abel and Ivan [2] for the differential mean value theorem of divided differences, Xu, Cui and Hu [13] for the differential mean value theorem of divided differences with repetitions, Trif [12] for the Pawlikowska mean value theorem).

The purpose of the present paper is to continue our investigations started in [12]. But unlike the paper [12], here we deal with the asymptotic behavior of the

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intermediate points in the mean value theorems for integrals as the involved interval shrinks to zero. For the reader's convenience we recall first the two mean value theorems for integrals.

Theorem 1.1 (first mean value theorem for integrals). *If $f:[a,b]\to\mathbb{R}$ is a continuous function and $g:[a,b]\to[0,\infty)$ is a nonnegative Riemann integrable function, then there is a number $c\in[a,b]$ such that*

$$\int_a^b f(t)g(t)dt = f(c)\int_a^b g(t)dt.$$

Corollary 1.2. *If $f:[a,b]\to\mathbb{R}$ is a continuous function, then there is a number $c\in[a,b]$ such that*

$$\int_a^b f(t)dt = f(c)(b-a).$$

Theorem 1.3 (second mean value theorem for integrals). *If $f:[a,b]\to\mathbb{R}$ is monotone and $g:[a,b]\to\mathbb{R}$ is Riemann integrable on $[a,b]$, then there is a number $c\in[a,b]$ such that*

$$\int_a^b f(t)g(t)dt = f(a)\int_a^c g(t)dt + f(b)\int_c^b g(t)dt.$$

The second mean value theorem for integrals is instrumental in theories like trigonometric series or Laplace transforms (see [8] for a proof and [11] for an interesting application of Theorem 1.3).

If $x\in(a,b)$, then Theorem 1.1, Corollary 1.2 and Theorem 1.3 applied to the interval $[a,x]$ instead of $[a,b]$ yield the existence of numbers $c_x\in[a,b]$ as functions of x on (a,b) such that

$$\int_a^x f(t)g(t)dt = f(c_x)\int_a^x g(t)dt, \tag{1.1}$$

$$\int_a^x f(t)dt = f(c_x)(x-a), \tag{1.2}$$

and

$$\int_a^x f(t)g(t)dt = f(a)\int_a^{c_x} g(t)dt + f(x)\int_{c_x}^x g(t)dt, \tag{1.3}$$

respectively.

Zhang [14, Theorem 4] proved that the point c_x in (1.2) satisfies

$$\lim_{x\to a}\frac{c_x-a}{x-a} = \frac{1}{\sqrt[n]{n+1}}, \tag{1.4}$$

provided that f is continuous on $[a,b]$ and n times differentiable at a with $f^{(j)}(a)=0$ $(1 \le j \le n-1)$ and $f^{(n)}(a) \ne 0$. In the special case when $n=1$, an earlier result obtained by Jacobson [7] is recovered.

In section 2 of our paper we obtain a formula which is similar to (1.4), but involves the asymptotic behavior of the point c_x in the mean value formula (1.1). The asymptotic behavior of the point c_x in the mean value formula (1.3) is investigated in section 3.

2. Asymptotic behavior of the intermediate point in the first mean value theorem for integrals

In the proofs of the main results in this and the next section we need the following

Lemma 2.1. *If p is a nonnegative integer and $\omega : [a,b] \to \mathbb{R}$ is a continuous function such that $\omega(t) \to 0$ as $t \searrow a$, then*

$$\int_a^x \omega(t)(t-a)^p dt = o((x-a)^{p+1}) \qquad (x \searrow a).$$

Proof. Indeed, for every $x \in (a,b)$ by Theorem 1.1 there exists $c_x \in [a,x]$ such that

$$\int_a^x \omega(t)(t-a)^p dt = \omega(c_x) \int_a^x (t-a)^p dt = \frac{\omega(c_x)}{p+1}(x-a)^{p+1}.$$

Since $\omega(c_x) \to 0$ as $x \searrow a$, we obtain the conclusion. □

Theorem 2.2. *Suppose that $f, g : [a,b] \to \mathbb{R}$ are two functions satisfying the following conditions:*

(i) *f is continuous on $[a,b]$ and there is a positive integer n such that f is n times differentiable at a with $f^{(j)}(a) = 0$ for $1 \le j \le n-1$ and $f^{(n)}(a) \ne 0$;*

(ii) *g is nonnegative, Riemann integrable on $[a,b]$ and there is a nonnegative integer k such that g is k times differentiable at a with $g^{(j)}(a) = 0$ for $0 \le j \le k-1$ and $g^{(k)}(a) \ne 0$.*

Then the point c_x *in* (1.1) *satisfies*

$$\lim_{x\to a}\frac{c_x-a}{x-a}=\sqrt[n]{\frac{k+1}{n+k+1}}\,. \tag{2.1}$$

Proof. Without loosing the generality we may assume that $f(a)=0$. Indeed, otherwise we replace f by the function $t\in[a,b]\mapsto f(t)-f(a)$. Note that if c_x satisfies (1.1), then c_x satisfies also

$$\int_a^x (f(t)-f(a))g(t)dt=(f(c_x)-f(a))\int_a^x g(t)dt.$$

By the Taylor expansions of f and g we have

$$\begin{aligned} f(t) &= \frac{f^{(n)}(a)}{n!}\,(t-a)^n+\omega(t)(t-a)^n,\\ g(t) &= \frac{g^{(k)}(a)}{k!}\,(t-a)^k+\varepsilon(t)(t-a)^k, \end{aligned}$$

where ω and ε are continuous functions on $[a,b]$ satisfying $\omega(t)\to 0$ and $\varepsilon(t)\to 0$ as $t\searrow a$. Therefore we have

$$f(t)g(t)=\frac{f^{(n)}(a)g^{(k)}(a)}{n!\,k!}\,(t-a)^{n+k}+\gamma(t)(t-a)^{n+k},$$

where γ is continuous on $[a,b]$ and $\gamma(t)\to 0$ as $t\searrow a$. By Lemma 2.1 we deduce that

$$\int_a^x f(t)g(t)dt=\frac{f^{(n)}(a)g^{(k)}(a)}{n!\,k!\,(n+k+1)}\,(x-a)^{n+k+1}+o((x-a)^{n+k+1}) \tag{2.2}$$

as $x\searrow a$. By Lemma 2.1 we have also

$$\int_a^x g(t)dt=\frac{g^{(k)}(a)}{(k+1)!}\,(x-a)^{k+1}+o((x-a)^{k+1}) \qquad (x\searrow a).$$

Since

$$f(c_x)=\frac{f^{(n)}(a)}{n!}\,(c_x-a)^n+\omega(c_x)(c_x-a)^n$$

and $0\le c_x-a\le x-a$, it follows that

$$f(c_x)\int_a^x g(t)dt=\frac{f^{(n)}(a)g^{(k)}(a)}{n!\,(k+1)!}\,(x-a)^{k+1}(c_x-a)^n+o((x-a)^{n+k+1}) \tag{2.3}$$

as $x \searrow a$. By (1.1), (2.2) and (2.3) we conclude that

$$\begin{aligned}\frac{f^{(n)}(a)g^{(k)}(a)}{n!\,(k+1)!}\,&(x-a)^{k+1}(c_x-a)^n \\ &= \frac{f^{(n)}(a)g^{(k)}(a)}{n!\,k!\,(n+k+1)}\,(x-a)^{n+k+1}+o((x-a)^{n+k+1}) \qquad (x\searrow a).\end{aligned}$$

Multiplying both sides by $n!\,(k+1)!(x-a)^{-(n+k+1)}/(f^{(n)}(a)g^{(k)}(a))$ we get

$$\left(\frac{c_x-a}{x-a}\right)^n = \frac{k+1}{n+k+1}+o(1) \qquad (x\searrow a),$$

whence the conclusion (2.1). □

Note that if $g(t) = 1$ for all $t \in [a,b]$, then (ii) is satisfied for $k = 0$. In this case (1.1) becomes (1.2) and (2.1) becomes (1.4), i.e., we recover Zhang's result mentioned in the introduction as a special case of Theorem 2.2.

3. Asymptotic behavior of the intermediate point in the second mean value theorem for integrals

Theorem 3.1. *Suppose that $f, g : [a,b] \to \mathbb{R}$ are two functions satisfying the following conditions:*

(i) *f is monotone and there is a positive integer n such that f is n times differentiable at a with $f^{(j)}(a) = 0$ for $1 \le j \le n-1$ and $f^{(n)}(a) \neq 0$;*

(ii) *g is Riemann integrable on $[a,b]$ and there is a nonnegative integer k such that g is k times differentiable at a with $g^{(j)}(a) = 0$ for $0 \le j \le k-1$ and $g^{(k)}(a) \neq 0$.*

Then the point c_x in (1.3) *satisfies*

$$\lim_{x\to a}\frac{c_x-a}{x-a} = \sqrt[k+1]{\frac{n}{n+k+1}}\,.$$

Proof. Note that (1.3) is equivalent to

$$\int_a^x (f(t)-f(a))g(t)dt = (f(x)-f(a))\int_{c_x}^x g(t)dt.$$

So, without loosing the generality we may assume that $f(a) = 0$ (otherwise we replace f by the function $t \in [a, b] \mapsto f(t) - f(a)$). Under the assumption that $f(a) = 0$ equality (1.3) becomes

$$\int_a^x f(t)g(t)dt = f(x) \int_{c_x}^x g(t)dt. \tag{3.1}$$

By using the Taylor expansions of f and g and proceeding as in the proof of Theorem 2.2 we deduce that (2.2) holds and that

$$\begin{aligned} f(x) \int_{c_x}^x g(t)dt &= \frac{f^{(n)}(a)g^{(k)}(a)}{n!\,(k+1)!}(x-a)^n \left[(x-a)^{k+1} - (c_x - a)^{k+1}\right] \\ &\quad + o((x-a)^{n+k+1}) \qquad (x \searrow a). \end{aligned} \tag{3.2}$$

By (3.1), (2.2) and (3.2) we conclude that

$$\begin{aligned} &\frac{f^{(n)}(a)g^{(k)}(a)}{n!\,(k+1)!}(x-a)^n \left[(x-a)^{k+1} - (c_x - a)^{k+1}\right] \\ &\quad = \frac{f^{(n)}(a)g^{(k)}(a)}{n!\,k!\,(n+k+1)}(x-a)^{n+k+1} + o((x-a)^{n+k+1}) \qquad (x \searrow a). \end{aligned}$$

Multiplying both sides by $n!\,(k+1)!(x-a)^{-(n+k+1)}/(f^{(n)}(a)g^{(k)}(a))$ we get

$$1 - \left(\frac{c_x - a}{x - a}\right)^{k+1} = \frac{k+1}{n+k+1} + o(1) \qquad (x \searrow a),$$

whence the conclusion. □

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