STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LV**, Number 3, September 2010

# DUALITY FOR HADAMARD PRODUCTS APPLIED TO CERTAIN CONDITION FOR $\alpha$-STARLIKENESS

JANUSZ SOKÓŁ

*Dedicated to Professor Grigore Ştefan Sălăgean on his* $60^{th}$ *birthday*

**Abstract**. Let $\mathcal{P}(\alpha,\beta)$, $\alpha > 0$, $\beta < 1$, denote the class of all analytic functions $f$ in the unit disc with the normalization $f(0) = 1$, $f'(0) = 1$ and satisfying the condition

$$\mathfrak{Re}[e^{i\varphi}(f'(z) + \frac{1}{\alpha}zf''(z) - \beta)] > 0, \quad |z| < 1$$

for some $\varphi \in \mathbb{R}$. In this paper we find conditions on $\alpha, \beta$ so that $\mathcal{P}(\alpha,\beta) \subseteq \mathcal{S}^*(\mu)$, where $\mu < 1$ is given and $\mathcal{S}^*(\mu)$ denote the class of starlike function of order $\mu$. We take advantage of the Ruscheweh's Duality theory.

## 1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the open unit disc

$$U = \{z : \ |z| < 1\}$$

of the complex plane $\mathbb{C}$. Everywhere in this paper $z \in U$ unless we make a note. We say that $f \in \mathcal{H}$ is convex when $f(U)$ is a convex set. Let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ consisting of functions normalized by $f(0) = 0$, $f'(0) = 1$. For $\mu < 1$, by $\mathcal{S}^*(\mu)$ we denote the well known subclass of $\mathcal{A}$ consisting of starlike function of order $\mu$. As is well known

$$\mathcal{S}^*(\mu) = \left\{ f \in \mathcal{A} : \ \mathfrak{Re}\left[\frac{zf'(z)}{f(z)}\right] > \mu \text{ for } z \in \ U \right\}$$

Received by the editors: 26.04.2010.

2000 *Mathematics Subject Classification.* 30C45, 33C05.

*Key words and phrases.* Duality principle, dual set, Hadamard product, Starlike functions, convex functions, functions with positive real part, Gaussian hypergeometric function.

$\mathcal{S}^*(0) = \mathcal{S}^*$ is the class of starlike functions which map $U$ onto a starlike domain with respect to the origin. For $\alpha > 0$ and $\beta < 1$ given, define

$$\mathcal{P}(\alpha, \beta) = \left\{ f \in \mathcal{A} : \exists \varphi \in \mathbb{R} \text{ s. t. } \mathfrak{Re} \left[ e^{i\varphi} \left( f'(z) + \frac{1}{\alpha} z f''(z) - \beta \right) \right] > 0, z \in U \right\}.$$

In the geometric theory of function, a variety of sufficient conditions for starlikeness have been considered. We refer to the monographs [4], [5] for details. In the present work we tray to find conditions on $\alpha, \beta$ so that $\mathcal{P}(\alpha, \beta) \subseteq S^*(\mu)$, where $\mu < 1$ is given. If $f$ and $g$ are analytic in U with $f(z) = a_0 + a_1 z + a_2 z^2 + \ldots$ and $g(z) = b_0 + b_1 z + b_2 z^2 + \ldots$ then the Hadamard product (or convolution) of $f$ and $g$ is defined by

$$(f * g)(z) = a_0 b_0 + a_1 b_1 z + a_2 b_2 z^2 + \ldots.$$

The convolution has the algebraic properties of ordinary multiplication. In convolution theory, the concept of duality is central. For a set

$$V \subseteq \mathcal{A}_0 = \left\{ g : g(z) = \frac{f(z)}{z}, f \in \mathcal{A} \right\}$$

the dual set $V^*$ is defined as

$$V^* = \{ g \in \mathcal{A}_0 : (f * g)(z) \neq 0 \text{ for all } f \in V, z \in U \}.$$

In this paper we use the powerful method of duality principle in geometric function theory developed by Ruscheweyh [8]. The basic results of Ruscheweyh's duality theory one can find in the book [9]. The duality principle states that, under certain conditions on $V$, the range of a continuous linear functional on $V$ equals the range of the same linear functional on $(V^*)^* = V^{**}$. This is a useful information since in many cases of interest $V^{**}$ is much larger than $V$. Then by investigating the small set we can get results about the large set. One such pair of the sets is described in the theorem below.

**Theorem 1.1.** *Let*

$$V_\beta = \left\{ \beta + \frac{(1-\beta)(1+xz)}{1+yz} : \quad |x| = |y| = 1 \right\}, \beta \in \mathbb{R}, \beta \neq 1.$$

*Then*

$$V_\beta^{**} = \left\{ g \in \mathcal{A}_0 : \exists \varphi \in \mathbb{R} \text{ such that } \mathfrak{Re} \left[ e^{i\varphi} \left( g(z) - \beta \right) \right] > 0, z \in U \right\}.$$

Theorem 1.1 with $\beta = 0$ one can find in [9, p. 22]. Notice that if $h \in V_\beta$, $h(z) = \beta + (1-\beta)\frac{1+xz}{1+yz}$ with $|x| = |y| = 1$, $\beta \in \mathbb{R}$, $\beta \neq 1$, then

$$h(z) = 1 + (1-\beta)\left(1 - \frac{x}{y}\right)\frac{yz}{1-yz} = 1 + (1-\beta)(1-e^{i\psi})\sum_{k=1}^{\infty}(yz)^k \tag{1.1}$$

for some $\psi \in \mathbb{R}$. A subset $V \subseteq \mathcal{A}_0$ is said to be complete if it has the following property:

$$f \in V \Rightarrow f(xz) \in V \ \forall |x| \leq 1.$$

**Theorem 1.2.** *(Duality principle, see* [8]*) Let $V \subseteq \mathcal{A}_0$ be compact and complete. If $\lambda$ is a continuous linear functional on $\mathcal{H}$, then*

$$\lambda(V) = \lambda\left(V^{**}\right), \quad \overline{co}(V) = \overline{co}\left(V^{**}\right).$$

The sets $V_\beta$ and $V_\beta^{**}$ in Theorem 1.1 are compact and complete. The following Theorem 1.3 one can find in [9, p. 23] and in [10].

**Theorem 1.3.** *(see* [10]*) Let $f \in \mathcal{A}$. Then $f$ belongs to the class $\mathcal{S}^*(\mu)$ of starlike function of order $\mu$ if and only if*

$$\frac{f(z)}{z} * \frac{1 + \frac{\varepsilon+2\mu-1}{2(1-\mu)}z}{(1-z^2)} \neq 0 \quad \forall\ |\varepsilon| = 1, \quad \forall\ z \in U.$$

## 2. Main results

**Theorem 2.1.** *Suppose that $\alpha > 0$, $\beta < 1$, $\mu < 1$. Then $\mathcal{P}(\alpha,\beta) \subseteq \mathcal{S}^*(\mu)$ if and only if*

$$\mathfrak{Re}\left[H(\varepsilon;z)\right] > -\frac{1-\mu}{1-\beta} \quad \forall |\varepsilon| = 1, \quad \forall z \in U, \tag{2.1}$$

*where*

$$H(\varepsilon;z) = \alpha\sum_{k=1}^{\infty}\frac{k(1+\varepsilon)+2(1-\mu)}{(k+1)(k+\alpha)}z^k. \tag{2.2}$$

*Proof.* Let a function $f$ be in the class $\mathcal{P}(\alpha,\beta)$. If we denote $f'(z) + \frac{z}{\alpha}f''(z) = g_\alpha(z)$, then we have $g_\alpha \in V_\beta^{**}$. If $f(z) = \sum\limits_{k=1}^{\infty} a_k z^k$, $a_1 = 1$, then

$$f'(z) + \frac{z}{\alpha}f''(z) = \sum_{k=1}^{\infty}\frac{k(k-1+\alpha)}{\alpha}a_k z^{k-1} = g_\alpha(z)$$

so

$$\frac{f(z)}{z} = \sum_{k=1}^{\infty} a_k z^{k-1} = g_\alpha(z) * \sum_{k=1}^{\infty} \frac{\alpha z^{k-1}}{k(k-1+\alpha)},$$

and we obtain one-to-one correspondence between $\mathcal{P}(\alpha,\beta)$ and $V_\beta^{**}$. Thus, by Theorem 1.3, $\mathcal{P}(\alpha,\beta) \subseteq \mathcal{S}^*(\mu)$ if and only if

$$g_\alpha(z) * \sum_{k=1}^{\infty} \frac{\alpha z^{k-1}}{k(k-1+\alpha)} * \frac{1+\frac{\varepsilon+2\mu-1}{2(1-\mu)}z}{(1-z)^2} \neq 0 \quad \forall g_\alpha \in V_\beta^{**},\ \forall\ |\varepsilon| = 1, \forall\ z \in U. \tag{2.3}$$

Let us consider for $z \in U$ the continuous linear functional $\lambda_z : \mathcal{A}_0 \to \mathbb{C}$, such that

$$\lambda_z(h) := h(z) * \sum_{k=1}^{\infty} \frac{\alpha z^{k-1}}{k(k-1+\alpha)} * \frac{1+\frac{\varepsilon+2\mu-1}{2(1-\mu)}z}{(1-z)^2},$$

By Duality principle we have $\lambda_z(V) = \lambda_z(V_\beta^{**})$. Therefore (2.3) holds if and only if

$$\left[1+(1-\beta)(1-e^{i\psi})\sum_{k=1}^{\infty} z^k\right] * \left[1+\sum_{k=1}^{\infty} \frac{\alpha z^k}{(k+1)(k+\alpha)}\right] * \left[\frac{1+\frac{\varepsilon+2\mu-1}{2(1-\mu)}z}{(1-z)^2}\right] \neq 0 \tag{2.4}$$

for all $\psi \in \mathbb{R}$, $|\varepsilon| = 1$, $z \in U$. Using the properties of convolution we can reformulate (2.4) as

$$\alpha \sum_{k=1}^{\infty} \frac{k(1+\varepsilon)+2(1-\mu)}{(k+1)(k+\alpha)} z^k \neq -\frac{2(1-\mu)}{(1-e^{i\psi})(1-\beta)}. \tag{2.5}$$

For $\psi \in \mathbb{R}$ the quantity on the right site of (2.5) takes its values on the line $\mathfrak{Re}\, w = -\frac{1-\mu}{1-\beta}$ so (2.5) is equivalent to (2.1) . □

Starlikeness of functions in $\mathcal{P}(\alpha,\beta)$ has been investigated. For example we have the reformulated version from [3].

**Theorem 2.2.** *(see [3]) If $f \in \mathcal{P}(\alpha,\beta)$ and $\alpha \le 3$ and $\beta(\alpha)$ be given by*

$$\frac{\beta(\alpha)}{1-\beta(\alpha)} = \alpha \int_0^1 \frac{t^{\alpha-1}(t-1)}{t+1}\ \mathrm{d}t,$$

*then $f \in \mathcal{S}^*(0)$ and the value of $\beta(\alpha)$ is sharp.*

Note that Fournier and Ruscheweyh introduced in [3] the integral transform

$$V_\lambda : \mathcal{A} \to \mathcal{A}$$

such that

$$V_\lambda(f)(z) = \int_0^1 \lambda(t) \frac{f(tz)}{t}\ \mathrm{d}t,$$

where $\lambda(t)$is real valued integrable function satisfying the normalizing condition

$$\int_0^1 \lambda(t)\ \mathrm{d}t = 1.$$

This operator was introduced mainly to find conditions on $\lambda(t)$ and $\beta$ so that $V_\lambda(f)$ maps $\mathcal{P}(\alpha,\beta)$ into $S^*(0)$, when $\alpha \to \infty$. Recently Balasubramanian, Ponnusamy and Prabhakaran in [2] and Ponnusamy and Rønning in [7] extended this considerations to find conditions on $\lambda(t)$ and $\beta$ such that $V_\lambda(f)$ is starlike of order $\mu$, $(0 \le \mu \le 1/2)$ when $f \in \mathcal{P}(\alpha,\beta)$. For convexity of this integral transform see [1].

While Theorem 2.1 precisely answers when $\mathcal{P}(\alpha,\beta) \subseteq \mathcal{S}^*(\mu)$ it is difficult to answer when the condition (2.1) is satisfied in general. It seems that $\mathfrak{Re} H(\varepsilon; z)$ attains its minimum at $z = -1$ and $\varepsilon = 1$ but it is hard to show.

**Conjecture 2.3.** *Let $f$ be given by (2.2). Then*

$$\min\{\mathfrak{Re} H(\varepsilon;z) : |\varepsilon| = 1,\ |z| < 1\} = H(1;-1).$$

In [11] we apply the general theory of differential subordinations to obtain several weaker but simple sufficient conditions for $\mu$-starlikeness while Owa and Sălăgean in [6] considered a sufficient condition and a necessary condition for starlikeness of complex order of functions with negative coefficients. One can expressed the function $H(\varepsilon;z)$ in terms of the Gaussian hypergeometric function

$${}_2F_1(a,b,c;z) \quad = \quad \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k,$$

where $(x)_k$ denotes the Pochhammer symbol defined by

$$(x)_k = x(x+1)(x+2)\cdots(x+k-1) \ \text{ for } \ k \in \mathbb{N} \ \text{ and } \ (x)_0 = 1.$$

Then for $\alpha \ne 1$ we have

$$\begin{aligned} H(\varepsilon;z) &= \alpha \sum_{k=1}^{\infty} \frac{k(1+\varepsilon) + 2(1-\mu)}{(k+1)(k+\alpha)} z^k \\ &= \frac{\alpha(\varepsilon + 2\mu - 1)}{1-\alpha} \sum_{k=1}^{\infty} \frac{z^k}{k+1} + \frac{2(1-\mu) - \alpha(\varepsilon+1)}{1-\alpha} \sum_{k=1}^{\infty} \frac{\alpha z^k}{k+\alpha} \end{aligned}$$

$$= \frac{\alpha(\varepsilon+2\mu-1)\,[{}_2F_1(1,1,2;z)-1]+[2(1-\mu)-\alpha(\varepsilon+1)]\,[{}_2F_1(1,\alpha,\alpha+1;z)-1]}{1-\alpha}$$

$$= 2(\mu-1)+\frac{\alpha(\varepsilon+2\mu-1)}{1-\alpha}{}_2F_1(1,1,2;z)+\frac{2(1-\mu)-\alpha(\varepsilon+1)}{1-\alpha}{}_2F_1(1,\alpha,\alpha+1;z)$$

$$= 2(\mu-1)+\frac{\alpha(\varepsilon+2\mu-1)}{1-\alpha}\frac{1}{z}\ln\frac{1}{1-z}+\frac{2(1-\mu)-\alpha(\varepsilon+1)}{1-\alpha}{}_2F_1(1,\alpha,\alpha+1;z).$$

We can rewrite the inequality (2.1) in the form

$$\frac{1-\mu}{\alpha(1-\beta)}+\mathfrak{Re}\left[\sum_{k=1}^{\infty}\frac{kz^k}{(k+1)(k+\alpha)}\right]+2(1-\mu)\mathfrak{Re}\left[\sum_{k=1}^{\infty}\frac{z^k}{(k+1)(k+\alpha)}\right] \tag{2.6}$$

$$> \mathfrak{Re}\left[-\varepsilon\sum_{k=1}^{\infty}\frac{kz^k}{(k+1)(k+\alpha)}\right] \quad \forall|\varepsilon|=1, \quad \forall z\in U,$$

thus we can see that (2.6) is satisfied when

$$\frac{1-\mu}{\alpha(1-\beta)}+\mathfrak{Re}\left[\sum_{k=1}^{\infty}\frac{kz^k}{(k+1)(k+\alpha)}\right]+2(1-\mu)\mathfrak{Re}\left[\sum_{k=1}^{\infty}\frac{z^k}{(k+1)(k+\alpha)}\right] \tag{2.7}$$

$$> \left|\sum_{k=1}^{\infty}\frac{kz^k}{(k+1)(k+\alpha)}\right| \qquad \forall z\in U.$$

**Conjecture 2.4.** *Let the function $G$ be given by*

$$G(z)=2(1+\alpha)\sum_{k=1}^{\infty}\frac{1}{(k+1)(k+\alpha)}z^k$$

*Then the function $zG'(z)$ is a convex function when $-1<\alpha$.*

Note tat it is known that $G$ is a convex while $zG'$ is a starlike function. With this notation (2.7) becomes

$$\frac{2(1+\alpha)(1-\mu)}{\alpha(1-\beta)}+\mathfrak{Re}zG'(z)+2(1-\mu)\mathfrak{Re}G(z)>|zG'(z)|\forall z\in U. \tag{2.8}$$

If Conjecture 2.4 is true, then we have $G'(-1)<\mathfrak{Re}G'(z)<G'(1)$ so for (2.8)it suffices that

$$\frac{1-\mu}{\alpha(1-\beta)}+\sum_{k=1}^{\infty}\frac{k(-1)^k}{(k+1)(k+\alpha)}+2(1-\mu)\sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)(k+\alpha)} \tag{2.9}$$

$$> \sum_{k=1}^{\infty}\frac{k}{(k+1)(k+\alpha)}.$$

While (2.9) is not a necessary for (2.8) it still remains hard to verify.

## References

[1] Balasubramanian, R., Ponnusamy, M., Prabhakaran, D. J., *Convexity of integral transforms and function spaces*, Integral Transforms and Special Functions, **18(1)** (2007), 1-14.

[2] Balasubramanian, R., Ponnusamy, M., Prabhakaran, D. J., *On extremal problems related to integral trnsforms of a class of analytic functions*, J. Math. Anal. Appl., **336** (2007), 542-555.

[3] Fournier, R., Ruscheweyh, S., *On two extremal problems related to univalent functions*, Rocky Mountain I. Math., **24(2)** (1994), 529-538.

[4] Goodman, A. W., *Univalent Functions*, vol. I and II, Mariner Publishing Co., Tampa, Florida (1983).

[5] Miller, S. S., Mocanu, P. T., *Differential subordinations: theory and applications*, Series of Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225, Marcel Dekker Inc., New York / Basel 2000.

[6] Owa, S., Sălăgean, G. S., *On an open problem of S. Owa*, J. Math. Anal. Appl., **218(2)** (1998), 453-457.

[7] Ponnusamy, S., Rønning, F., *Integral transforms of a class of analytic functions*, Complex Var. Elliptic. Eq., **53(5)** (2008), 423-434.

[8] Ruscheweyh, S., *Duality for Hadamard products with applications to extremal problems for functions regular in the unit disc*, Trans. Amer. Math Soc., **210** (1975), 63-74.

[9] Ruscheweyh, S., *Convolutions in Geometric Function Theory*, Sem. Math. Sup. 83, Press Univ. Montreal 1982.

[10] Silverman, H., Silvia, E. M., Telage, D., *Convolutions conditions for convexity starlikeness and spiral-likeness*, Math. Z., **162(2)** (1978), 125-130.

[11] Sokół, J., *On a condition for alpha-starlikeness*, J. Math. Anal. Appl., **352** (2009), 696-701.

Department of Mathematics
Rzeszów University of Technology
ul. W. Pola 2, 35-959 Rzeszów, Poland
*E-mail address*: jsokol@prz.edu.pl