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ON ARGUMENT PROPERTY OF CERTAIN ANALYTIC FUNCTIONS

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Dedicated to Professor Grigore Ştefan Sălăgean on his 60^{th} *birthday*

Abstract. In this paper we generalize the results of Libera and McGregor concerning argument property of analytic functions. We use the result in [3] to prove the following:

Theorem. *Let*

$$f(z) = z + \sum_{n=p+K}^{\infty} a_n z^n, \qquad g(z) = z + \sum_{n=p+K}^{\infty} b_n z^n$$

be analytic in Δ, $f(z) \neq 0$ *in* $0 < |z| < 1$, *and suppose that for some* α, β $(0 < \alpha < 1,\ 0 < \beta < 1)$

$$\left|\arg\left(\frac{f'(z)}{g'(z)}\right)\right| < \frac{\pi}{2}\alpha + \mathrm{Tan}^{-1}\frac{2\alpha\beta}{1-\beta^2} - \mathrm{Tan}^{-1}\frac{2\alpha\beta}{(1-\beta^2)\sqrt{1+\alpha^2}}$$

in Δ, *and that* $\dfrac{g'(z)}{zg(z)} \prec \dfrac{1+\beta z}{1-\beta z}$ *where* $\prec$ *means subordination. Then we have*

$$\left|\arg\left(\frac{f(z)}{g(z)}\right)\right| < \frac{\pi}{2}\alpha \quad in \quad \Delta.$$

1. **Introduction**

Let f and g be analytic in the unit disk $\Delta = \{z : |z| < 1\}$ $f(0) = g(0) = 0$, g maps Δ onto a many sheeted domain which is starlike with respect to the origin, and

$$\mathrm{Re}\frac{f'(z)}{g'(z)} > 0 \quad \text{in } \Delta\ .$$

Then Libera [1] proved

$$\mathrm{Re}\frac{f(z)}{g(z)} > 0 \quad \text{in } \Delta\ .$$

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The essential ideas of the proof of the above result are the same as given by Sakaguchi [6].

On the other hand, MacGregor [2] proved that for real β,

$$\mathrm{Re}\frac{f'(z)}{g'(z)} > \beta \quad \text{in } \Delta.$$

implies

$$\mathrm{Re}\frac{f(z)}{g(z)} > \beta \quad \text{in } \Delta.$$

Ponnusamy and Karunakaran [4] generalized the above results as the following:

Theorem 1.1. *Let α be a complex number satisfying $Re\alpha > 0$ and $\beta < 1$. Let*

$$f(z) = z^p + \sum_{n=p+K}^{\infty} a_n z^n, \quad g(z) = z^p + \sum_{n=p+K}^{\infty} b_n z^n$$

are analytic in Δ for $1 \leq p$, $1 \leq K$ and that g satisfies

$$\mathrm{Re}\left(\alpha\frac{g(z)}{g'(z)}\right) > \delta \quad \text{in } \Delta$$

where

$$0 \leq \delta < \frac{\mathrm{Re}\alpha}{p}.$$

If

$$\mathrm{Re}\left\{(1-\alpha)\frac{f(z)}{g(z)} + \alpha\frac{f'(z)}{g'(z)}\right\} > \beta \quad \text{in } \Delta.$$

Then

$$\mathrm{Re}\frac{f(z)}{g(z)} > \frac{2\beta + K\delta}{2 + K\delta} \quad \text{in } \Delta.$$

Putting $\alpha = 1$ in Theorem 1.1, it follows that

Corollary 1.2. *If*

$$f(z) = z^p + \sum_{n=p+K}^{\infty} a_n z^n \quad , \quad 1 \leq p, \quad 1 \leq pK \text{ and } \quad g(z) = z^p + \sum_{n=p+K}^{\infty} b_n z^n$$

are analytic in Δ and g satisfies

$$\mathrm{Re}\frac{g(z)}{zg'(z)} > \delta \quad \text{in } \Delta$$

where $0 \leq \delta < \frac{1}{p}$ *then for β real*

$$\frac{f'(z)}{g'(z)}) > \beta \quad \text{in } \Delta$$

implies

$$\mathrm{Re}\frac{f(z)}{g(z)} > \frac{2\beta + K\delta}{2 + K\delta} \quad \text{in } \Delta.$$

For a argument properties of analytic functions, Pommerenke [5] obtained the following result. If f is analytic in Δ and h is convex in Δ and

$$\left|\arg\left(\frac{f'(z)}{h'(z)}\right)\right| < \frac{\alpha\pi}{2} \quad (0 \le \alpha \le 1)$$

then

$$\left|\arg\left(\frac{f(z_2) - f(z_1)}{h(z_2) - h(z_1)}\right)\right| < \frac{\alpha\pi}{2}$$

where

$$|z_1| < 1 \quad \text{and} \quad |z_2| < 1.$$

2. Main theorem

In this short paper, we will obtain a generalization of Libera's result by applying Nunokawa's result [3].

Lemma 2.1. *Let p be analytic in Δ, $p(0) = 1$, $p(z) \neq 0$ in Δ and suppose that there exists a point $z_0 \in \Delta$ such that*

$$|\arg p(z_0)| < \frac{\pi}{2}\alpha \qquad for\ |z| < |z_0|$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}\alpha$$

where $0 < \alpha$.
Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = iK\alpha$$

where

$$K \ge \frac{1}{2}\left(a + \frac{1}{a}\right) \quad \text{when} \quad \arg p(z_0) = \frac{\pi}{2}\alpha$$

and

$$K \le -\frac{1}{2}\left(a + \frac{1}{a}\right) \quad \text{when} \quad \arg p(z_0) = -\frac{\pi}{2}\alpha$$

where

$$\arg p(z_0)^{\frac{1}{\alpha}} = \pm ia \quad \text{and} \quad 0 < a$$

Theorem 2.2. *Let*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

be analytic in Δ $\quad f(z) \neq 0 \quad$ *in* $\quad 0 < |z| < 1$,

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

be analytic in Δ *and suppose*

$$|\arg(\frac{f'(z)}{g'(z)})| < \frac{\pi}{2}\alpha + \mathrm{Tan}^{-1}\frac{2\alpha\beta}{1-\beta^2} - \mathrm{Tan}^{-1}\frac{2\alpha\beta}{(1-\beta^2)\sqrt{1+\alpha^2}}$$

in Δ *where* $\quad 0 < \alpha < 1, \quad 0 < \beta < 1$ *and*

$$\frac{zg'(z)}{g(z)} \prec \frac{1+\beta z}{1-\beta z}$$

where $\prec$ *means the subordination. Then we have*

$$\left|\arg\left(\frac{f(z)}{g(z)}\right)\right| < \frac{\pi}{2}\alpha \qquad \text{in } \Delta.$$

Proof. Let us put

$$p(z) = \frac{f(z)}{g(z)}, \quad p(0) = 1$$

Then it follows that

$$\frac{f'(z)}{g'(z)} = p(z) + \frac{g(z)}{g'(z)}p'(z)$$

$$= p(z)\left(1 + \frac{g(z)}{zg'(z)}.\frac{zp'(z)}{p(z)}\right).$$

If there exist a point z_0, $|z_0| < 1$ such that

$$|\arg p(z)| < \frac{\pi}{2}\alpha \quad \text{for } |z| < |z_0|$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}\alpha$$

then from Lemma 2.1 we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = iK\alpha.$$

From the hypothesis, we have the image of the circle Δ under the mapping $w = \frac{1+\beta z}{1-\beta z}$ is contained in the circle whose center is $\frac{1+\beta^2}{1-\beta^2}$ with radius $\frac{2\beta}{1-\beta^2}$. Applying the above properties, for the case

$$\arg p(z_0) = \frac{\pi}{2}\alpha,$$

we have

$$\begin{aligned}\arg \frac{f'(z_0)}{g'(z_0} &= \arg p(z_0) + \arg\left(1 + i\alpha K \frac{g(z_0)}{g'(z_0)}\right)\\ &\geq \frac{\pi}{2}\alpha + \mathrm{Tan}^{-1}\delta K - \mathrm{Tan}^{-1}\frac{\delta K}{\sqrt{1+(\rho^2-\delta^2)K^2}}\end{aligned}$$

where $\rho = \frac{\alpha(1+\beta^2)}{1-\beta^2}$, $\delta = \frac{2\alpha\beta}{1-\beta^2}$ and then it follows that

$$\rho^2 - \delta^2 = \alpha^2.$$

Now let us put

$$\mathrm{F(K)} = \mathrm{Tan}^{-1}\delta K - Tan^{-1}\frac{\delta K}{\sqrt{1+(\rho^2-\delta^2)K^2}} \qquad , \quad 1 \leq K.$$

Then we have

$$\begin{aligned}\mathrm{F'(K)} &= \frac{\delta}{1+\delta^2K^2} - \left(\frac{\delta(1-(\rho^2-\delta^2)K^2) - \delta(\rho^2-\delta^2)K^2}{(1+(\rho^2-\delta^2)K^2)^{\frac{3}{2}}}\right)\frac{(1+(\rho^2-\delta^2)K^2)}{1+\rho^2K^2}\\ &= \frac{\delta}{1+\delta^2K^2} - \frac{\delta}{(1+\rho^2K^2)\sqrt{1+(\rho^2-\delta^2)K^2}}\\ &> \frac{\delta}{1+\delta^2K^2} - \frac{\delta}{1+\rho^2K^2}\\ &= \frac{\delta(\rho^2-\delta^2)}{(1+\delta^2K^2)(1+\rho^2K^2)} > 0.\end{aligned}$$

This shows that $F(K)$ takes the minimum value at $K = 1$. Therefore we have

$$\arg\frac{f'(z_0)}{g'(z_0)} \geq \frac{\pi}{2}\alpha + \mathrm{Tan}^{-1}\frac{2\alpha\beta}{1-\beta^2} - \mathrm{Tan}^{-1}\frac{2\alpha\beta}{(1-\beta^2)\sqrt{1+\alpha^2}}$$

This contradicts the hypothesis and for the case $\arg p(z_0) = -\frac{\pi}{2}\alpha$, applying the same method as the above, we have

$$\arg\frac{f'(z_0)}{g'(z_0)} \leq -\left(\frac{\pi}{2}\alpha + Tan^{-1}\frac{2\alpha\beta}{1-\beta^2} - Tan^{-1}\frac{2\alpha\beta}{(1-\beta^2)\sqrt{1+\alpha^2}}\right).$$

This is also contradiction and it completes the proof. □

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