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ON THE PROPERTIES OF A SUBCLASS OF ANALYTIC FUNCTIONS

DORINA RĂDUCANU

Dedicated to Professor Grigore Ştefan Sălăgean on his 60^{th} *birthday*

Abstract. In this paper we consider a new class of analytic functions defined by a generalized differential operator. Inclusion results, structural formula, coefficient estimates and other properties of this class of functions are obtained.

1. **Introduction**

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$.

The Hadamard product or convolution of the functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \ \text{ and } \ g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

is given by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \ , \ z \in \mathbb{U}.$$

Let $f \in \mathcal{A}$. We consider the following differential operator introduced by Răducanu and Orhan in [7]:

$$D^0_{\lambda\mu} f(z) = f(z)$$

$$D^1_{\lambda\mu} f(z) = D_{\lambda\mu} f(z) = \lambda\mu z^2 f''(z) + (\lambda - \mu) z f'(z) + (1 - \lambda + \mu) f(z)$$

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$$D_{\lambda\mu}^{m}f(z)=D_{\lambda\mu}\left(D_{\lambda\mu}^{m-1}f(z)\right) \quad (1.2)$$

where $0\leq\mu\leq\lambda$ and $m\in\mathbb{N}:=\{1,2,\ldots\}$.

If the function f is given by (1.1), then from (1.2) we see that:

$$D_{\lambda\mu}^{m}f(z)=z+\sum_{n=2}^{\infty}A_n(\lambda,\mu,m)a_nz^n \quad (1.3)$$

where

$$A_n(\lambda,\mu,m)=\left[1+(\lambda\mu n+\lambda-\mu)(n-1)\right]^m. \quad (1.4)$$

If $\lambda=1$ and $\mu=0$, we get Sălăgean differential operator [9] and if $\mu=0$, we obtain the differential operator defined by Al-Oboudi [1].

From (1.3) it follows that $D_{\lambda\mu}^{m}f(z)$ can be written in terms of convolution as

$$D_{\lambda\mu}^{m}f(z)=(f*g)(z) \quad (1.5)$$

where

$$g(z)=z+\sum_{n=2}^{\infty}A_n(\lambda,\mu,m)z^n. \quad (1.6)$$

Definition 1.1. We say that a function $f\in\mathcal{A}$ is in the class $\mathcal{R}_{\lambda\mu}^{m}(\alpha,\gamma)$ if

$$\Re\left\{(1-\alpha)\frac{D_{\lambda\mu}^{m}f(z)}{z}+\alpha(D_{\lambda\mu}^{m}f(z))'\right\}>\gamma\ ,\ z\in\mathbb{U}$$

for $\alpha\geq0$, $0\leq\gamma<1$, $0\leq\mu\leq\lambda$ and $m\in\mathbb{N}_0:=\{0,1,2,\ldots\}$.

Note that:

i. $\mathcal{R}_{\lambda\mu}^{0}(1,\gamma)$ is the subclass of $\mathcal{A}$ consisting of functions with $\Re f'(z)>\gamma$.
ii. $\mathcal{R}_{\lambda 0}^{m}(1,\gamma)$ is the class of functions investigated in [1].
iii. $\mathcal{R}_{\lambda\mu}^{m}(1,\gamma)$ reduces to the class of functions considered in [8].
iv. $\mathcal{R}_{\lambda\mu}^{0}(\alpha,\gamma)$ is the class of functions studied by G. Chunyi and S. Owa in [4].

The main object of this paper is to present a systematic investigation for the class $\mathcal{R}_{\lambda\mu}^{m}(\alpha,\gamma)$. In particular, for this class of functions we obtain some inclusion results, structural formula, extreme points and other properties.

2. **Inclusion results**

In order to prove our inclusion results we need the following lemmas.

Lemma 2.1. *([4]) Let $\alpha \geq 0$ and $\gamma \geq 0$. Let $D(z)$ be a starlike function in $\mathbb{U}$ and let $N(z)$ be an analytic function in $\mathbb{U}$ such that $N(0) = D(0) = 0$ and $N'(0) = D'(0) = 1$. If*

$$\Re\left[(1-\alpha)\frac{N(z)}{D(z)} + \alpha\frac{N'(z)}{D'(z)}\right] > \gamma\ ,\ z \in \mathbb{U}$$

then

$$\Re\frac{N(z)}{D(z)} > \gamma\ ,\ z \in \mathbb{U}.$$

Lemma 2.2. *([6]) Let $h(z)$ be a convex function in $\mathbb{U}$ and let $A \geq 0$. Suppose that $B(z)$ and $C(z)$ are analytic in $\mathbb{U}$ with $C(0) = 0$ and*

$$\Re B(z) \geq A + 4\left|\frac{C(z)}{h'(0)}\right|\ ,\ z \in \mathbb{U}.$$

If p is an analytic function , with $p(0) = h(0)$, which satisfies

$$Az^2p''(z) + B(z)zp'(z) + p(z) + C(z) \prec h(z)\ ,\ z \in \mathbb{U}$$

then $p(z) \prec h(z),\ z \in \mathbb{U}$.

Note that the symbol " $\prec$ " stands for subordination.

Theorem 2.3. *Let $\alpha \geq 0$, $0 \leq \gamma < 1$, $0 \leq \mu \leq \lambda$ and $m \in \mathbb{N}_0$. Then*

$$\mathcal{R}^m_{\lambda\mu}(\alpha,\gamma) \subset \mathcal{R}^m_{\lambda\mu}(0,\gamma).$$

Proof. Suppose $f \in \mathcal{R}^m_{\lambda\mu}(\alpha,\gamma)$. Then, from Definition 1.1, we have

$$\Re\left\{(1-\alpha)\frac{D^m_{\lambda\mu}f(z)}{z} + \alpha(D^m_{\lambda\mu}f(z))'\right\} > \gamma\ ,\ z \in \mathbb{U}.$$

Consider $N(z) = D^m_{\lambda\mu}f(z)$. Making use of (1.3) we have $N(0) = 0$ and $N'(0) = 1$. Let $D(z) = z$. Since $D(z)$ is starlike in $\mathbb{U}$ and $D(0) = 0 = D'(0) - 1$, from Lemma 2.1, we obtain

$$\Re\left\{\frac{D^m_{\lambda\mu}f(z)}{z}\right\} > \gamma\ ,\ z \in \mathbb{U}$$

which implies $f \in \mathcal{R}^m_{\lambda\mu}(0,\gamma)$. Thus $\mathcal{R}^m_{\lambda\mu}(\alpha,\gamma) \subset \mathcal{R}^m_{\lambda\mu}(0,\gamma)$ and the proof of the theorem is completed. $\square$

Theorem 2.4. *Let $0 \le \beta < \alpha$, $0 \le \gamma < 1$, $0 \le \mu \le \lambda$ and $m \in \mathbb{N}_0$. Then*

$$\mathcal{R}^m_{\lambda\mu}(\alpha,\gamma) \subset \mathcal{R}^m_{\lambda\mu}(\beta,\gamma).$$

Proof. If $\beta = 0$, from Theorem 2.3, we have $\mathcal{R}^m_{\lambda\mu}(\alpha,\gamma) \subset \mathcal{R}^m_{\lambda\mu}(0,\gamma)$.
Let $f \in \mathcal{R}^m_{\lambda\mu}(\alpha,\gamma)$ and assume $\beta \neq 0$. Then

$$(1-\beta)\frac{D^m_{\lambda\mu}f(z)}{z} + \beta(D^m_{\lambda\mu}f(z))' =$$
$$\frac{\beta}{\alpha}\left[\left(\frac{\alpha}{\beta}-1\right)\frac{D^m_{\lambda\mu}f(z)}{z} + (1-\alpha)\frac{D^m_{\lambda\mu}f(z)}{z} + \alpha(D^m_{\lambda\mu}f(z))'\right].$$

Since $f \in \mathcal{R}^m_{\lambda\mu}(\alpha,\gamma)$, making use of Definition 1.1 and Theorem 2.3, we obtain

$$\Re\left\{(1-\beta)\frac{D^m_{\lambda\mu}f(z)}{z} + \beta(D^m_{\lambda\mu}f(z))'\right\} =$$
$$\frac{\beta}{\alpha}\left[\left(\frac{\alpha}{\beta}-1\right)\Re\frac{D^m_{\lambda\mu}f(z)}{z} + \Re\left\{(1-\alpha)\frac{D^m_{\lambda\mu}f(z)}{z} + \alpha(D^m_{\lambda\mu}f(z))'\right\}\right]$$
$$> \frac{\beta}{\alpha}\left(\frac{\alpha}{\beta}-1\right)\gamma + \frac{\beta}{\alpha}\gamma = \gamma.$$

It follows that $f \in \mathcal{R}^m_{\lambda\mu}(\beta,\gamma)$ and thus, $\mathcal{R}^m_{\lambda\mu}(\alpha,\gamma) \subset \mathcal{R}^m_{\lambda\mu}(\beta,\gamma)$. □

Another inclusion result is given in the next theorem.

Theorem 2.5. *Let $\alpha \ge 0$, $0 \le \gamma < 1$, $0 \le \mu \le \lambda$ and $m \in \mathbb{N}_0$. Then*

$$\mathcal{R}^{m+1}_{\lambda\mu}(\alpha,\gamma) \subset \mathcal{R}^m_{\lambda\mu}(\alpha,\gamma).$$

Proof. Suppose $f \in \mathcal{R}^{m+1}_{\lambda\mu}(\alpha,\gamma)$. Then

$$\Re\left\{(1-\alpha)\frac{D^{m+1}_{\lambda\mu}f(z)}{z} + \alpha(D^{m+1}_{\lambda\mu}f(z))'\right\} > \gamma$$

which is equivalent to

$$(1-\alpha)\frac{D^{m+1}_{\lambda\mu}f(z)}{z} + \alpha(D^{m+1}_{\lambda\mu}f(z))' \prec h(z)\ ,\ z \in \mathbb{U} \tag{2.1}$$

where

$$h(z) = \frac{1+(1-2\gamma)z}{1-z}\ ,\ z \in \mathbb{U}. \tag{2.2}$$

From (1.2), we have

$$D^{m+1}_{\lambda\mu}f(z) = \lambda\mu z^2[D^m_{\lambda\mu}f(z)]'' + (\lambda-\mu)z[D^m_{\lambda\mu}f(z)]' + (1-\lambda+\mu)D^m_{\lambda\mu}f(z).$$

It follows that

$$R(z) := (1-\alpha)\frac{D_{\lambda\mu}^{m+1}f(z)}{z} + \alpha(D_{\lambda\mu}^{m+1}f(z))'$$

$$= \lambda\mu\left\{(1-\alpha)\frac{z^2(D_{\lambda\mu}^{m}f(z))''}{z} + \alpha[z^2(D_{\lambda\mu}^{m}f(z))'']'\right\}$$

$$+(\lambda-\mu)\left\{(1-\alpha)\frac{z(D_{\lambda\mu}^{m}f(z))'}{z} + \alpha[z(D_{\lambda\mu}^{m}f(z))']'\right\}$$

$$+(1-\lambda+\mu)\left\{(1-\alpha)\frac{D_{\lambda\mu}^{m}f(z)}{z} + \alpha(D_{\lambda\mu}^{m}f(z))'\right\}.$$

Denote

$$p(z) = (1-\alpha)\frac{D_{\lambda\mu}^{m}f(z)}{z} + \alpha(D_{\lambda\mu}^{m}f(z))'\ ,\ z\in\mathbb{U}. \tag{2.3}$$

Simple calculations show that

$$R(z) = \lambda\mu z^2 p''(z) + (2\lambda\mu+\lambda-\mu)zp'(z) + p(z). \tag{2.4}$$

Making use of (2.4), the differential subordination (2.1) becomes

$$\lambda\mu z^2 p''(z) + (2\lambda\mu+\lambda-\mu)zp'(z) + p(z) \prec h(z)\ ,\ z\in\mathbb{U}.$$

It is easy to check that conditions of Lemma 2.2 with $h(z)$ given by (2.2), $p(z)$ given by (2.3), $A=\lambda\mu$, $B(z)\equiv 2\lambda\mu+\lambda-\mu$ and $C(z)\equiv 0$ are satisfied. Thus, we obtain $p(z)\prec h(z)$ which implies that

$$\Re\left\{(1-\alpha)\frac{D_{\lambda\mu}^{m}f(z)}{z} + \alpha(D_{\lambda\mu}^{m}f(z))'\right\} > \gamma\ ,\ z\in\mathbb{U}.$$

Therefore, $f\in\mathcal{R}_{\lambda\mu}^{m}(\alpha,\gamma)$ and the proof of our theorem is completed. □

3. **Structural formula**

In this section a structural formula, extreme points, coefficient bounds for functions in $\mathcal{R}_{\lambda\mu}^{m}(\alpha,\gamma)$ are given.

Theorem 3.1. *A function $f\in\mathcal{A}$ is in the class $\mathcal{R}_{\lambda\mu}^{m}(\alpha,\gamma)$ if and only if it can be expressed as*

$$f(z) = \left[z+\sum_{n=2}^{\infty}\frac{z^n}{A_n(\lambda,\mu,m)}\right] * \int_{|\zeta|=1}\left[z+2(1-\gamma)\bar{\zeta}\sum_{n=2}^{\infty}\frac{(\zeta z)^n}{1+(n-1)\alpha}\right]d\mu(\zeta) \tag{3.1}$$

where $\mu(\zeta)$ is the probability measure defined on the unit circle

$$\mathbb{T} = \{\zeta \in \mathbb{C} : |\zeta| = 1\}.$$

Proof. Definition 1.1 implies that $f \in \mathcal{R}^m_{\lambda\mu}(\alpha, \gamma)$ if and only if

$$\frac{(1-\alpha)\frac{D^m_{\lambda\mu}f(z)}{z} + \alpha(D^m_{\lambda\mu}f(z))' - \gamma}{1-\gamma} = p(z)\ ,\ z \in \mathbb{U} \tag{3.2}$$

where $p(z)$ belongs to the class $\mathcal{P}$ consisting of normalized analytic functions which have positive real part in $\mathbb{U}$.

From (3.2) we have

$$(1-\alpha)\frac{D^m_{\lambda\mu}f(z) - \gamma z}{z} + \alpha[(D^m_{\lambda\mu}f(z))' - \gamma] = (1-\gamma)p(z). \tag{3.3}$$

If $\alpha \neq 0$, multiplying both sides of (3.3) by $\frac{1}{\alpha}z^{\frac{1}{\alpha}-1}$, we obtain

$$\left[z^{\frac{1}{\alpha}-1}(D^m_{\lambda\mu}f(z) - \gamma z)\right]' = z^{\frac{1}{\alpha}-1}\frac{1-\gamma}{\alpha}p(z).$$

Using Herglotz expression of functions in the class $\mathcal{P}$, we have

$$\left[z^{\frac{1}{\alpha}-1}(D^m_{\lambda\mu}f(z) - \gamma z)\right]' = z^{\frac{1}{\alpha}-1}\frac{1-\gamma}{\alpha}\int_{|\zeta|=1}\frac{1+\zeta z}{1-\zeta z}d\mu(\zeta).$$

Integrating both sides of this equality we get

$$z^{\frac{1}{\alpha}-1}(D^m_{\lambda\mu}f(z) - \gamma z) = \int_0^z\left[u^{\frac{1}{\alpha}-1}\frac{1-\gamma}{\alpha}\int_{|\zeta|=1}\frac{1+\zeta u}{1-\zeta u}d\mu(\zeta)\right]du$$

which is equivalent to

$$D^m_{\lambda\mu}f(z) = \frac{1}{\alpha}\int_{|\zeta|=1}\left[z^{1-\frac{1}{\alpha}}\int_0^z u^{\frac{1}{\alpha}-1}\frac{1+\zeta u(1-2\gamma)}{1-\zeta u}du\right]d\mu(\zeta).$$

So we have

$$D^m_{\lambda\mu}f(z) = \int_{|\zeta|=1}\left[z + 2(1-\gamma)\bar{\zeta}\sum_{n=2}^{\infty}\frac{(\zeta z)^n}{1+(n-1)\alpha}\right]d\mu(\zeta). \tag{3.4}$$

From (1.5), (1.6) and (3.4) it follows that

$$f(z) = \left[z + \sum_{n=2}^{\infty}\frac{z^n}{A_n(\lambda,\mu,m)}\right] * \int_{|\zeta|=1}\left[z + 2(1-\gamma)\bar{\zeta}\sum_{n=2}^{\infty}\frac{(\zeta z)^n}{1+(n-1)\alpha}\right]d\mu(\zeta).$$

Since this deductive process can be converse, we have proved our theorem. □

Remark 3.2. If $\alpha = 0$, the expression (3.1) is also true and it says that if $f \in \mathcal{A}$ satisfies $\Re \frac{D^m_{\lambda\mu} f(z)}{z} > \gamma$, then f can be expressed as

$$f(z) = \left[z + \sum_{n=2}^{\infty} \frac{z^n}{A_n(\lambda, \mu, m)} \right] * \int_{|\zeta|=1} \left[z + 2(1-\gamma)\bar{\zeta} \sum_{n=2}^{\infty} (\zeta z)^n \right] d\mu(\zeta).$$

Corollary 3.3. *The extreme points of the class* $\mathcal{R}^m_{\lambda\mu}(\alpha, \gamma)$ *are*

$$f_\zeta(z) = z + 2(1-\gamma)\bar{\zeta} \sum_{n=2}^{\infty} \frac{(\zeta z)^n}{[1+(n-1)\alpha] A_n(\lambda, \mu, m)}\ ,\ z \in \mathbb{U}\ ,\ |\zeta| = 1. \tag{3.5}$$

Proof. Denote

$$[D^m_{\lambda\mu} f(z)]_\zeta = z + 2(1-\gamma)\bar{\zeta} \sum_{n=2}^{\infty} \frac{(\zeta z)^n}{1+(n-1)\alpha}.$$

Then, equality (3.4) can be written as

$$[D^m_{\lambda\mu} f(z)]_\mu = \int_{|\zeta|=1} [D^m_{\lambda\mu} f(z)]_\zeta d\mu(\zeta).$$

Since probability measures $\{\mu\}$ and class $\mathcal{P}$ are one-to-one it follows that the map $\mu \to [D^m_{\lambda\mu} f(z)]_\mu$ is one-to-one and the assertion follows (see [5]). □

Making use of Corollary 3.3 we can obtain coefficients bounds for the functions in the class $\mathcal{R}^m_{\lambda\mu}(\alpha, \gamma)$.

Corollary 3.4. *If* $f \in \mathcal{R}^m_{\lambda\mu}(\alpha, \gamma)$, *then*

$$|a_n| \le \frac{2(1-\gamma)}{[1+(n-1)\alpha] A_n(\lambda, \mu, m)}\ ,\ n \ge 2.$$

The result is sharp.

Proof. The coefficient bounds are maximized at an extreme point so, the result follows from (3.5). □

Corollary 3.5. *If* $f \in \mathcal{R}^m_{\lambda\mu}(\alpha, \gamma)$, *then for* $|z| = r < 1$

$$|f(z)| \ge r - 2(1-\gamma) r^2 \sum_{n=2}^{\infty} \frac{1}{[1+(n-1)\alpha] A_n(\lambda, \mu, m)}$$

$$|f(z)| \le r + 2(1-\gamma) r^2 \sum_{n=2}^{\infty} \frac{1}{[1+(n-1)\alpha] A_n(\lambda, \mu, m)}$$

and

$$|f'(z)| \geq 1 - 2(1-\gamma)r \sum_{n=2}^{\infty} \frac{n}{[1+(n-1)\alpha]A_n(\lambda,\mu,m)}$$

$$|f'(z)| \leq 1 + 2(1-\gamma)r \sum_{n=2}^{\infty} \frac{n}{[1+(n-1)\alpha]A_n(\lambda,\mu,m)}$$

4. **Convolution property**

In order to prove a convolution property for the class $\mathcal{R}^m_{\lambda\mu}(\alpha,\gamma)$ we need the following result.

Lemma 4.1. *([10]) If $p(z)$ is analytic in $\mathbb{U}$, $p(0) = 1$ and $\Re p(z) > \frac{1}{2}$, then for any analytic function F in $\mathbb{U}$, the function $F * p$ takes values in the convex hull of $F(\mathbb{U})$.*

Theorem 4.2. *The class $\mathcal{R}^m_{\lambda\mu}(\alpha,\gamma)$ is closed under the convolution with a convex function. That is, if $f \in \mathcal{R}^m_{\lambda\mu}(\alpha,\gamma)$ and g is convex in $\mathbb{U}$, then $f * g \in \mathcal{R}^m_{\lambda\mu}(\alpha,\gamma)$.*

Proof. It is known that, if g is a convex function in $\mathbb{U}$, then

$$\Re \frac{g(z)}{z} > \frac{1}{2}.$$

Suppose that $f \in \mathcal{R}^m_{\lambda\mu}(\alpha,\gamma)$. Making use of the convolution properties, we have

$$\Re\left\{(1-\alpha)\frac{D^m_{\lambda\mu}(f*g)(z)}{z} + \alpha[D^m_{\lambda\mu}(f*g)(z)]'\right\} =$$

$$\Re\left\{\left[(1-\alpha)\frac{D^m_{\lambda\mu}f(z)}{z} + \alpha[D^m_{\lambda\mu}f(z)]'\right] * \frac{g(z)}{z}\right\}.$$

Using Lemma 4.1, the result follows. □

Corollary 4.3. *The class $\mathcal{R}^m_{\lambda\mu}(\alpha,\gamma)$ is invariant under Bernardi integral operator* [3] *defined by*

$$F_c(f)(z) = \frac{1+c}{z^c}\int_0^z t^{c-1}f(t)dt\ ,\ \Re c > 0.$$

Proof. Assume $f \in \mathcal{R}^m_{\lambda\mu}(\alpha,\gamma)$. It is easy to check that $F_c(f)(z) = (f*g)(z)$, where

$$g(z) = \sum_{n=1}^{\infty} \frac{1+c}{n+c} z^n.$$

Since the function g is convex (see [2]), the result follows by applying Theorem 4.2. Therefore, $F_c[\mathcal{R}^m_{\lambda\mu}(\alpha,\gamma)] \subset \mathcal{R}^m_{\lambda\mu}(\alpha,\gamma)$. □

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Faculty of Mathematics and Computer Science
"Transilvania" University of Braşov
50091, Iuliu Maniu, 50, Braşov, Romania
E-mail address: dorinaraducanu@yahoo.com