STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LV**, Number 3, September 2010

ORDER OF CLOSE-TO-CONVEXITY FOR ANALYTIC FUNCTIONS OF COMPLEX ORDER

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Dedicated to Professor Grigore Ştefan Sălăgean on his 60^{th} *birthday*

Abstract. The aim of this paper is to find the order of close-to-convexity for certain analytic functions of complex order.

1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. A function $f(z)$ in $\mathcal{A}$ is said to be starlike function of complex order $\gamma(\gamma \in \mathbb{C}-\{0\})$, if and only if

$$\operatorname{Re}\left\{1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > 0, \qquad (z \in \mathcal{U}). \tag{1.2}$$

We denote by $\mathcal{S}(\gamma)$ the class of all such functions. Also, a function $f(z)$ in $\mathcal{A}$ is said to be convex function of complex order $\gamma(\gamma \in \mathbb{C}-\{0\})$, that is, $f \in \mathcal{C}(\gamma)$, if and only if

$$\operatorname{Re}\left\{1 + \frac{1}{\gamma}\frac{zf''(z)}{f'(z)}\right\} > 0, \qquad (z \in \mathcal{U}). \tag{1.3}$$

The class $\mathcal{S}(\gamma)$ was introduced by Nasr and Aouf [7] and the class $\mathcal{C}(\gamma)$ was introduced by Wiatrowski [15] and considered in[6] (see also [5], [10], [13] and [2]).

Received by the editors: 01.03.2010.

2000 *Mathematics Subject Classification.* 30C45.

Key words and phrases. Analytic, starlike and convex functions of complex order, close-to-convex functions.

We note that $f(z) \in \mathcal{C}(\gamma) \Leftrightarrow zf'(z) \in \mathcal{S}(\gamma)$ and $\mathcal{S}(1-\alpha) = \mathcal{S}^*(\alpha)$, $\mathcal{C}(1-\alpha) = \mathcal{C}(\alpha)$ where $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ denote, respectively, the familiar classes of starlike and convex functions of a real order $\alpha(0 \leq \alpha < 1)$ in $\mathcal{U}$ (see, for example, [14]).

A function $f(z)$ in $\mathcal{A}$ is said to be close-to-convex of complex order $\gamma(\gamma \in \mathbb{C}-\{0\})$, and type $\delta \in \mathbb{R}$ if there exists a function $g(z)$ belonging to $\mathcal{S}(\gamma)$ such that

$$\mathrm{Re}\left\{1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{g(z)} - 1\right)\right\} > \delta, \qquad (z \in \mathcal{U}). \tag{1.4}$$

We denote by $\mathcal{K}(\gamma, \delta)$ the subclass of $\mathcal{A}$ consisting of functions which are close-to-convex of complex order γ and type β in $\mathcal{U}$. We note that the class $\mathcal{K}(1, 0)$ is the class of close-to-convex functions introduced by Kaplan [4] and Ozaki [11].

Pfaltzgraff *et al.*[12] have proved that if $f(z)$ in $\mathcal{A}$ satisfies the condition

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (\frac{1}{2} \leq \alpha < 1), \tag{1.5}$$

then $f(z)$ in the class $\mathcal{S}$ (and convex in at least one direction in $\mathcal{U}$). Furthermore, Cerebiez-Tarabicka et al. [1] have shown that if $f(z)$ in $\mathcal{A}$ satisfies the condition

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > -\frac{1}{2} \qquad (\frac{1}{2} \leq \alpha < 1), \tag{1.6}$$

then

$$\mathrm{Re}\left(\frac{zf'(z)}{g(z)}\right) > 0, \qquad (z \in \mathcal{U}). \tag{1.7}$$

Recently, Owa [9] proved that if $f(z)$ in $\mathcal{A}$ satisfies the condition

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \qquad (z \in \mathcal{U}) \tag{1.8}$$

then

$$\mathrm{Re}\left(\frac{zf'(z)}{g(z)}\right) > \frac{3}{5} \qquad (z \in \mathcal{U}) \tag{1.9}$$

where $g(z) \in \mathcal{S}^*(\alpha/(\alpha+1))$, $\alpha \geq 0$.

Also, Frasin and Oros [3] proved that if the function $f(z)$ in $\mathcal{A}$ satisfies the condition

$$\mathrm{Re}\left(\frac{zf''(z)}{f'(z)} - \beta\right) > 0 \qquad (z \in \mathcal{U}) \tag{1.10}$$

then

$$\mathrm{Re}\left(\frac{zf'(z)}{g(z)}\right) > \frac{1}{2\beta - 1} \qquad (z \in \mathcal{U}) \tag{1.11}$$

where $g(z) \in \mathcal{S}^*$and $1 < \beta \leq 3/2$.

In order to show our results, we shall need the following lemma due to Obradovič *et al.*[8].

Lemma 1.1. *Let* $f \in \mathcal{S}(b)$, $b \in \mathbb{C}-\{0\}$, *and let* $a \in \mathbb{C}-\{0\}$ *with* $0 < 2ab \leq 1$. *Then*

$$Re\left\{\left(\frac{f(z)}{z}\right)^{a}\right\} > 2^{-2ab} \qquad (z \in \mathcal{U}). \tag{1.12}$$

2. Main results

With the aid of Lemma 1.1, we can prove the following result.

Theorem 2.1. *If the functions* $f(z)$*and* $g(z)$ *are in* $\mathcal{A}$ *and satisfies the conditions*

$$Re\left\{1+\frac{1}{b}\left(\frac{zf''(z)}{f'(z)}\right)\right\} > 0 \qquad (z \in \mathcal{U}), \tag{2.1}$$

with $0 < 2a\gamma \leq 1$, $\gamma = b/\ (a+1)$; $a, b \in \mathbb{C}-\{0\}$; $a \neq -1$, *and*

$$Im\left(\frac{a+1}{b}\right) \leq 0 \ \textit{or} \ Im\left(\frac{zf'(z)}{g(z)}\right) \leq 0, \tag{2.2}$$

then $f(z)$ *belongs to the class* $\mathcal{K}(\gamma, \delta)$, *where*

$$\delta = 1 + \left(2^{\frac{-2ab}{a+1}} - 1\right) Re\left(\frac{a+1}{b}\right).$$

Proof. If we define $g(z)$ by

$$1 + \frac{a+1}{b}\left(\frac{zg'(z)}{g(z)} - 1\right) = 1 + \frac{1}{b}\left(\frac{zf''(z)}{f'(z)}\right) \tag{2.3}$$

then from the condition (2.1) and (2.3), we have $g(z) \in \mathcal{S}(\gamma)$, with $\gamma = b/\ (a+1)$. It is easy to see that (2.3) implies

$$f'(z) = \left(\frac{g(z)}{z}\right)^{a+1} \tag{2.4}$$

or

$$\frac{zf'(z)}{g(z)} = \left(\frac{g(z)}{z}\right)^{a} \tag{2.5}$$

Applying Lemma 1.1 to $g(z)$, we obtain

$$\begin{aligned}
\mathrm{Re}\left\{1+\frac{a+1}{b}\left(\frac{zf'(z)}{g(z)}-1\right)\right\} &= \mathrm{Re}\left\{1+\frac{a+1}{b}\left(\left(\frac{g(z)}{z}\right)^{a}-1\right)\right\} \\
&= 1+\mathrm{Re}\left(\frac{a+1}{b}\right)\mathrm{Re}\left\{\left(\frac{g(z)}{z}\right)^{a}-1\right\} \\
&- \mathrm{Im}\left(\frac{a+1}{b}\right)\mathrm{Im}\left\{\left(\frac{g(z)}{z}\right)^{a}-1\right\} \\
&\geq 1+\mathrm{Re}\left(\frac{a+1}{b}\right)\mathrm{Re}\left\{\left(\frac{g(z)}{z}\right)^{a}-1\right\} \\
&> 1+\left(2^{-2a\gamma}-1\right)\mathrm{Re}\left(\frac{a+1}{b}\right) \\
&= 1+\left(2^{\frac{-2ab}{a+1}}-1\right)\mathrm{Re}\left(\frac{a+1}{b}\right).
\end{aligned}$$

This completes the proof of Theorem 2.1. □

Letting $a=1$ in Theorem 2.1, we have

Corollary 2.2. *If the function* $f\in\mathcal{C}(b)$ *with* $0<b\leq 2$*, then* $f\in\mathcal{K}(b/2,\delta)$*, where*

$$\delta=1+\frac{2^{1-b}-2}{b}.$$

Letting $b=1$ in Theorem 2.1, we have

Corollary 2.3. *If the functions* $f(z)$*and* $g(z)$ *are in* $\mathcal{A}$ *and satisfies the conditions*

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0 \qquad (z\in\mathcal{U}), \tag{2.6}$$

with $0<2a\gamma\leq 1$, $\gamma=1/\ (a+1)$*;* $a\in\mathbb{C}-\{0\}$*;* $a\neq -1$*, and*

$$Im\,(a+1)\leq 0 \ \ or \ \ Im\left(\frac{zf'(z)}{g(z)}\right)\leq 0, \tag{2.7}$$

then $f(z)$ *belongs to the class* $\mathcal{K}(\gamma,\delta)$*, where*

$$\delta=1+\left(2^{\frac{-2a}{a+1}}-1\right)Re\,(a+1)\,.$$

Letting $b=1$ in Corollary 2.2 or $a=1$ in Corollary 2.3, we have

Corollary 2.4. *Let the functions* $f(z)$*and* $g(z)$ *be in* $\mathcal{A}$ *. If*

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0 \qquad (z\in\mathcal{U}), \tag{2.8}$$

then

$$Re\left(\frac{zf'(z)}{g(z)}\right)>\frac{1}{2} \qquad (z\in\mathcal{U}), \tag{2.9}$$

Therefore, if $f(z)$ *is convex in* U *then* $f(z)$ *is close-to-convex of order* $1/2$ *in* $\mathcal{U}$.

Letting $b = a + 1$ in in Theorem 2.1, we have

Corollary 2.5. *Let the functions* $f(z)$*and* $g(z)$ *be in* $\mathcal{A}$ *. If*

$$Re\left\{1+\frac{1}{a+1}\left(\frac{zf''(z)}{f'(z)}\right)\right\}>0 \qquad (z\in\mathcal{U}), \tag{2.10}$$

where $0 < a \leq 1/2$,*then*

$$Re\left\{\frac{zf'(z)}{g(z)}\right\}>\frac{1}{4^a}, \qquad (z\in\mathcal{U}). \tag{2.11}$$

Letting $a = 1/2$ in Corollary 2.5, we have

Corollary 2.6. *Let the functions* $f(z)$*and* $g(z)$ *be in* $\mathcal{A}$ *. If*

$$Re\left\{1+\frac{2}{3}\left(\frac{zf''(z)}{f'(z)}\right)\right\}>0 \qquad (z\in\mathcal{U}), \tag{2.12}$$

then

$$Re\left\{\frac{zf'(z)}{g(z)}\right\}>\frac{1}{2}, \qquad (z\in\mathcal{U}), \tag{2.13}$$

That is, $f(z)$ *is close-to-convex of order* $1/2$ *in* $\mathcal{U}$.

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