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SOME STRONG DIFFERENTIAL SUBORDINATIONS OBTAINED BY SĂLĂGEAN DIFFERENTIAL OPERATOR

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Dedicated to Professor Grigore Ştefan Sălăgean on his 60^{th} *birthday*

Abstract. S. S. Miller and P. T. Mocanu introduced the notion of differential superordination as a dual concept of differential subordination . The notion of strong differential subordination was introduced by J. A. Antonino and S. Romaguera. By using the Sălăgean differential operator we introduce a class of holomorphic functions denoted by $S_n^m(\alpha)$, and obtain some strong subordinations results.

1. **Introduction and preliminaries**

Denote by U the unit disc of the complex plane,

$$U = \{z \in \mathbb{C};\ |z| < 1\} \tag{1.1}$$

$$\overline{U} = \{z \in \mathbb{C};\ |z| \leq 1\} \tag{1.2}$$

the closed unit disc of the complex plane.

In the paper [3], Georgia I. Oros defined the classes $\mathcal{H}(U \times \overline{U})$ denote the class of analytic functions in $U \times \overline{U}$,

$$A^*_{\zeta} = \{f \in \mathcal{H}(U \times \overline{U}) \mid f(z,\zeta) = z + a_2(\zeta)z^2 + \dots,\ z \in U,\ \zeta \in \overline{U}\}, \tag{1.3}$$

$$A^*_{n\zeta} = \{f \in \mathcal{H}(U \times \overline{U}) \mid f(z,\zeta) = z + a_{n+1}(\zeta)z^{n+1} + \dots,\ z \in U,\ \zeta \in \overline{U}\}, \tag{1.4}$$

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for $n = 1$, $A^*_{n\zeta} = A^*_\zeta$, with $a_k(\zeta)$ holomorphic functions in $\overline{U}$, $k \geq 2$,

$$\mathcal{H}^*[a,n,\zeta] = \{f \in \mathcal{H}(U \times \overline{U}) \mid f(z,\zeta) = a + a_n(\zeta)z^n + a_{n+1}(\zeta)z^{n+1} + \dots,\ z \in U,\ \zeta \in \overline{U}\} \tag{1.5}$$

where $a_k(\zeta)$ holomorphic functions in $\overline{U}$, $k \geq n$, and let

$$\mathcal{H}_u(U) = \{f \in \mathcal{H}^*[a,n,\zeta] \mid f(z,\zeta) \text{ univalent in } U \text{ for all } \zeta \in \overline{U}\}, \tag{1.6}$$

$$K^* = \left\{ f \in \mathcal{H}^*[a,n,\zeta] \mid \operatorname{Re} \frac{zf''(z,\zeta)}{f'(z,\zeta)} + 1 > 0,\ z \in U \text{ for all } \zeta \in \overline{U} \right\} \tag{1.7}$$

the class of convex functions,

$$S^* = \left\{ f \in \mathcal{H}^*[a,n,\zeta] \mid \operatorname{Re} \frac{zf'(z,\zeta)}{f(z,\zeta)} > 0,\ z \in U \text{ for all } \zeta \in \overline{U} \right\} \tag{1.8}$$

the class of starlike functions.

Definition 1.1. [4] Let $f(z,\zeta)$, $H(z,\zeta)$ analytic in $U \times \overline{U}$. The function $f(z,\zeta)$ is said to be strongly subordinate to $H(z,\zeta)$, or $H(z,\zeta)$ is said to be strongly superordinate to $f(z,\zeta)$, if there exists a function w analytic in U, with $w(0) = 0$, and $|w(z)| < 1$ such that $f(z,\zeta) = H(w(z),\zeta)$ for all $\zeta \in \overline{U}$. In such a case we write $f(z,\zeta) \prec\prec H(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.

Remark 1.2. [4] (i) Since $f(z,\zeta)$ is analytic in $U \times \overline{U}$, for all $\zeta \in \overline{U}$ and univalent in U, for all $\zeta \in \overline{U}$, Definition 1.1 is equivalent to $f(0,\zeta) = H(0,\zeta)$ for all $\zeta \in \overline{U}$ and $f(U \times \overline{U}) \subset H(U \times \overline{U})$.

(ii) If $H(z,\zeta) \equiv H(z)$ and $f(z,\zeta) \equiv f(z)$ then strong subordination becomes usual notion of subordination.

Lemma 1.3. [2, page 71] *Let $h(z,\zeta)$ be a convex function with $h(0,\zeta) = a$ for every $\zeta \in \overline{U}$ and let $\gamma \in \mathbb{C}^*$ be a complex number with* $\operatorname{Re} \gamma \geq 0$. *If $p \in \mathcal{H}^*[a,n,\zeta]$ and*

$$p(z,\zeta) + \frac{1}{\gamma} zp'(z,\zeta) \prec\prec h(z,\zeta) \tag{1.9}$$

then $p(z,\zeta) \prec\prec q(z,\zeta) \prec\prec h(z,\zeta)$ where

$$g(z,\zeta) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t,\zeta) t^{(\gamma/n)-1} dt. \tag{1.10}$$

The function $g(z,\zeta)$ is convex and is the best dominant.

Lemma 1.4. [1] *Let $g(z,\zeta)$ be a convex function in U ,for all $\zeta \in \overline{U}$ and let*

$$h(z,\zeta) = g(z,\zeta) + n\alpha g'(z,\zeta), \tag{1.11}$$

where $\alpha > 0$ and n is a positive integer. If

$$p(z,\zeta) = g(0,\zeta) + p_n(\zeta)z^n + \dots$$

is holomorphic in U, for all $\zeta \in \overline{U}$ and

$$p(z,\zeta) + \alpha z p'(z,\zeta) \prec\prec h(z,\zeta) \tag{1.12}$$

then

$$p(z,\zeta) \prec\prec g(z,\zeta) \tag{1.13}$$

and this result is sharp.

Definition 1.5. [5] For $f \in A^*_\zeta$, $n \in \mathbb{N}^* \cup \{0\}$, the operator $S^n f$ is defined by

$$\begin{aligned}
&S^n : A^*_\zeta \to A^*_\zeta \\
&S^0 f(z,\zeta) = f(z,\zeta) \\
&S^1 f(z,\zeta) = z f'(z,\zeta) \\
&\dots \\
&S^{n+1} f(z,\zeta) = z[S^n f(z,\zeta)]',\ z \in U,\ \zeta \in \overline{U}.
\end{aligned}$$

Remark 1.6. If $f \in A^*_\zeta$,

$$f(z,\zeta) = z + \sum_{j=2}^{\infty} a_j(\zeta) z^j$$

then

$$S^n f(z,\zeta) = z + \sum_{j=2}^{\infty} j^n a_j(\zeta) z^j, \quad z \in U,\ \zeta \in \overline{U}.$$

2. Main results

Definition 2.1. If $\alpha < 1$ and $m, n \in \mathbb{N}$, let $S_m^n(\alpha)$ denote the class of functions $f \in A_{n\zeta}^*$ which satisfy the inequality

$$\text{Re } [S^m f(z,\zeta)]' > \alpha. \tag{2.1}$$

Theorem 2.2. *If $\alpha < 1$ and $m, n \in \mathbb{N}$, then*

$$S_n^{m+1}(\alpha) \subset S_n^m(\delta) \tag{2.2}$$

where

$$\begin{aligned} \delta = \delta(\alpha, n, m) &= (2\alpha - 1) + 1 - (2\alpha - 1)\frac{1}{n}\beta\left(\frac{1}{n}\right), \\ \beta(x) &= \int_0^1 \frac{t^{x-1}}{1+t} dt. \end{aligned} \tag{2.3}$$

Proof. Let $f \in S_n^{m+1}(\alpha)$. By using the properties of the operator $S^m f(z,\zeta)$, we have

$$S^{m+1} f(z,\zeta) = z[S^m f(z,\zeta)]', \; z \in U, \; \zeta \in \overline{U}. \tag{2.4}$$

Differentiating (2.4) we obtain

$$[S^{m+1} f(z,\zeta)]' = [S^m f(z,\zeta)]' + z[S^m f(z,\zeta)]'', \quad z \in U, \; \zeta \in \overline{U}. \tag{2.5}$$

If we let $p(z,\zeta) = [S^m f(z,\zeta)]'$, then

$$p'(z,\zeta) = [S^m f(z,\zeta)]''$$

and (2.5) becomes

$$[S^{m+1} f(z,\zeta)]' = p(z,\zeta) + zp'(z,\zeta). \tag{2.6}$$

Since $f \in S_n^{m+1}(\alpha)$, by using Definition 2.1, we have

$$\text{Re } [p(z,\zeta) + zp'(z,\zeta)] > \alpha \tag{2.7}$$

which is equivalent to

$$p(z,\zeta) + zp'(z,\zeta) \prec\prec \frac{1 + (2\alpha - 1)z}{1 + z} \equiv h(z,\zeta). \tag{2.8}$$

By using Lemma 1.3, we have

$$p(z,\zeta) \prec\prec g(z,\zeta) \prec\prec h(z,\zeta) \tag{2.9}$$

where

$$g(z,\zeta) = \frac{1}{nz^{1/n}} \int_0^z \frac{1-(2\alpha-1)t}{1+t} t^{(1/n)-1} dt. \tag{2.10}$$

The function $g(z,\zeta)$ is convex and is the best dominant.

From $p(z,\zeta) \prec\prec g(z,\zeta)$, it results that

$$\text{Re } p(z,\zeta) > \delta = g(1,\zeta) = \delta(\alpha, n, m) \tag{2.11}$$

where

$$g(1,\zeta) = \frac{1}{n}\int_0^1 t^{\frac{1}{n}-1} \cdot \frac{1+(2\alpha-1)t}{1+t} dt \tag{2.12}$$

$$= \frac{1}{n}\int_0^1 t^{\frac{1}{n}-1} \cdot \frac{1+(2\alpha-1)t+(2\alpha-1)-(2\alpha-1)}{1+t} dt$$

$$= \frac{1}{n}\int_0^1 t^{\frac{1}{n}-1} \left[\frac{(2\alpha-1)(t+1)}{1+t} + \frac{1-2\alpha+1}{1+t}\right] dt$$

$$= (2\alpha-1)\frac{1}{n}\int_0^1 t^{\frac{1}{n}-1} dt + \frac{1}{n}\int_0^1 t^{\frac{1}{n}-1} \cdot \frac{1-(2\alpha-1)}{1+t} dt$$

$$= (2\alpha-1)\frac{1}{n} \cdot \frac{t^{\frac{1}{n}}}{\frac{1}{n}}\Big|_0^1 + \frac{1-(2\alpha-1)}{n}\int_0^1 \frac{t^{\frac{1}{n}-1}}{1+t} dt$$

$$= (2\alpha-1) + \frac{1-(2\alpha-1)}{n}\beta\left(\frac{1}{n}\right) \tag{2.13}$$

from which we deduce that $S_n^{m+1}(\alpha) \subset S_n^m(\delta)$. □

Theorem 2.3. *Let $g(z,\zeta)$ be a convex function $g(0,\zeta) = 1$ and let $h(z,\zeta)$ be a function such that*

$$h(z,\zeta) = g(z,\zeta) + zg'(z,\zeta). \tag{2.14}$$

If $f \in A^*_{n\zeta}$ and verifies the strong differential subordination

$$[S^{m+1} f(z,\zeta)]' \prec\prec h(z,\zeta) \tag{2.15}$$

then

$$[S^m f(z,\zeta)]' \prec\prec g(z,\zeta). \tag{2.16}$$

Proof. From

$$S^{m+1}f(z,\zeta) = z[S^m f(z,\zeta)]' \tag{2.17}$$

we obtain

$$[S^{m+1}f(z,\zeta)]' = [S^m f(z,\zeta)]' + z[S^m f(z,\zeta)]''. \tag{2.18}$$

If we let $p(z,\zeta) = [S^m f(z,\zeta)]'$, then we obtain

$$[S^{m+1}f(z,\zeta)]' = p(z,\zeta) + zp'(z,\zeta) \tag{2.19}$$

and (2.15) becomes

$$p(z,\zeta) + zp'(z,\zeta) \prec\prec g(z,\zeta) + zg'(z,\zeta) \equiv h(z,\zeta). \tag{2.20}$$

Using Lemma 1.4, we have

$$p(z,\zeta) \prec\prec g(z,\zeta), \text{ i.e., } S^m f(z,\zeta) \prec\prec g(z,\zeta). \tag{2.21}$$

□

Theorem 2.4. *Let $h \in \mathcal{H}^*[a,n,\zeta]$, with $h(0,\zeta) = 1$, $h'(0,\zeta) \neq 0$ which verifies the inequality*

$$\operatorname{Re}\left[1 + \frac{zh''(z,\zeta)}{h'(z,\zeta)}\right] > -\frac{1}{2(m+1)}, \quad m \geq 0. \tag{2.22}$$

If $f \in A^*_{n\zeta}$ and verifies the strong differential subordination

$$[S^{m+1}f(z,\zeta)]' \prec\prec h(z,\zeta), \quad z \in U \tag{2.23}$$

then

$$[S^m f(z,\zeta)]' \prec\prec g(z,\zeta), \tag{2.24}$$

where

$$g(z,\zeta) = \frac{1}{nz^{1/n}} \int_0^z t^{(1/n)-1} h(t,\zeta)dt. \tag{2.25}$$

The function g is convex and is the best dominant.

Proof. From

$$S^{m+1}f(z,\zeta) = z[S^m f(z,\zeta)]' \tag{2.26}$$

we obtain

$$[S^{m+1}f(z,\zeta)]' = [S^m f(z,\zeta]' + z[S^m f(z,\zeta)]''. \tag{2.27}$$

If we let $p(z,\zeta) = [S^m f(z,\zeta)]'$, then we obtain

$$[S^{m+1} f(z,\zeta)]' = p(z,\zeta) + zp'(z,\zeta) \tag{2.28}$$

and (2.23) becomes

$$p(z,\zeta) + zp'(z,\zeta) \prec\prec h(z,\zeta). \tag{2.29}$$

By using Lemma 1.3 we have

$$p(z,\zeta) \prec\prec g(z,\zeta) = \frac{1}{nz^{1/n}} \int_0^z h(t,\zeta) t^{\frac{1}{n}-1} dt. \tag{2.30}$$

□

Theorem 2.5. *Let $g(z,\zeta)$ be a convex function with $g(0,\zeta) = 1$ and*

$$h(z,\zeta) = g(z,\zeta) + zg'(z,\zeta). \tag{2.31}$$

If $f \in A^*_{n\zeta}$ and verifies the differential subordination

$$[S^m f(z,\zeta)]' \prec\prec h(z,\zeta), \quad z \in U,\ \zeta \in \overline{U} \tag{2.32}$$

then

$$\frac{S^m f(z,\zeta)}{z} \prec\prec g(z,\zeta). \tag{2.33}$$

Proof. We let

$$p(z,\zeta) = \frac{S^m f(z,\zeta)}{z}, \quad z \in U,\ \zeta \in \overline{U},$$

we obtain

$$S^m f(z,\zeta) = zp(z,\zeta). \tag{2.34}$$

By differentiating, we obtain

$$[S^m f(z,\zeta)]' = p(z,\zeta) + zp'(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}. \tag{2.35}$$

Then (2.32) becomes

$$p(z,\zeta) + zp'(z,\zeta) \prec\prec h(z,\zeta) = g(z,\zeta) + zg'(z,\zeta). \tag{2.36}$$

Using Lemma 1.4 we have

$$p(z,\zeta) \prec\prec g(z,\zeta).$$

□

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