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STRONG DIFFERENTIAL SUBORDINATIONS OBTAINED BY THE MEDIUM OF AN INTEGRAL OPERATOR

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Dedicated to Professor Grigore Ştefan Sălăgean on his 60^{th} *birthday*

Abstract. The concept of differential subordination was introduced in [2] by S. S. Miller and P. T .Mocanu and developed in [3], and the concept of strong differential subordination was introduced in [1] by J. A. Antonino and S. Romaquera and developed in [4], [5] by Georgia Irina Oros and Gheorghe Oros. In this paper we define the class $S_n^m(\alpha)$, and we study strong differential subordination.

1. Introduction and preliminaries

Let U denote the unit disc of the complex plane :

$$U = \{z \in \mathbb{C} : \ |z| < 1\}$$

and

$$\overline{U} = \{z \in \mathbb{C} : \ |z| \leq 1\}.$$

Let $\mathcal{H}(U \times \overline{U})$ denote the class of analytic functions in $U \times \overline{U}$. In [4], the author has defined the class

$$\mathcal{H}\zeta[a, n] = \{f \in \mathcal{H}(U \times \overline{U}) : \ f(z, \zeta) = a + a_n(\zeta)z^n + a_{n+1}(\zeta)z^{n+1} + \cdots , \ z \in U, \zeta \in \overline{U}\}$$

with $a_k(\zeta)$ holomorphic functions in $\overline{U}$, $k \geq n$,

$$\mathcal{H}_n(U) = \{f \in \mathcal{H}\zeta[a, n] : \ f(z, \zeta) \text{ univalent in U for all } \zeta \in \overline{U}\},$$

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$$\mathcal{A}\zeta_n = \{f \in \mathcal{H}\zeta[a,n] : \ f(z,\zeta) = z + a_2(\zeta)z^2 + \cdots + a_n(\zeta)z^n + \cdots, \ z \in U, \zeta \in \overline{U}\}$$

with $\mathcal{A}\zeta_1 = \mathcal{A}\zeta$,

$$K\zeta = \left\{ f \in \mathcal{H}\zeta[a,n] : \ \mathrm{Re}\, \frac{zf''(z,\zeta)}{f'(z,\zeta)} + 1 > 0, \ z \in U, \text{for all}\, \zeta \in \overline{U} \right\}.$$

Definition 1.1. [4] Let $H(z,\zeta)$, $f(z,\zeta)$ be analytic in $U \times \overline{U}$. The function $f(z,\zeta)$ is said to be strongly subordinate to $H(z,\zeta)$, or $H(z,\zeta)$ is said to be strongly superordinate to $f(z,\zeta)$, if there exists a function ω analytic in U, $\omega(0) = 0$, $|\omega(z)| < 1$, such that $f(z,\zeta) = H[\omega(z),\zeta]$, for all $\zeta \in \overline{U}$. In such a case we write $f(z,\zeta) \prec\prec H(z,\zeta), \quad z \in U, \zeta \in \overline{U}$.

Remark 1.2. (i) If $H(z,\zeta)$ is analytic in $U \times \overline{U}$ and univalent in U for all $\zeta \in \overline{U}$, Definition (1.1) is equivalent to $f(0,\zeta) = H[0,\zeta]$, for all $\zeta \in \overline{U}$ and

$$f(U \times \overline{U}) \subset H(U \times \overline{U}).$$

(ii) If $H(z,\zeta) \equiv H(z)$ and $f(z,\zeta) \equiv f(z)$ then the strong subordination becomes the usual notion of subordination.

Definition 1.3. [6] For $f(z,\zeta) \in \mathcal{A}\zeta_n$, $n \in \mathbb{N}^* \cup \{0\}$, we define the integral operator: $I^n : \mathcal{A}\zeta_n \to \mathcal{A}\zeta_n$

$$\begin{aligned}
&I^0 f(z,\zeta) = f(z,\zeta)\\
&I^1 f(z,\zeta) = If(z,\zeta) = \textstyle\int_0^z f(t,\zeta)t^{-1}dt\\
&\dots\\
&I^n f(z,\zeta) = I(I^{n-1} f(z,\zeta)) \qquad (z \in U, \zeta \in \overline{U}).
\end{aligned}$$

Property 1.4. *For $f(z,\zeta) \in \mathcal{A}\zeta_n$, $n \in \mathbb{N}^* \cup \{0\}$, with the integral operator I^n : $\mathcal{A}\zeta_n \to \mathcal{A}\zeta_n$ we have:*

$$z[I^{n+1} f(z,\zeta)]' = I^n f(z,\zeta) \qquad (z \in U, \zeta \in \overline{U}).$$

In order to prove the main results we use the following definitions and lemmas, adapted to the class defined in [4]:

Lemma 1.5. [2, 3] *(Miller and Mocanu) Let $h(z,\zeta)$ be a convex function, with $h(0,\zeta) = a$ and let $\gamma \in \mathbb{C}^*$ be a complex number with* $\operatorname{Re}\gamma \geq 0$. *If $p \in \mathcal{H}\zeta[a,n]$ and*

$$p(z,\zeta) + \frac{1}{\gamma} z p'(z,\zeta) \prec\prec h(z,\zeta)$$

then

$$p(z,\zeta) \prec\prec g(z,\zeta) \prec\prec h(z,\zeta),$$

where

$$g(z,\zeta) = \frac{\gamma}{n z^{\gamma/n}} \int_0^z h(t,\zeta) t^{\frac{\gamma}{n}-1} dt \quad (z \in U, \zeta \in \overline{U}).$$

The function g is convex and is the best (a,n) dominant.

Lemma 1.6. [2, 3] *(Miller and Mocanu) Let $h(z,\zeta)$ be a convex function in U and let*

$$h(z,\zeta) = g(z,\zeta) + n\alpha z g'(z,\zeta), \quad z \in U, \zeta \in \overline{U}$$

where $\alpha > 0$ and n is a positive integer. If

$$p(z,\zeta) = g(0,\zeta) + p_n(\zeta) z^n + p_{n+1}(\zeta) z^{n+1} + \cdots$$

is holomorphic in $U \times \overline{U}$ and

$$p(z,\zeta) + \alpha z p'(z,\zeta) \prec\prec h(z,\zeta),$$

then

$$p(z,\zeta) \prec\prec g(z,\zeta)$$

and this result is sharp.

2. **Main results**

Definition 2.1. Let $\alpha > 1$ and $m, n \in \mathbb{N}$. We denote by $S_n^m(\alpha)$ the set of functions $f \in A\zeta_n$ that satisfy the inequality

$$Re[I^m f(z,\zeta)]' > \alpha, \quad z \in U, \zeta \in \overline{U}.$$

Theorem 2.2. *If $\alpha < 1$, and $m, n \in \mathbb{N}$, then*

$$S_n^m(\alpha) \subset S_n^{m+1}(\delta),$$

where

$$\delta = \delta(\alpha, \zeta, n) = 2\alpha - \zeta + \frac{2(\zeta - \alpha)}{n}\sigma\left(\frac{1}{n}\right)$$

and

$$\sigma(x) = \int_0^1 \frac{t^{x-1}}{1+t}dt. \tag{2.1}$$

Proof. Let $f(z, \zeta) \in S_n^m(\alpha)$. From Definition 2.1 we have

$$Re[I^m f(z, \zeta)]' > \alpha, \quad z \in U, \zeta \in \overline{U}. \tag{2.2}$$

Using Property 1.4, we have

$$I^m f(z, \zeta) = z[I^{m+1} f(z, \zeta)]', \quad z \in U, \zeta \in \overline{U}. \tag{2.3}$$

Differentiating (2.3), with respect to z, we obtain

$$[I^m f(z, \zeta)]' = [I^{m+1} f(z, \zeta)]' + z[I^{m+1} f(z, \zeta)]'', \quad z \in U, \zeta \in \overline{U}. \tag{2.4}$$

We denote by

$$p(z, \zeta) = [I^{m+1} f(z, \zeta)]', \quad z \in U, \zeta \in \overline{U}, p(0, \zeta) = 1, \zeta \in \overline{U}. \tag{2.5}$$

Using (2.5), the relation (2.3) becomes

$$[I^m f(z, \zeta)]' = p(z, \zeta) + zp'(z, \zeta), \quad z \in U, \zeta \in \overline{U} \tag{2.6}$$

and replacing in (2.2), we obtain

$$Re[p(z, \zeta) + zp'(z, \zeta)] > \alpha, \quad z \in U, \zeta \in \overline{U}$$

equivalent to

$$p(z, \zeta) + zp'(z, \zeta) \prec\prec \frac{\zeta + (2\alpha - \zeta)z}{1+z} = h(z, \zeta). \tag{2.7}$$

Using Lemma 1.5, we obtain

$$p(z, \zeta) \prec\prec q(z, \zeta) \prec\prec h(z, \zeta)$$

where

$$q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z \frac{\zeta+(2\alpha-\zeta)t}{1+t}t^{\frac{1}{n}-1}dt = 2\alpha-\zeta+\frac{2(\zeta-\alpha)}{n}\sigma(x),$$

where $\sigma(x)$ is given by (2.1). The function $q(z,\zeta)$ is convex and is the best dominant. With $p(z,\zeta) \prec\prec q(z,\zeta)$ and $q(z,\zeta)$ being convex, and the fact that the image of $U\times\overline{U}$ through $g(z,\zeta)$ is symmetric with respect to the real axis, we deduce that

$$Re\ p(z,\zeta) > g(1,\zeta) = 2\alpha-\zeta+\frac{2(\zeta-\alpha)}{n}\sigma(\frac{1}{n}) = \delta(\alpha,\zeta,n) = \delta, \tag{2.8}$$

equivalent to

$$Re[I^{m+1}f(z,\zeta)]' > \delta, \qquad z\in U, \zeta\in\overline{U}. \tag{2.9}$$

Using Definition 2.1 we obtain $f\in S_n^{m+1}(\delta)$. Since $f\in S_n^m(\alpha)$, we obtain that

$$S_n^m(\alpha)\subset S_n^{m+1}(\delta).$$

□

Theorem 2.3. *Let $h(z,\zeta)$ an analytic function from $U\times\overline{U}$, with $h(0,\zeta)=1$, $h'(0,\zeta)\neq 0$, $\zeta\in\overline{U}$, that satisfies inequality*

$$Re[1+\frac{zh''(z,\zeta)}{h'(z,\zeta)}] > -\frac{1}{2}.$$

If $f(z,\zeta)\in A\zeta_n$ and verify the strong differential subordination

$$[I^m f(z,\zeta)]' \prec\prec h(z,\zeta), \tag{2.10}$$

then

$$[I^{m+1}f(z,\zeta)]' \prec\prec g(z,\zeta)$$

where

$$g(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt, \qquad z\in U, \zeta\in\overline{U}.$$

The function g is convex and is the best dominant.

Proof. A simple application of the differential subordination technique [1, 2], shows that the function $g(z,\zeta)$ is convex. By using (2.6), the strong differential subordination (2.10) becomes

$$p(z,\zeta) + zp'(z,\zeta) \prec\prec h(z,\zeta). \tag{2.11}$$

Using Lemma 1.5, we have

$$p(z,\zeta) \prec\prec g(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t,\zeta) t^{\frac{1}{n}-1} dt.$$

Using (2.5), we obtain

$$[I^{m+1} f(z,\zeta)]' \prec\prec \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t,\zeta) t^{\frac{1}{n}-1} dt.$$

□

Theorem 2.4. *Let $g(z,\zeta)$ be a convex function with $g(0,\zeta) = 1$ and suppose that*

$$h(z,\zeta) = g(z,\zeta) + zg'(z,\zeta), \qquad z \in U, \zeta \in \overline{U}.$$

If $f(z,\zeta) \in A\zeta_n$ and verify the strong differential subordination

$$[I^m f(z,\zeta)]' \prec\prec h(z,\zeta), \tag{2.12}$$

then

$$[I^{m+1} f(z,\zeta)]' \prec\prec g(z,\zeta).$$

Proof. By using (2.6), the strong differential subordination (2.12) becomes

$$p(z,\zeta) + zp'(z,\zeta) \prec\prec g(z,\zeta) + zg'(z,\zeta) \equiv h(z,\zeta).$$

Using Lemma 1.6, we have

$$p(z,\zeta) \prec\prec g(z,\zeta)$$

and using the notation (2.5), we obtain

$$[I^{m+1} f(z,\zeta)]' \prec\prec g(z,\zeta).$$

□

Theorem 2.5. *Let $g(z,\zeta)$ be a convex function with $g(0,\zeta) = 1$ and the function $h(z,\zeta)$, given by*

$$h(z,\zeta) = g(z,\zeta) + nzg'(z,\zeta).$$

If $f(z,\zeta) \in A\zeta_n$ and verify the strong differential subordination

$$[I^m f(z,\zeta)]' \prec\prec h(z,\zeta), \tag{2.13}$$

then

$$\frac{I^m f(z,\zeta)}{z} \prec\prec g(z,\zeta).$$

Proof. We denote with

$$p(z,\zeta) = \frac{I^m f(z,\zeta)}{z}, \qquad z \in U, \zeta \in \overline{U}, p(0,\zeta) = 1. \tag{2.14}$$

Using (2.14), we obtain

$$I^m f(z,\zeta) = zp(z,\zeta), \qquad z \in U, \zeta \in \overline{U}. \tag{2.15}$$

Differentiating (2.15), with respect to z, we obtain

$$[I^m f(z,\zeta)]' = p(z,\zeta) + zp'(z,\zeta), \qquad z \in U, \zeta \in \overline{U}. \tag{2.16}$$

Using (2.16), the strong differential subordination (2.13) becomes

$$p(z,\zeta) + zp'(z,\zeta) \prec\prec g(z,\zeta) + nzg'(z,\zeta).$$

Using Lemma 1.6, we have

$$p(z,\zeta) \prec\prec g(z,\zeta), \quad i.e. \quad \frac{I^m f(z,\zeta)}{z} \prec\prec g(z,\zeta).$$

□

Example 2.6. Let $g(z,\zeta)$ be the function

$$g(z,\zeta) = \frac{1 + (2\alpha - \zeta)z}{1+z}, \qquad z \in U, \zeta \in \overline{U}, g(0,\zeta) = 1, \alpha \in \mathbb{R}, \alpha < 1. \tag{2.17}$$

We verify that $g(z,\zeta)$ is a convex function. Differentiating (2.17), with respect to z, we obtain

$$\mathrm{Re}\left[\frac{zg''(z,\zeta)}{g'(z,\zeta)} + 1\right] = \mathrm{Re}\left[\frac{1-z}{1+z}\right] > 0.$$

From the Theorem (2.4), and using (2.17) we obtain that

$$h(z,\zeta) = g(z,\zeta) + zg'(z,\zeta) = \frac{1+(2\alpha-\zeta)z(2+z)}{(1+z)^2}, \qquad z \in U, \zeta \in \overline{U}. \quad (2.18)$$

For $\alpha = 0$ we obtain

$$h(z,\zeta) = \frac{1-\zeta z(2+z)}{(1+z)^2}.$$

We consider the function

$$g(z,\zeta) = \frac{z-\zeta\frac{z^2}{2}}{1+\frac{z}{2}}.$$

By Theorem (2.4) we obtain that, the strong differential subordination

$$\frac{1-\zeta z-\zeta\frac{z^2}{4}}{(1+\frac{z}{2})^2} \prec\prec \frac{1-\zeta z(2+z)}{(1+z)^2}$$

implies

$$\frac{1-\zeta\frac{z}{2}}{1+\frac{z}{2}} \prec\prec \frac{1-\zeta z}{1+z}.$$

Example 2.7. Let $h(z,\zeta)$ be the function

$$h(z,\zeta) = \frac{\zeta+z}{\zeta-z}, \qquad z \in U, \zeta \in \overline{U}, h(0,\zeta) = 1. \quad (2.19)$$

Let $g(z,\zeta)$ be a convex function with $g(0,\zeta) = 1$ and

$$h(z,\zeta) = g(z,\zeta) + zg'(z,\zeta), \qquad z \in U, \zeta \in \overline{U}.$$

That implies

$$g(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt = \frac{1}{z}\int_0^z \frac{\zeta+t}{\zeta-t}dt$$

and

$$g(z,\zeta) = \frac{-2\zeta}{z}log(\zeta-z) + \frac{2\zeta}{z}log(\zeta) - 1.$$

By Theorem (2.4) we obtain that, the strong differential subordination

$$\frac{2\zeta+z}{2\zeta-z} \prec\prec \frac{\zeta+z}{\zeta-z}$$

implies

$$\frac{-4\zeta}{z}log(2\zeta-z) + \frac{4\zeta}{z}log(2\zeta) - 1 \prec\prec \frac{-2\zeta}{z}log(\zeta-z) + \frac{2\zeta}{z}log(\zeta) - 1.$$

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