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A CONVEXITY PROPERTY FOR AN INTEGRAL OPERATOR F_m

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Dedicated to Professor Grigore Ştefan Sălăgean on his 60^{th} birthday

Abstract. In this paper we define an integral operator denoted by $F_m(z)$ using the Ruscheweyh derivative of order n applied to the functions $f_i(z) \in \mathcal{A}$, $i = \{1, 2, \dots, m\}$, $z \in U$. We determine conditions on the functions $R^n f_i(z)$, where R^n is the Ruscheweyh operator (Definition 1.1), in order for $F_m(z)$ to be convex.

1. Introduction and preliminaries

Let U be the unit disk of the complex plane:

$$U = \{z \in \mathbb{C} : \ |z| < 1\}.$$

Let $\mathcal{H}(U)$ be the space of holomorphic functions in U. Also, let

$$A_n = \{f \in \mathcal{H}(U), \ f(z) = z + a_{n+1}z^{n+1} + \dots, \ z \in U\}$$

with $A_1 = A$ and

$$S = \{f \in A : \ f \text{ is univalent in } U\}.$$

Let

$$K(\alpha) = \left\{ f \in A : \ \operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > \alpha, \ z \in U \right\}$$

denote the class of normalized convex functions of order α, where $0 \le \alpha < 1$,

$$K(0) = K,$$

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$$S^*(\alpha) = \left\{ f \in A : \ \mathrm{Re}\, \frac{zf'(z)}{f(z)} > \alpha,\ z \in U \right\}$$

denote the class of starlike functions of order α, with $0 \le \alpha < 1$, $S^*(0) = S^*$.

In the papers [9], [10], F. Ronning introduces two classes of univalent functions denoted by SP and $SP(\alpha, \beta)$, respectively. The class SP consists of those functions $f \in S$ which satisfy the condition

$$\mathrm{Re}\, \frac{zf'(z)}{f(z)} > \left| \frac{zf'(z)}{f(z)} - 1 \right|, \ \text{for all } z \in U. \tag{1.1}$$

The class $SP(\alpha, \beta)$, $\alpha > 0$, $\beta \in [0, 1)$ consists of the functions $f \in S$ which satisfy the condition

$$\left| \frac{zf'(z)}{f(z)} - (\alpha + \beta) \right| \le \mathrm{Re}\, \frac{zf'(z)}{f(z)} + \alpha - \beta, \ \text{for all } z \in U. \tag{1.2}$$

In [12], the authors introduce the class denoted by $SD(\alpha, \beta)$ consisting of the functions $f \in A$ which satisfy the inequality

$$\mathrm{Re}\, \frac{zf'(z)}{f(z)} > \alpha \left| \frac{zf'(z)}{f(z)} - 1 \right| + \beta, \tag{1.3}$$

for $\alpha \ge 0$ and $\beta \in [0, 1)$.

Definition 1.1. (St. Ruscheweyh [11]). For $f \in A$, $n \in \mathbb{N} \cup \{0\}$, let R^n be the operator defined by $R^n : A \to A$

$$R^0 f(z) = f(z)$$
$$(n+1) R^{n+1} f(z) = z[R^n f(z)]' + n R^n f(z),\ z \in U.$$

Remark 1.2. If $f \in A$

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j$$

then

$$R^n f(z) = z + \sum_{j=1}^{\infty} C_{n+j-1}^{n} a_j z^j, \quad z \in U,$$

with

$$R^n f(0) = 0 \quad \text{and} \quad [R^n f(0)]' = 1.$$

2. Main results

By using the Ruscheweyh differential operator (Definition 1.1) we introduce the following integral operator.

Definition 2.1. Let $n, m \in \mathbb{N} \cup \{0\}$, $i \in \{1, 2, 3, \dots, m\}$, $\alpha_i \in \mathbb{R}$ with $\alpha_i \geq 0$,

$$A^m = \underbrace{A \times A \times \cdots \times A}_{m}.$$

We define the integral operator $I : A^m \to A$

$$I(f_1, f_2, \dots, f_m)(z) = F_m(z) \tag{2.1}$$

$$= \int_0^z \left[\frac{R^n f_1(t)}{t}\right]^{\alpha_1} \cdots \left[\frac{R^n f_m(t)}{t}\right]^{\alpha_m} dt, \quad z \in U$$

where $f_i(z) \in A$, $i \in \{1, 2, 3, \dots, m\}$ and R^n is the Ruscheweyh differential operator given by Definition 1.1.

Remark 2.2. (i) For $n = 0$, $m = 1$, $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = \cdots = \alpha_m = 0$,

$$R^0 f(z) = f(z) \in A$$

and we obtain Alexander integral operator introduced in 1915 in [1]:

$$I(z) = \int_0^z \frac{f(t)}{t} dt, \quad z \in U.$$

(ii) For $n = 0$, $m = 1$, $\alpha_1 = \alpha \in [0, 1]$, $\alpha_2 = \alpha_3 = \cdots = \alpha_m = 0$, $R^0 f(z) = f(z) \in S$ and we obtain the integral operator

$$I(z) = \int_0^z \left[\frac{f(t)}{t}\right]^{\alpha} dt, \quad z \in U$$

which was studied in several papers such as [6]. For $\alpha \in \mathbb{C}$, $|\alpha| \leq \frac{1}{4}$ the operator was studied in [4], [5] and for $|\alpha| \leq \frac{1}{3}$ in [8].

(iii) For $n = 1$, $m = 1$, $\alpha_1 = \alpha \in \mathbb{C}$, $|\alpha| \leq \frac{1}{4}$, $\alpha_2 = \cdots = \alpha_m = 0$, $R^1 f(z) = z f'(z)$, $z \in U$, $f \in S$, and we obtain the integral operator

$$I(z) = \int_0^z [f'(t)]^{\alpha} dt$$

which was studied in [7].

(iv) For $n = 0$, $m \in \mathbb{N} \cup \{0\}$, $\alpha_i > 0$, $i \in \{1, 2, \ldots, m\}$ we obtain the integral operator defined by D. Breaz and N. Breaz in [3] given by

$$F(z) = \int_0^z \left[\frac{f_1(t)}{t}\right]^{\alpha_1} \left[\frac{f_2(t)}{t}\right]^{\alpha_2} \cdots \left[\frac{f_m(t)}{t}\right]^{\alpha_m} dt.$$

Property 2.3. *Let $m \in \mathbb{N} \cup \{0\}$, $i \in \{1, 2, \ldots, m\}$. If $f_i(z) \in A$ then $F_m(z)$ given by (2.1) belongs to the class A.*

Proof. From (2.1) we have

$$\begin{aligned} F_m(z) &= \int_0^z \left[\frac{R^n f_1(t)}{t}\right]^{\alpha_1} \cdots \left[\frac{R^n f_m(t)}{t}\right]^{\alpha_m} dt \\ &= \int_0^z \left[\frac{t + \sum_{k=2}^{\infty} a_{k,1} t^k}{t}\right]^{\alpha_1} \cdots \left[\frac{t + \sum_{k=2}^{\infty} a_{k,m} t^k}{t}\right]^{\alpha_m} dt \\ &= \int_0^z \left[1 + \sum_{k=2}^{\infty} a_{k,1} t^{k-1}\right]^{\alpha_1} \cdots \left[1 + \sum_{k=2}^{\infty} a_{k,m} t^{k-1}\right]^{\alpha_m} dt \\ &= \int_0^z \left(1 + \sum_{k=2}^{\infty} \gamma_k t\right) dt = t\Big|_0^z + \sum_{k=2}^{\infty} \gamma_k \frac{t^k}{k}\Big|_0^z \\ &= z + \sum_{k=2}^{\infty} \delta_k t^k \in A, \end{aligned}$$

hence $F_m(z) \in A$. □

Definition 2.4. Let $\mathcal{R}(\beta)$ be the subclass of functions $f \in A$ which satisfy the condition

$$\operatorname{Re} \frac{z[R^n f(z)]'}{R^n f(z)} > \beta, \quad 0 \le \beta < 1, \ z \in U. \tag{2.2}$$

Remark 2.5. (i) For $n = 0$, $\mathcal{R}(\beta)$ becomes the class of starlike functions of order β denoted by $S^*(\beta)$.

(ii) For $n = 0$, $\beta = 0$, $\mathcal{R}(\beta)$ becomes $\mathcal{R}(0) = S^*$, the class of starlike functions.

Definition 2.6. Let $K(\beta) \subset A^m = \underbrace{A \times A \times \cdots \times A}_{m}$ denote the subclass of functions $(f_1, f_2, \ldots, f_m) \in A^m$ which satisfy the condition

$$\operatorname{Re} \left[1 + \frac{zF_m''(z)}{F_m'(z)}\right] > \beta, \quad \beta < 1, \ z \in U, \tag{2.3}$$

where $F_m(z)$ is given by (2.1).

Theorem 2.7. *Let $n, m \in \mathbb{N} \cup \{0\}$, $i \in \{1, 2, \ldots, m\}$, $\alpha_i \in \mathbb{R}$ with $\alpha_i \geq 0$, $\beta_i \in \mathbb{R}$, $0 \leq \beta_i < 1$ and $\sum_{i=1}^{m} \alpha_i(\beta_i - 1) \geq -1$. If $f_i \in \mathcal{R}(\beta_i)$ then $F_m(z) \in K(\delta)$ where F_m is given by (4) and*

$$\delta = 1 + \sum_{i=1}^{m} \alpha_i(\beta_i - 1).$$

Proof. By differentiating (2.1), we obtain

$$F'_m(z) = \left[\frac{R^n f_1(z)}{z}\right]^{\alpha_1} \ldots \left[\frac{R^n f_m(z)}{z}\right]^{\alpha_m}, \quad z \in U. \tag{2.4}$$

Using (2.4) we obtain:

$$\text{Log } F'_m(z) = \alpha_1[\text{Log } R^n f_1(z) - \text{Log } z] + \cdots + \tag{2.5}$$
$$+ \alpha_m[\text{Log } R^n f_m(z) - \text{Log } z], \quad z \in U.$$

By differentiating (2.5) we have

$$\frac{F''_m(z)}{F'_m(z)} = \alpha_1 \left[\frac{(R^n f_1(z))'}{R^n f_1(z)} - \frac{1}{z}\right] + \cdots + \alpha_m \left[\frac{(R^n f_m(z))'}{R^n f_m(z)} - \frac{1}{z}\right] \tag{2.6}$$

and after a short calculation we obtain

$$1 + \frac{zF''_m(z)}{F'_m(z)} = \alpha_1 \frac{z(R^n f_1(z))'}{R^n f_1(z)} + \cdots + \alpha_m \frac{z(R^n f_m(z))'}{R^n f_m(z)} \tag{2.7}$$
$$+1 - (\alpha_1 + \cdots + \alpha_m).$$

Since $f_i \in \mathcal{R}(\beta_i)$ we have

$$\text{Re } \left[1 + \frac{zF''_m(z)}{F'_m(z)}\right] = \alpha_1 \text{Re } \frac{z(R^n f_1(z))'}{R^n f_1(z)} + \cdots + \alpha_m \text{Re } \frac{z(R^n f_m(z))'}{R^n f_m(z)}$$
$$+1 - \sum_{i=1}^{m} \alpha_i > \alpha_1\beta_1 + \cdots + \alpha_m\beta_m + 1 - \sum_{i=1}^{m} \alpha_i$$
$$> \sum_{i=1}^{m} \alpha_i\beta_i + 1 - \sum_{i=1}^{m} \alpha_i > 1 + \sum_{i=1}^{m} \alpha_i(\beta_i - 1).$$

□

If $f_i \in \mathcal{R}(\beta)$ then Theorem 2.7 can be rewritten as the following:

Corollary 2.8. *Let $n, m \in \mathbb{N} \cup \{0\}$, $i \in \{1, 2, \ldots, m\}$, $\alpha_i \in \mathbb{R}$ with $\alpha_i \geq 0$, $\beta \in \mathbb{R}$, $\beta < 1$. If $f_i \in \mathcal{R}(\beta)$ then $F_m(z) \in K(\delta')$ where*

$$\delta' = 1 + (\beta - 1) \sum_{i=1}^{m} \alpha_i.$$

If $\beta = 0$ and $\sum_{i=1}^{m} \alpha_i = 1$ then Theorem 2.7 can be rewritten as the following:

Corollary 2.9. *Let $n, m \in \mathbb{N} \cup \{0\}$, $i \in \{1, 2, \ldots, m\}$, $\alpha_i \in \mathbb{R}$ with $\alpha_i \geq 0$ and $\sum_{i=1}^{m} \alpha_i = 1$, and $\beta = 0$. If $f_i \in \mathcal{R}(0)$ then $F_m(z) \in K$.*

Theorem 2.10. *Let $n, m \in \mathbb{N} \cup \{0\}$, $i \in \{1, 2, \ldots, m\}$, $\alpha_i \in \mathbb{R}$ with $0 \leq \alpha_i < 1$ and $\sum_{i=1}^{m} \frac{\alpha_i}{2} < 1$. If $R^n f_i \in SP$ and*

$$\left| \frac{z(R^n f_i(z))'}{R^n f_i(z)} - 1 \right| \geq \frac{1}{2}, \quad z \in U \tag{2.8}$$

then

$$F_m(k) \in K(\omega),$$

where F_m is given by (4) and

$$\omega = 1 - \sum_{i=1}^{m} \frac{\alpha_i}{2}.$$

Proof. Since $R^n f_i \in SP$, using (2.7) and (2.8) we have:

$$\begin{aligned}
\operatorname{Re}\left[1 + \frac{zF_m''(z)}{F_m'(z)}\right] &= \alpha_1 \operatorname{Re} \frac{z(R^n f_1(z))'}{R^n f_1(z)} + \cdots + \alpha_m \operatorname{Re} \frac{z(R^n f_m(z))'}{R^n f_m(z)} \\
+1 - \sum_{i=1}^{m} \alpha_i &\geq \alpha_1 \left| \frac{z(R^n f_1(z))'}{R^n f_1(z)} - 1 \right| + \cdots + \alpha_m \left| \frac{z(R^n f_m(z))'}{R^n f_m(z)} - 1 \right| \\
&\quad +1 - \sum_{i=1}^{m} \alpha_i > 1 - \sum_{i=1}^{m} \alpha_i + \sum_{i=1}^{m} \frac{\alpha_i |z|}{2} \\
&\quad = 1 - \sum_{i=1}^{m} \alpha_i \left(1 - \frac{|z|}{2}\right).
\end{aligned}$$

□

If $\sum_{i=1}^{m} \alpha_i = 1$ then Theorem 2.10 can be rewritten as the following:

Corollary 2.11. *Let $n, m \in \mathbb{N} \cup \{0\}$, $i \in \{1, 2, \ldots, m\}$, $\alpha_i \in \mathbb{R}$ with $0 \le \alpha_i < 1$ and $\sum_{i=1}^{m} \alpha_i \left(1 - \frac{|z|}{2}\right) = 1$. If $R^n f_i \in SP$ then $F_m(z)$ is a convex function.*

Example 2.12. Let $n \in \mathbb{N} \cup \{0\}$, $m = 2$, $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{1}{3}$,

$$f_1(z) = z + a_2 z^2, \quad f_2(z) = z + b_2 z^2,$$

$$R^n f_1(z) = z + (n+1) a_2 z^2, \quad R^n f_2(z) = z + (n+1) b_2 z^2,$$

where $a_2, b_2 \in \mathbb{C}$, $|a_2| \ge \frac{1}{(n+1)(2-|z|)}$, $|b_2| \ge \frac{1}{(n+1)(2-|z|)}$, $z \in U$.

Evaluate

$$\left| \frac{z[z + (n+1) a_2 z^2]'}{z[1 + (n+1) a_2 z]} - 1 \right| = \left| \frac{(n+1) a_2 z}{1 + (n+1) a_2 z} \right|$$

$$= \sqrt{\frac{(n+1)^2 |a_2|^2 |z|}{(1 + (n+1)|a_2||z|)^2}} = \frac{(n+1)|a_2||z|}{1 + (n+1)|a_2||z|} \ge \frac{1}{2}$$

$$\left| \frac{z[z + (n+1) b_2 z^2]'}{z(1 + (n+1) b_2 z^2]} - 1 \right| = \left| \frac{1 + 2(n+1) b_2 z}{1 + (n+1) b_2 z} - 1 \right| = \left| \frac{(n+1) b_2 z}{1 + (n+1) b_2 z} \right|$$

$$= \sqrt{\frac{(n+1)^2 |b_2|^2 |z|^2}{[1 + (n+1)|b_2||z|]^2}} = \frac{(n+1)|b_2||z|}{1 + (n+1)|b_2||z|} \ge \frac{1}{2}$$

Using Theorem 2.10, we have

$$F_2(z) = \int_0^z [1 + (n+1) a_2 t]^{\frac{1}{2}} [1 + (n+1) b_2 t]^{\frac{1}{3}} dt \in K\left(\frac{1}{2}\right), \quad z \in U.$$

Theorem 2.13. *Let $n, m \in \mathbb{N} \cup \{0\}$, $i \in \{1, 2, \ldots, m\}$, $\alpha_i \in \mathbb{R}$ with $\alpha_i \ge 0$, $\lambda \in \mathbb{R}$ with $\lambda > 0$, $\mu \in \mathbb{R}$ with $\mu \in [0, 1)$ and $(\lambda - \mu + 1) \sum_{i=1}^{m} \alpha_i \le 1$. If $R^n f_i \in SP(\lambda, \mu)$ then $F_m \in K(\omega)$, where F_m is given by (4) and*

$$\omega = 1 - (\lambda - \mu + 1) \sum_{i=1}^{m} \alpha_i.$$

Proof. Since $R^n f_i \in SP(\lambda, \mu)$, using (2.7) we have:

$$\mathrm{Re}\left[1 + \frac{z F_m''(z)}{F_m'(z)}\right] = \alpha_1 \mathrm{Re}\, \frac{z(R^n f_1(z))'}{R^n f_1(z)} + \cdots + \alpha_m \mathrm{Re}\, \frac{z(R^n f_m(z))'}{R^n f_m(z)}$$

$$+1 - \sum_{i=1}^{m} \alpha_i \ge \alpha_1 \left[\left| \frac{z(R^n f_1(z))'}{R^n f_1(z)} - (\lambda + \mu) \right| - (\lambda - \mu) \right] + \cdots +$$

$$+\alpha_m\left[\left|\frac{z(R^nf_m(z))'}{R^nf_m(z)}-(\lambda+\mu)\right|-(\lambda-\mu)\right]+1-\sum_{i=1}^m\alpha_i$$

$$\geq\sum_{i=1}^m\alpha_i\left[\left|\frac{z(R^nf_i(z))'}{R^nf_1(z)}-(\lambda+\mu)\right|\right]-\sum_{i=1}^m\alpha_i(\lambda-\mu)$$

$$+1-\sum_{i=1}^m\alpha_i\geq 1-\sum_{i=1}^m\alpha_i(\lambda-\mu)-\sum_{i=1}^m\alpha_i$$

$$=1-\sum_{i=1}^m\alpha_i(1-\mu+1)=1-(\lambda-\mu+1)\sum_{i=1}^m\alpha_i.$$

□

If $\sum_{i=1}^m\alpha_i(\lambda-\mu+1)=1$ then Theorem 2.13 can be rewritten as the following:

Corollary 2.14. *Let $n,m\in\mathbb{N}\cup\{0\}$, $i\in\{1,2,\ldots,m\}$, $\alpha_i\in\mathbb{R}$ with $\alpha_i\geq 0$, $\lambda>0$, $\mu\in[0,1)$ with $(\lambda-\mu+1)\sum_{i=1}^m\alpha_i=1$. If $R^nf_i\in SP(\lambda,\mu)$ then $F_m(z)\in K$.*

Theorem 2.15. *Let $n,m\in\mathbb{N}\cup\{0\}$, $i\in\{1,2,\ldots,m\}$, $\alpha_i\in\mathbb{R}$, $\alpha_i\geq 0$, $\gamma\in\mathbb{R}$, $\gamma\geq 0$, $\delta\in(0,1)$ with $\left(1-\frac{\gamma}{4}-\delta\right)\sum_{i=1}^m\alpha_i<1$. If $R^nf_i\in SD(\gamma,\delta)$ and*

$$\left|\frac{z(R^nf_i(z))'}{R^nf_i(z)}-1\right|\geq\frac{1}{4},\quad z\in U,\ i\in\{1,2,3,\ldots,m\},\tag{2.9}$$

then $F_m\in K(\xi)$, where F_m is given by (4) and

$$\xi=1-\left(1-\frac{\gamma}{4}-\delta\right)\sum_{i=1}^m\alpha_i.$$

Proof. Since $R^nf_i\in SD(\gamma,\delta)$, using (10) and (12), we have

$$\text{Re}\left[1+\frac{zF_m''(z)}{F_m'(z)}\right]=\alpha_1\text{Re}\,\frac{z(R^nf_1(z))'}{R^nf_1(z)}+\cdots+\alpha_m\text{Re}\,\frac{z(R^nf_m(z))'}{R^nf_m(z)}+1-\sum_{i=1}^m\alpha_i$$

$$\geq\alpha_1\left[\gamma\left|\frac{z(R^nf_1(z))'}{R^nf_1(z)}-1\right|+\delta\right]+\cdots+\alpha_m\left[\gamma\left|\frac{z(R^nf_m(z))'}{R^nf_m(z)}-1\right|+\delta\right]+1-\sum_{i=1}^m\alpha_i$$

$$\geq\gamma\left[\alpha_1\left|\frac{z(R^nf_1(z))'}{R^nf_1(z)}-1\right|+\cdots+\alpha_m\left|\frac{z(R^nf_m(z))'}{R^nf_m(z)}-1\right|\right]+\delta\sum_{i=1}^m\alpha_i+1-\sum_{i=1}^m\alpha_i$$

$$\geq\frac{\gamma|z|}{4}\left(\frac{\alpha_1}{4}+\cdots+\frac{\alpha_m}{4}\right)+\delta\sum_{i=1}^m\alpha_i+1-\sum_{i=1}^m\alpha_i$$

$$=1-\left(1-\frac{\gamma}{4}-\delta\right)\sum_{i=1}^m\alpha_i.$$

□

If $1-\left(1-\frac{\gamma}{4}-\delta\right)\sum_{i=1}^{m}\alpha_i = 0$, then Theorem 2.15 can be rewritten as the following.

Corollary 2.16. *Let $n, m \in \mathbb{N}\cup\{0\}$, $i \in \{1,2,3,\dots,m\}$, $\alpha_i \in \mathbb{R}$, $\alpha_i \geq 0$, $\gamma \geq 0$, $\delta \in (0,1)$ with $1-\left(1-\frac{\gamma}{4}-\delta\right)\sum_{i=1}^{m}\alpha_i = 0$. If $R^n f_i \in SD(\gamma,\delta)$ and*

$$\left|\frac{z(R^n f_i(z))'}{R^n f_i(z)}-1\right| \geq \frac{1}{4}, \quad z \in U,\ i \in \{1,2,\dots,m\}$$

then $F_m(z)$ is convex.

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