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AN APPLICATION OF MILLER AND MOCANU LEMMA

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Dedicated to Professor Grigore Ştefan Sălăgean on his 60^{th} *birthday*

Abstract. Let $\mathcal{H}[a,n]$ be the class of functions $f(z) = a + a_n z^n + \dots$ which are analytic in the open unit disk $\mathbb{U}$. For $f(z) \in \mathcal{H}[a,n]$, S. S. Miller and P. T. Mocanu (J. Math. Anal. Appl. **65**(1978), 289-305) have shown Miller and Mocanu lemma which is the generalization of Jack lemma by I. S. Jack (J. London Math. Soc. **3**(1971), 469-474). Applying Miller and Mocanu lemma, an interesting property for $f(z) \in \mathcal{H}[a,n]$ and an example are discussed.

1. Introduction

Let $\mathcal{H}[a,n]$ denote the class of functions $f(z)$ of the form

$$f(z) = a + \sum_{k=n}^{\infty} a_k z^k \qquad (n = 1, 2, 3, \dots)$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, where $a \in \mathbb{C}$. Jack [1] has shown the result for analytic functions $w(z)$ in $\mathbb{U}$ with $w(0) = 0$, which is called Jack's lemma. In 1978, Miller and Mocanu [2] have given the generalization theorem for Jack's lemma, which was called Miller and Mocanu lemma.

Lemma 1.1 (Miller and Mocanu lemma)**.** *Let* $f(z) \in \mathcal{H}[a,n]$ *with* $f(z) \not\equiv a$*. If there exists a point* $z_0 \in \mathbb{U}$ *such that*

$$\max_{|z| \leqq |z_0|} |f(z)| = |f(z_0)|,$$

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then

$$\frac{z_0 f'(z_0)}{f(z_0)} = m$$

and

$$\mathrm{Re}\frac{z_0 f''(z_0)}{f'(z_0)} + 1 \geqq m,$$

where m is real and

$$m \geqq n\frac{|f(z_0) - a|^2}{|f(z_0)|^2 - |a|^2} \geqq n\frac{|f(z_0)| - |a|}{|f(z_0)| + |a|}.$$

If $a = 0$, then the above lemma becomes Jack's lemma due to Jack [1].

2. **Main theorem**

Applying Miller and Mocanu lemma, we derive

Theorem 2.1. *Let $f(z) \in \mathcal{H}[a, n]$ with $f(z) \neq 0$ for $z \in \mathbb{U}$. If there exists a point $z_0 \in \mathbb{U}$ such that*

$$\min_{|z| \leqq |z_0|} |f(z)| = |f(z_0)|,$$

then

$$\frac{z_0 f'(z_0)}{f(z_0)} = -m \tag{2.1}$$

and

$$\mathrm{Re}\frac{z_0 f''(z_0)}{f'(z_0)} + 1 \geqq -m, \tag{2.2}$$

where

$$m \geqq n\frac{|a - f(z_0)|^2}{|a|^2 - |f(z_0)|^2} \geqq n\frac{|a| - |f(z_0)|}{|a| + |f(z_0)|}.$$

Proof. We defined the function $g(z)$ by

$$\begin{aligned} g(z) &= \frac{1}{f(z)} \\ &= c + c_n z^n + c_{n+1} z^{n+1} + \dots \qquad \left(c = \frac{1}{a}\right). \end{aligned}$$

Then, $g(z)$ is analytic in $\mathbb{U}$ and $g(0) = c \neq 0$. Furthermore, by the assumtion of the theorem, $|g(z)|$ takes its maximum value at $z = z_0$ in the closed disk $|z| \leqq |z_0|$. It follows from this that

$$|g(z_0)| = \frac{1}{|f(z_0)|} = \frac{1}{\min\limits_{|z| \leqq |z_0|} |f(z)|} = \max_{|z| \leqq |z_0|} |g(z)|.$$

Therefore, applying Lemma 1.1 to $g(z)$, we observe that

$$\frac{z_0 g'(z_0)}{g(z_0)} = -\frac{z_0 f'(z_0)}{f(z_0)} = m$$

which shows (2.1) and

$$\begin{aligned} \mathrm{Re}\frac{z_0 g''(z_0)}{g'(z_0)} + 1 &= \mathrm{Re}\left(\frac{z_0 f''(z_0)}{f'(z_0)} - 2\frac{z_0 f'(z_0)}{f(z_0)}\right) + 1 \\ &= \mathrm{Re}\frac{z_0 f''(z_0)}{f'(z_0)} + 2m + 1 \\ &\geqq m \end{aligned}$$

which implies (2.2), where

$$m \geqq n\frac{|g(z_0) - c|^2}{|g(z_0)|^2 - |c|^2} = n\frac{|a - f(z_0)|^2}{|a|^2 - |f(z_0)|^2} \geqq n\frac{|a| - |f(z_0)|}{|a| + |f(z_0)|}.$$

This completes the assertion of Theorem 2.1. □

Example 2.2. Let us consider the function $f(z)$ given by

$$\begin{aligned} f(z) &= \frac{a + \left(e^{i \arg(a)} - a\right) z^n}{1 - z^n} \\ &= a + e^{i \arg(a)} z^n + e^{i \arg(a)} z^{2n} + \dots \qquad (z \in \mathbb{U}) \end{aligned}$$

for some complex number a with $|a| > \frac{1}{2}$.
Then, $f(z)$ maps the disk $\mathbb{U}_r = \{z : |z| < r \leqq 1\}$ onto the domain

$$\left| f(z) - \left(a + \frac{e^{i \arg(a)} r^{2n}}{1 - r^{2n}}\right)\right| \leqq \frac{r^n}{1 - r^{2n}}.$$

Thus, we know that there exists a point $z_0 = re^{i\frac{\pi}{n}} \in \mathbb{U}$ such that

$$\min_{|z| \leqq |z_0|} |f(z)| = |f(z_0)| = |a| - \frac{r^n}{1 - r^{2n}}.$$

For such a point z_0, we obtain that

$$\frac{z_0 f'(z_0)}{f(z_0)} = -\frac{nr^n}{(1+r^n)(|a|-(1-|a|)r^n)} = -m$$

where

$$m = \frac{nr^n}{(1+r^n)(|a|-(1-|a|)r^n)} > 0.$$

Therefore, we get that

$$\mathrm{Re}\frac{z_0 f''(z_0)}{f'(z_0)} + 1 = n\frac{1-r^n}{1+r^n} > 0 > -m.$$

Furthermore, we obtain that

$$n\frac{|a-f(z_0)|^2}{|a|^2-|f(z_0)|^2} = \frac{nr^n}{2|a|+(2|a|-1)r^n} = \frac{nr^n}{2\left(|a|-(1-|a|)r^n+\frac{1}{2}r^n\right)} < m.$$

Putting a with a real number in Example 2.2, we get Example 2.3.

Example 2.3. Let us consider the function

$$\begin{aligned} f(z) &= \frac{a+(1-a)z^n}{1-z^n} \\ &= a + z^n + z^{2n} + \dots \qquad (z \in \mathbb{U}) \end{aligned}$$

for $a > \frac{1}{2}$. Then, it follows that the function $f(z)$ maps the disk $\mathbb{U}_r$ onto the domain

$$\left| f(z) - \left(a + \frac{r^{2n}}{1-r^{2n}}\right)\right| \leqq \frac{r^n}{1-r^{2n}}.$$

Thus, there exists a point $z_0 = re^{i\frac{\pi}{n}} \in \mathbb{U}$ such that

$$\min_{|z| \leqq |z_0|} |f(z)| = |f(z_0)| = a - \frac{r^n}{1-r^{2n}}.$$

For such a point z_0, we obtain

$$\frac{z_0 f'(z_0)}{f(z_0)} = -\frac{nr^n}{(1+r^n)(a-(1-a)r^n)} = -m$$

where

$$m = \frac{nr^n}{(1+r^n)(a-(1-a)r^n)} > 0.$$

Therefore, we see that

$$\mathrm{Re}\frac{z_0 f''(z_0)}{f'(z_0)} + 1 = n\frac{1-r^n}{1+r^n} > 0 > -m.$$

Moreover, we have that

$$n\frac{|a-f(z_0)|^2}{|a|^2-|f(z_0)|^2} = \frac{nr^n}{2a+(2a-1)r^n} = \frac{nr^n}{2\left(a-(1-a)r^n+\frac{1}{2}r^n\right)} < m.$$

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