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ON STARLIKENESS OF A CLASS OF INTEGRAL OPERATORS FOR MEROMORPHIC STARLIKE FUNCTIONS

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Dedicated to Professor Grigore Ştefan Sălăgean on his 60^{th} *birthday*

Abstract. Let M_0 be the class of meromorphic functions in $\dot{U}$ of the form $g(z) = \frac{1}{z} + \alpha_0 + \alpha_1 z + \cdots$, $z \in \dot{U}$. For $\Phi, \varphi \in H[1,1]$, $\Phi(z)\varphi(z) \neq 0$, $z \in U$, $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $\beta \neq 0$ and $g \in M_0$, we consider the integral operator $J^{\Phi,\varphi}_{\alpha,\beta,\gamma,\delta} : K \subset M_0 \to M_0$ defined by

$$J^{\Phi,\varphi}_{\alpha,\beta,\gamma,\delta}(g)(z) = \left[\frac{\gamma-\beta}{z^\gamma \Phi(z)} \int_0^z g^\alpha(t)\varphi(t)t^{\delta-1}dt\right]^{\frac{1}{\beta}}, \ z \in \dot{U}.$$

The first result of this paper gives us the conditions for which $J^{\Phi,\varphi}_{\alpha,\beta,\gamma,\delta}$ will be well-defined. Furthermore, we study the properties of a function $G = J_{\beta,\gamma}(g)$, where $J_{\beta,\gamma} = J^{1,1}_{\beta,\beta,\gamma,\gamma}$, when $g \in M_0^*(\alpha,\delta)$. For the second result we consider $\beta < 0$, $\gamma - \beta > 0$, $\alpha \in [\alpha_0, 1)$, where $\alpha_0 = \max\left\{\frac{\beta+\gamma+1}{2\beta}, \frac{\gamma}{\beta}\right\}$ and we find the order of starlikeness of the class $J_{\beta,\gamma}(M_0^*(\alpha))$. For the third result we consider $0 \le \alpha < 1$, $0 < \beta < \gamma$ and we find some conditions for α, β, γ and $\delta = \delta(\alpha,\beta,\gamma)$ such that

$$J_{\beta,\gamma}[M_0^*(\alpha) \cap K_{\beta,\gamma}] \subset M_0^*(\delta).$$

1. Introduction and preliminaries

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc in the complex plane, $\dot{U} = U \setminus \{0\}$ and $H(U) = \{f : U \to \mathbb{C} : f \text{ is holomorphic in } U\}$.

We will also use the following notations:

$$H[a,n] = \{f \in H(U) : f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \ldots\} \text{ for } a \in \mathbb{C},\ n \in \mathbb{N}^*,$$

$$A_n = \{f \in H(U) : f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \ldots\},\ n \in \mathbb{N}^*,$$

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and for $n = 1$ we denote A_1 by A and this set is called *the class of analytic functions normalized at the origin*.

Let S^* be the class of normalized starlike functions on U, i.e.

$$S^* = \left\{ f \in A : \operatorname{Re} \frac{zf'(z)}{f(z)} > 0,\ z \in U \right\}.$$

We denote by M_0 the class of meromorphic functions in $\dot{U}$ of the form

$$g(z) = \frac{1}{z} + \alpha_0 + \alpha_1 z + \cdots,\ z \in \dot{U}.$$

Let

$$M_0^* = \left\{ g \in M_0 : \operatorname{Re} \left[-\frac{zg'(z)}{g(z)} \right] > 0,\ z \in U \right\}$$

be called the class of meromorphic starlike functions in $\dot{U}$.

We note that if f is a normalized starlike function in U, then the function $g = \dfrac{1}{f}$ belongs to the class M_0^*.

For $\alpha < 1$, $\delta > 1$ let

$$M_0^*(\alpha) = \left\{ g \in M_0 : \operatorname{Re} \left[-\frac{zg'(z)}{g(z)} \right] > \alpha,\ z \in U \right\},$$

$$M_0^*(\alpha, \delta) = \left\{ g \in M_0 : \alpha < \operatorname{Re} \left[-\frac{zg'(z)}{g(z)} \right] < \delta,\ z \in U \right\}.$$

Definition 1.1. [3, p.4], [4, p.45] Let $f, g \in H(U)$. We say that the function f is subordinate to the function g, and we denote this by $f(z) \prec g(z)$, if there is a function $w \in H(U)$, with $w(0) = 0$ and $|w(z)| < 1$, $z \in U$, such that

$$f(z) = g[w(z)],\ z \in U.$$

Remark 1.2. If $f(z) \prec g(z)$, then $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

Theorem 1.3. [3, p.4], [4, p.46] *Let $f, g \in H(U)$ and let g be a univalent function in U. Then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.*

Definition 1.4. [3, p. 46], [4, p.228] Let $c \in \mathbb{C}$ with $\operatorname{Re} c > 0$ and $n \in \mathbb{N}^*$. We consider

$$C_n = C_n(c) = \frac{n}{\operatorname{Re} c} \left[|c| \sqrt{1 + \frac{2 \operatorname{Re} c}{n}} + \operatorname{Im} c \right].$$

If the univalent function $R: U \to \mathbb{C}$ is given by $R(z) = \dfrac{2C_n z}{1-z^2}$, then we will denote by $R_{c,n}$ the "Open Door" function, defined as

$$R_{c,n}(z) = R\left(\frac{z+b}{1+\bar{b}z}\right) = 2C_n \frac{(z+b)(1+\bar{b}z)}{(1+\bar{b}z)^2-(z+b)^2} \ ,$$

where $b = R^{-1}(c)$.

Theorem 1.5. [3, Theorem 2.5c.] *Let $\Phi, \varphi \in H[1,n]$ with $\Phi(z) \neq 0, \varphi(z) \neq 0$, for $z \in U$. Let $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $\beta \neq 0$, $\alpha + \delta = \beta + \gamma$ and $\mathrm{Re}\,(\alpha+\delta) > 0$. Let the function $f(z) = z + a_{n+1}z^{n+1} + \cdots \in A_n$ and suppose that*

$$\alpha \frac{zf'(z)}{f(z)} + \frac{z\varphi'(z)}{\varphi(z)} + \delta \prec R_{\alpha+\delta,n}(z).$$

If $F = I^{\Phi,\varphi}_{\alpha,\beta,\gamma,\delta}(f)$ is defined by

$$F(z) = I^{\Phi,\varphi}_{\alpha,\beta,\gamma,\delta}(f)(z) = \left[\frac{\beta+\gamma}{z^\gamma \Phi(z)} \int_0^z f^\alpha(t)\varphi(t)t^{\delta-1}dt\right]^{\frac{1}{\beta}}, \qquad (1.1)$$

then $F \in A_n$ with $\dfrac{F(z)}{z} \neq 0$, $z \in U$, and

$$\mathrm{Re}\left[\beta \frac{zF'(z)}{F(z)} + \frac{z\Phi'(z)}{\Phi(z)} + \gamma\right] > 0,\ z \in U.$$

All powers in (1.1) are principal ones.

Lemma 1.6. [3, Theorem 2.3i.], [4, p.209] *Let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ be a function that satisfies the condition*

$$\mathrm{Re}\,\psi(\rho i, \sigma; z) \leq 0\,, \qquad (1.2)$$

when $\rho, \sigma \in \mathbb{R}, \sigma \leq -\dfrac{n}{2}(1+\rho^2)$, $z \in U, n \geq 1$.

If $p \in H[1,n]$ and

$$\mathrm{Re}\,\psi(p(z), zp'(z); z) > 0, \quad z \in U,$$

then

$$\mathrm{Re}\,p(z) > 0, \quad z \in U.$$

Theorem 1.7. [3, Theorem 3.2a.], [4, p.247] *Let $\beta, \gamma \in \mathbb{C}$, $\beta \neq 0$ and let h be a convex function on U such that $\mathrm{Re}\,[\beta h(z) + \gamma] > 0$, $z \in U$. If $p \in H[h(0), n]$ and*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z),$$

then $p(z) \prec h(z)$.

Theorem 1.8. [5], [4, p.299]**(the order of starlikeness of the class** $I_{\beta,\gamma}(S^*(\alpha))$**)** *Let* $\beta > 0$, $\gamma + \beta > 0$ *and consider the integral operator* $I_{\beta,\gamma}$ *defined by*

$$I_{\beta,\gamma}(f)(z) = \left[\frac{\gamma+\beta}{z^\gamma}\int_0^z t^{\gamma-1}f^\beta(t)dt\right]^{\frac{1}{\beta}}.$$

If $\alpha \in [\alpha_0, 1)$ *where* $\alpha_0 = \max\left\{\frac{\beta-\gamma-1}{2\beta}, -\frac{\gamma}{\beta}\right\}$, *then the order of starlikeness of the class* $I_{\beta,\gamma}(S^*(\alpha))$ *is given by*

$$\delta(\alpha;\beta,\gamma) = \frac{1}{\beta}\left[\frac{\gamma+\beta}{{}_2F_1(1, 2\beta(1-\alpha), \gamma+1+\beta; \frac{1}{2})} - \gamma\right],$$

where ${}_2F_1$ *represents the hypergeometric function.*

2. Main results

Let $\Phi, \varphi \in H[1,1]$ with $\Phi(z)\varphi(z) \neq 0$, $z \in U$ and let $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $\beta \neq 0$. The first result of this section is a corollary of Theorem 1.5 and gives us the conditions for which the integral operator $J^{\Phi,\varphi}_{\alpha,\beta,\gamma,\delta} : K \subset M_0 \to M_0$,

$$J^{\Phi,\varphi}_{\alpha,\beta,\gamma,\delta}(g)(z) = \left[\frac{\gamma-\beta}{z^\gamma\Phi(z)}\int_0^z g^\alpha(t)\varphi(t)t^{\delta-1}dt\right]^{\frac{1}{\beta}},$$

is well-defined.

Theorem 2.1. *Let* $\Phi, \varphi \in H[1,1]$ *with* $\Phi(z)\varphi(z) \neq 0$, $z \in U$. *Let* $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ *with* $\beta \neq 0$, $\alpha + \gamma = \beta + \delta$ *and* $\mathrm{Re}\,(\gamma - \beta) > 0$. *If* $g \in M_0$ *and*

$$\alpha\frac{zg'(z)}{g(z)} + \frac{z\varphi'(z)}{\varphi(z)} + \delta \prec R_{\delta-\alpha,1}(z), \tag{2.1}$$

then

$$G(z) = J^{\Phi,\varphi}_{\alpha,\beta,\gamma,\delta}(g)(z) = \left[\frac{\gamma-\beta}{z^\gamma\Phi(z)}\int_0^z g^\alpha(t)\varphi(t)t^{\delta-1}dt\right]^{\frac{1}{\beta}} \in M_0,$$

with $zG(z) \neq 0$, $z \in U$, *and*

$$\mathrm{Re}\left[\beta\frac{zG'(z)}{G(z)} + \frac{z\Phi'(z)}{\Phi(z)} + \gamma\right] > 0,\ z \in U.$$

All powers are chosen as principal ones.

Proof. We denote $\alpha_1 = -\alpha$, $\beta_1 = -\beta$, so we have $\gamma+\beta_1 = \delta+\alpha_1$, and $\mathrm{Re}\,(\gamma+\beta_1) > 0$. We remark that from (2.1) we have $zg(z) \neq 0$, $z \in U$.

We know that $g \in M_0$ with $zg(z) \neq 0$, $z \in U$, if and only if $f = \dfrac{1}{g} \in A_1$ with $\dfrac{f(z)}{z} \neq 0$, $z \in U$. It is also easy to see that $\dfrac{zg'(z)}{g(z)} = -\dfrac{zf'(z)}{f(z)}$, $z \in U$.

Using these new notations we obtain

$$\alpha_1 \frac{zf'(z)}{f(z)} + \frac{z\varphi'(z)}{\varphi(z)} + \delta \prec R_{\delta+\alpha_1,1}(z),\ z \in U,$$

and applying Theorem 1.5 we have

$$F(z) = I^{\Phi,\varphi}_{\alpha_1,\beta_1,\gamma,\delta}(f)(z) = \left[\frac{\beta_1+\gamma}{z^\gamma \Phi(z)} \int_0^z f^{\alpha_1}(t)\varphi(t)t^{\delta-1}dt\right]^{\frac{1}{\beta_1}} \in A_1,$$

with $\dfrac{F(z)}{z} \neq 0$, $z \in U$, and

$$\mathrm{Re}\left[\beta_1 \frac{zF'(z)}{F(z)} + \frac{z\Phi'(z)}{\Phi(z)} + \gamma\right] > 0,\ z \in U.$$

Therefore, we have $G(z) = \dfrac{1}{F(z)} \in M_0$ with $zG(z) \neq 0$ and, because

$$\frac{zG'(z)}{G(z)} = -\frac{zF'(z)}{F(z)},\ z \in U,$$

we also have

$$\mathrm{Re}\left[\beta \frac{zG'(z)}{G(z)} + \frac{z\Phi'(z)}{\Phi(z)} + \gamma\right] > 0,\ z \in U.$$

□

We next consider a special case of Theorem 2.1. If we let $\Phi \equiv \varphi \equiv 1$, $\alpha = \beta$, $\gamma = \delta$ and if we use the notation $J_{\beta,\gamma}$ instead of $J^{1,1}_{\beta,\beta,\gamma,\gamma}$, we obtain:

Corollary 2.2. *Let $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$ and $\mathrm{Re}\,(\gamma - \beta) > 0$. If $g \in M_0$ and*

$$\beta \frac{zg'(z)}{g(z)} + \gamma \prec R_{\gamma-\beta,1}(z),$$

then

$$G(z) = J_{\beta,\gamma}(g)(z) = \left[\frac{\gamma-\beta}{z^\gamma} \int_0^z g^\beta(t)t^{\gamma-1}dt\right]^{\frac{1}{\beta}} \in M_0, \tag{2.2}$$

with $zG(z) \neq 0$, $z \in U$, and

$$\mathrm{Re}\left[\beta \frac{zG'(z)}{G(z)} + \gamma\right] > 0,\ z \in U.$$

Remark 2.3. 1. Let us define the classes $K_{\beta,\gamma}$ as

$$K_{\beta,\gamma}=\left\{g\in M_0:\gamma+\beta\frac{zg'(z)}{g(z)}\prec R_{\gamma-\beta,1}(z),\ z\in U\right\}.$$

From Corollary 2.2, we have $J_{\beta,\gamma}:K_{\beta,\gamma}\to M_0$ with $zJ_{\beta,\gamma}(g)(z)\neq 0$, $z\in U$, and

$$\operatorname{Re}\left[\gamma+\beta\frac{zJ'_{\beta,\gamma}(g)(z)}{J_{\beta,\gamma}(g)(z)}\right]>0,\ z\in U.$$

2. We denote

$$\tilde{K}_{\beta,\gamma}=\left\{g\in M_0:\operatorname{Re}\left[\gamma+\beta\frac{zg'(z)}{g(z)}\right]>0,\ z\in U\right\}.$$

Using the above corollary we have $J_{\beta,\gamma}(K_{\beta,\gamma})\subset\tilde{K}_{\beta,\gamma}$, so $J_{\beta,\gamma}(\tilde{K}_{\beta,\gamma})\subset\tilde{K}_{\beta,\gamma}$, where $\beta,\ \gamma\in\mathbb{C}$ with $\beta\neq 0$ and $\operatorname{Re}(\gamma-\beta)>0$.

3. Let $\beta<0$, $\gamma\in\mathbb{C}$ with $\operatorname{Re}\gamma>\beta$ and $\frac{\operatorname{Re}\gamma}{\beta}\leq\alpha<1$. Then, from $J_{\beta,\gamma}(\tilde{K}_{\beta,\gamma})\subset\tilde{K}_{\beta,\gamma}$, we deduce $J_{\beta,\gamma}(M_0^*(\alpha))\subset M_0^*\left(\frac{\operatorname{Re}\gamma}{\beta}\right)$.

It's easy to see that from

$$G(z)=\left[\frac{\gamma-\beta}{z^\gamma}\int_0^z t^{\gamma-1}g^\beta(t)dt\right]^{\frac{1}{\beta}},\ z\in\dot{U},$$

we obtain

$$p(z)+\frac{zp'(z)}{\gamma-\beta p(z)}=-\frac{zg'(z)}{g(z)},\quad\text{where}\quad p(z)=-\frac{zG'(z)}{G(z)},\ z\in U. \tag{2.3}$$

Next we will study the properties of the image of a function $g\in M_0^*(\alpha,\delta)$ through the integral operator $J_{\beta,\gamma}$ defined by (2.2).

Theorem 2.4. *Let* $\beta>0$, $\gamma\in\mathbb{C}$ *and* $0\leq\alpha<1<\delta\leq\frac{\operatorname{Re}\gamma}{\beta}$.
If $g\in M_0^*(\alpha,\delta)$, *then* $G=J_{\beta,\gamma}(g)\in M_0^*(\alpha,\delta)$.

Proof. We know that $g\in M_0^*(\alpha,\delta)$ is equivalent to

$$\alpha<\operatorname{Re}\left[-\frac{zg'(z)}{g(z)}\right]<\delta,\ z\in U,$$

so,

$$\operatorname{Re}\gamma-\beta\delta<\operatorname{Re}\left[\gamma+\beta\frac{zg'(z)}{g(z)}\right]<\operatorname{Re}\gamma-\beta\alpha,\ z\in U,\quad\text{when}\quad\beta>0.$$

Because $\delta\leq\frac{\operatorname{Re}\gamma}{\beta}$ we get $\operatorname{Re}\left[\gamma+\beta\frac{zg'(z)}{g(z)}\right]>0$, $z\in U$, and using Corollary 2.2, we obtain that $G=J_{\beta,\gamma}(g)\in M_0$, $zG(z)\neq 0$, $z\in U$, and $\operatorname{Re}\left[\gamma+\beta\frac{zG'(z)}{G(z)}\right]>0$, $z\in U$.

From (2.3) we know that

$$p(z)+\frac{zp'(z)}{\gamma-\beta p(z)}=-\frac{zg'(z)}{g(z)},\quad \text{where}\quad p(z)=-\frac{zG'(z)}{G(z)}.$$

Since $G\in M_0$ with $zG(z)\neq 0$, $z\in U$, we have $p(z)=-\frac{zG'(z)}{G(z)}\in H[1,1]$.
It's not difficult to see that there is a convex function q on U such that $q(U)=\{z\in \mathbb{C}:\alpha<\operatorname{Re} z<\delta\}$ and $q(0)=1$, so

$$g\in M_0^*(\alpha,\delta)\Rightarrow -\frac{zg'(z)}{g(z)}\prec q(z).$$

Now we have

$$p(z)+\frac{zp'(z)}{\gamma-\beta p(z)}\prec q(z)\,,\text{with } q \text{ convex on } U,\ q(0)=1.$$

We want to apply Theorem 1.7 to the above differential subordination, so we need to see that $\operatorname{Re}[\gamma-\beta q(z)]>0$, $z\in U$.
Since $\beta>0$, we obtain from $\alpha<\operatorname{Re} q(z)<\delta$, $z\in U$, that

$$\operatorname{Re}\gamma-\beta\delta<\operatorname{Re}[\gamma-\beta q(z)]<\operatorname{Re}\gamma-\beta\alpha,\ z\in U.$$

Because $\delta\le\frac{\operatorname{Re}\gamma}{\beta}$ we have $\operatorname{Re}[\gamma-\beta q(z)]>0$, $z\in U$, and using Theorem 1.7 we obtain $p(z)\prec q(z)$, which is equivalent to

$$-\frac{zG'(z)}{G(z)}\prec q(z),\ z\in U. \tag{2.4}$$

Since $G\in M_0$, we get from (2.4) that $G\in M_0^*(\alpha,\delta)$. □

Taking $\beta=1$ in the above theorem we obtain:

Corollary 2.5. *Let $\gamma\in\mathbb{C}$ and $0\le\alpha<1<\delta\le\operatorname{Re}\gamma$. If $g\in M_0^*(\alpha,\delta)$, then*

$$G=J_{1,\gamma}(g)=\frac{\gamma-1}{z^\gamma}\int_0^z t^{\gamma-1}g(t)dt\in M_0^*(\alpha,\delta).$$

Theorem 2.6. *Let $\beta<0$, $\gamma\in\mathbb{C}$ and $\frac{\operatorname{Re}\gamma}{\beta}\le\alpha<1<\delta$.*
If $g\in M_0^*(\alpha,\delta)$, then $G=J_{\beta,\gamma}\in M_0^*(\alpha,\delta)$.

Proof. From Remark 2.3 item 3., we have $J_{\beta,\gamma}(M_0^*(\alpha))\subset M_0^*\left(\frac{\operatorname{Re}\gamma}{\beta}\right)$, hence $G=J_{\beta,\gamma}(g)\in M_0^*\left(\frac{\operatorname{Re}\gamma}{\beta}\right)$. Since $G\in M_0^*\left(\frac{\operatorname{Re}\gamma}{\beta}\right)$, we have $G\in M_0$ and $zG(z)\neq 0$, $z\in U$, so $-\frac{zG'(z)}{G(z)}\in H[1,1]$.

Because $g \in M_0^*(\alpha, \delta)$ and

$$p(z) + \frac{zp'(z)}{\gamma - \beta p(z)} = -\frac{zg'(z)}{g(z)}, \quad \text{where} \quad p(z) = -\frac{zG'(z)}{G(z)},$$

we will use the same idea as at the proof of Theorem 2.4. So, we have to see that $\mathrm{Re}\,[\gamma - \beta q(z)] > 0$, $z \in U$, where q is convex on U, $q(0) = 1$, $q(U) = \{z \in \mathbb{C} : \alpha < \mathrm{Re}\, z < \delta\}$.

From $\mathrm{Re}\, q(z) > \alpha$, $z \in U$, we obtain $\mathrm{Re}\,\gamma - \beta \mathrm{Re}\, q(z) > \mathrm{Re}\,\gamma - \alpha\beta \geq 0$, $z \in U$, when $\alpha \geq \dfrac{\mathrm{Re}\,\gamma}{\beta}$, $\beta < 0$.

Applying Theorem 1.7 to the differential subordination

$$p(z) + \frac{zp'(z)}{\gamma - \beta p(z)} \prec q(z),\ z \in U,$$

we obtain $p(z) \prec q(z)$, which is equivalent to

$$-\frac{zG'(z)}{G(z)} \prec q(z),\ z \in U. \tag{2.5}$$

Since $G \in M_0$, we get from (2.5) that $G \in M_0^*(\alpha, \delta)$. □

Remark 2.7. If we consider $\delta \to \infty$ in the above theorem, we obtain that for $\beta < 0$, $\gamma \in \mathbb{C}$, $\beta < \mathrm{Re}\,\gamma$ and $\dfrac{\mathrm{Re}\,\gamma}{\beta} \leq \alpha < 1$,

$$g \in M_0^*(\alpha) \Rightarrow G = J_{\beta,\gamma}(g) \in M_0^*(\alpha).$$

Definition 2.8. For a given number $\alpha \in \left[\dfrac{\mathrm{Re}\,\gamma}{\beta}, 1\right)$, where $\beta < 0$, $\gamma \in \mathbb{C}$, $\beta < \mathrm{Re}\,\gamma$, we define the order of starlikeness of the class $J_{\beta,\gamma}(M_0^*(\alpha))$ as the biggest number $\mu = \mu(\alpha; \beta, \gamma)$ such that $J_{\beta,\gamma}(M_0^*(\alpha)) \subset M_0^*(\mu)$.

Theorem 2.9. (the order of starlikeness of the class $J_{\beta,\gamma}(M_0^*(\alpha))$) *Let $\beta < 0$, $\gamma - \beta > 0$ and let $J_{\beta,\gamma}$ be given by (2.2). If $\alpha \in [\alpha_0, 1)$, where $\alpha_0 = \max\left\{\dfrac{\beta + \gamma + 1}{2\beta}, \dfrac{\gamma}{\beta}\right\}$, then the order of starlikeness of the class $J_{\beta,\gamma}(M_0^*(\alpha))$ is given by*

$$\mu(\alpha; \beta, \gamma) = -\frac{1}{\beta}\left[\frac{\gamma - \beta}{{}_2F_1(1, 2\beta(\alpha - 1), \gamma + 1 - \beta; \frac{1}{2})} - \gamma\right],$$

where ${}_2F_1$ represents the hypergeometric function.

Proof. We know that if $g \in M_0$ with $zg(z) \neq 0$, $z \in U$, then $\frac{1}{g} \in A$.
It's not difficult to see that

$$J_{\beta,\gamma}(g) = \frac{1}{I_{-\beta,\gamma}\left(\frac{1}{g}\right)},\ \beta < 0,\ g \in M_0^*(\alpha).$$

Using the fact that $g \in M_0^*(\alpha)$ is equivalent to $\frac{1}{g} \in S^*(\alpha)$, we obtain from Theorem 1.8 that

$$I_{-\beta,\gamma}(S^*(\alpha)) \subset S^*(\delta(\alpha; -\beta, \gamma)),$$

so

$$J_{\beta,\gamma}(M_0^*(\alpha)) \subset M_0^*(\delta(\alpha; -\beta, \gamma)).$$

It's easy to prove that $\delta(\alpha; -\beta, \gamma)$ is the largest number μ such that $J_{\beta,\gamma}(M_0^*(\alpha)) \subset M_0^*(\mu)$, so the order of starlikeness of the class $J_{\beta,\gamma}(M_0^*(\alpha)))$ is $\mu(\alpha; \beta, \gamma) = \delta(\alpha; -\beta, \gamma)$. □

Further we will find some conditions for α, β, γ and $\delta = \delta(\alpha, \beta, \gamma)$ such that

$$J_{\beta,\gamma}[M_0^*(\alpha) \cap K_{\beta,\gamma}] \subset M_0^*(\delta).$$

Theorem 2.10. *Let* $0 \leq \alpha < 1$ *and* $0 < \beta < \gamma$. *Let's denote*

$$\begin{aligned}
\beta_1(\alpha,\gamma) &= \frac{2\sqrt{2\gamma(\alpha-1)^2+\alpha}-\alpha-1}{2(\alpha-1)^2},\\
\delta_1(\alpha,\beta,\gamma) &= \frac{2\alpha\beta+2\gamma+1-\sqrt{(1+2\alpha\beta-2\gamma)^2+8(\gamma-\beta)}}{4\beta},\\
\delta_2(\alpha,\beta,\gamma) &= \frac{2\alpha\beta+2\beta+1-\sqrt{(1+2\alpha\beta-2\beta)^2+8(\beta-\gamma)}}{4\beta}.
\end{aligned}$$

If $\gamma > \frac{1}{8}$ *and* $\beta < \beta_1(\alpha, \gamma)$, *then* $J_{\beta,\gamma}[M_0^*(\alpha) \cap K_{\beta,\gamma}] \subset M_0^*(\delta_1(\alpha, \beta, \gamma))$.
If $\gamma \leq \frac{1}{8}$ *or* $\begin{cases} \gamma > \frac{1}{8} \\ \beta \geq \beta_1(\alpha,\gamma) \end{cases}$, *then* $J_{\beta,\gamma}[M_0^*(\alpha) \cap K_{\beta,\gamma}] \subset M_0^*(\delta(\alpha, \beta, \gamma))$, *where*

$$\delta(\alpha,\beta,\gamma) = \min\{\delta_1(\alpha,\beta,\gamma), \delta_2(\alpha,\beta,\gamma)\}. \tag{2.6}$$

The operator $J_{\beta,\gamma}$ *is defined by (2.2).*

Proof. We remark that $\beta_1(\alpha, \gamma)$ is a real number and it is the greatest root for the equation

$$\Delta_2 = (1+2\alpha\beta-2\beta)^2+8(\beta-\gamma) = 4(\alpha-1)^2\beta^2+4\beta(\alpha+1)+1-8\gamma = 0,$$

hence $\Delta_2 \geq 0$, when $\beta \geq \beta_1(\alpha, \gamma)$.

It's not difficult to see that

$$\beta_1(\alpha, \gamma) \geq 0 \Leftrightarrow (8\gamma - 1)(\alpha - 1)^2 \geq 0 \Leftrightarrow \gamma \geq \frac{1}{8}.$$

We next verify that the number $\delta_1(\alpha, \beta, \gamma)$ is less than 1. It's obvious that $\delta_1(\alpha, \beta, \gamma)$ is a real number since $\gamma - \beta > 0$. Further we will use the notation δ_1 instead of $\delta_1(\alpha, \beta, \gamma)$.

We have $\delta_1 < 1$ if and only if

$$2\alpha\beta + 2\gamma + 1 - 4\beta < \sqrt{(1 + 2\alpha\beta - 2\gamma)^2 + 8(\gamma - \beta)}. \tag{2.7}$$

If $2\alpha\beta + 2\gamma + 1 - 4\beta < 0$ then the inequality (2.7) is fulfilled.
If $2\alpha\beta + 2\gamma + 1 - 4\beta \geq 0$, we use the square of the inequality (2.7) and after a simple computation, we obtain that (2.7) is equivalent to $(\beta - \gamma)(1 - \alpha) < 0$ which is true for $\beta < \gamma$ and $\alpha \in [0, 1)$. Thus, we have $\delta_1 < 1$.
Since $g \in K_{\beta,\gamma}$, with $\beta < \gamma$, we have from Corollary 2.2 that $zG(z) = zJ_{\beta,\gamma}(g)(z) \neq 0$, $z \in U$. Now let us put

$$-\frac{zG'(z)}{G(z)} = (1 - \delta)p(z) + \delta,\ z \in U, \tag{2.8}$$

where $p \in H(U)$ with $p(0) = 1$ and $\delta < 1$. We remark that the function p also depends on δ.
Using (2.8) and the logarithmic differential for (2.2), we obtain

$$-\frac{zg'(z)}{g(z)} - \alpha = (1 - \delta)p(z) + \delta - \alpha + \frac{(1 - \delta)zp'(z)}{\gamma - \beta\delta - (1 - \delta)\beta p(z)},\ z \in U.$$

Let us denote

$$\psi(p(z), zp'(z); z) = (1 - \delta)p(z) + \delta - \alpha + \frac{(1 - \delta)zp'(z)}{\gamma - \beta\delta - (1 - \delta)\beta p(z)},\ z \in U.$$

Since $g \in M_0^*(\alpha)$, we have $\mathrm{Re}\left[-\frac{zg'(z)}{g(z)}\right] > \alpha$, so

$$\mathrm{Re}\, \psi(p(z), zp'(z); z) > 0,\ z \in U.$$

To be able to use Lemma 1.6 we need to verify the condition (1.2) for $n = 1$.
For $\rho \in \mathbb{R}$, $z \in U$ and $\sigma \leq -\frac{1}{2}(1 + \rho^2)$, we have

$$\mathrm{Re}\, \psi(i\rho, \sigma; z) = \delta - \alpha + (1 - \delta)\sigma \mathrm{Re}\, \frac{1}{\gamma - \beta\delta - (1 - \delta)\beta\rho i} = \tag{2.9}$$

$$= \delta - \alpha + \frac{(\gamma - \beta\delta)(1-\delta)\sigma}{(\gamma - \beta\delta)^2 + (1-\delta)^2\beta^2\rho^2}.$$

Because $(\gamma - \beta\delta)(1-\delta) > 0$ and $\sigma \le -\frac{1}{2}(1+\rho^2)$, we obtain from (2.9) that

$$\mathrm{Re}\,\psi(i\rho, \sigma; z) \le \delta - \alpha - \frac{(\gamma - \beta\delta)(1-\delta)}{2[(\gamma - \beta\delta)^2 + (1-\delta)^2\beta^2\rho^2]}.$$

Thus,

$$\mathrm{Re}\,\psi(i\rho, \sigma; z) \le -\frac{1}{D}(A + B\rho^2),\ \rho \in \mathbb{R},$$

where

$$A = (\gamma - \beta\delta)[2\beta\delta^2 - (1 + 2\gamma + 2\alpha\beta)\delta + 2\alpha\gamma + 1],$$

$$B = (1-\delta)[2\beta^2\delta^2 - \beta(1 + 2\beta + 2\alpha\beta)\delta + 2\alpha\beta^2 + \gamma],$$

$$D = 2[(\gamma - \beta\delta)^2 + (1-\delta)^2\beta^2\rho^2] > 0.$$

If $\gamma > \frac{1}{8}$ and $0 < \beta < \beta_1(\alpha, \gamma)$, then $\Delta_2 < 0$, so $B > 0$ for every $\delta \in \mathbb{R}$. Moreover, since $\beta > 0$, we have $A \ge 0$ when $\delta \le \delta_1(\alpha, \beta, \gamma)$. Hence, the condition (1.2) is satisfied for $\delta \le \delta_1(\alpha, \beta, \gamma) < 1$ and applying Lemma 1.6 we obtain $\mathrm{Re}\,p(z) > 0$, $z \in U$, when $\delta \le \delta_1(\alpha, \beta, \gamma)$.

From (2.8) and $\mathrm{Re}\,p(z) > 0$, $z \in U$, when $\delta \le \delta_1(\alpha, \beta, \gamma)$, we get $G \in M_0^*(\delta_1(\alpha, \beta, \gamma))$.

If $\gamma \le \frac{1}{8}$ or $\begin{cases} \gamma > \frac{1}{8} \\ \beta \ge \beta_1(\alpha, \gamma) \end{cases}$ and $\delta \le \delta(\alpha, \beta, \gamma)$, where $\delta(\alpha, \beta, \gamma)$ is given by (2.6), then $A \ge 0$ and $B \ge 0$, therefore the condition (1.2) is satisfied. Applying Lemma 1.6 we obtain $\mathrm{Re}\,p(z) > 0$, $z \in U$, for all $\delta \le \delta(\alpha, \beta, \gamma)$, so $G \in M_0^*(\delta(\alpha, \beta, \gamma))$. □

We see that if we consider, in the above theorem, the condition $zG(z) = zJ_{\alpha,\beta}(g)(z) \ne 0$, $z \in U$, we get:

Theorem 2.11. *Let $0 \le \alpha < 1$, $0 < \beta < \gamma$, $g \in M_0^*(\alpha)$ and $G(z) = J_{\alpha,\beta}(g)(z)$, where the operator $J_{\beta,\gamma}$ is defined by (2.2). Suppose that $zG(z) \ne 0$, $z \in U$. Let's denote*

$$\beta_1(\alpha, \gamma) = \frac{2\sqrt{2\gamma(\alpha-1)^2 + \alpha} - \alpha - 1}{2(\alpha-1)^2},$$

$$\delta_1(\alpha, \beta, \gamma) = \frac{2\alpha\beta + 2\gamma + 1 - \sqrt{(1 + 2\alpha\beta - 2\gamma)^2 + 8(\gamma - \beta)}}{4\beta},$$

$$\delta_2(\alpha, \beta, \gamma) = \frac{2\alpha\beta + 2\beta + 1 - \sqrt{(1 + 2\alpha\beta - 2\beta)^2 + 8(\beta - \gamma)}}{4\beta}.$$

If $\gamma > \frac{1}{8}$ *and* $\beta < \beta_1(\alpha,\gamma)$*, then* $G \in M_0^*(\delta_1(\alpha,\beta,\gamma))$*.*

If $\gamma \le \frac{1}{8}$ *or* $\begin{cases} \gamma > \frac{1}{8} \\ \beta \ge \beta_1(\alpha,\gamma) \end{cases}$*, then* $G \in M_0^*(\delta(\alpha,\beta,\gamma))$*, where*

$$\delta(\alpha,\beta,\gamma) = \min\{\delta_1(\alpha,\beta,\gamma), \delta_2(\alpha,\beta,\gamma)\}.$$

The properties of the integral operator $J_{1,\gamma}$, were studied by many authors in different papers, from which we remember [1], [2], [6], [7], [8].

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