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CLASSES OF MEROMORPHIC FUNCTIONS DEFINED BY THE EXTENDED SĂLĂGEAN OPERATOR

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Dedicated to Professor Grigore Ştefan Sălăgean on his 60^{th} birthday

Abstract. In the paper, we define classes of meromorphic functions, in terms of the extended Sălăgean operator. By using Jack's Lemma and the Briot-Bouquet differential subordination we obtain some inclusion relations for defined classes.

1. **Introduction**

Let $\mathcal{A}$ denote the class of functions which are *analytic* in $\mathcal{U} := \mathcal{U}(1)$, where $\mathcal{U}(R) := \{z : |z| < R\}$, $0 < R \leq 1$. By Ω we denote the class of the Schwarz functions, i.e. the class of functions $\omega \in \mathcal{A}$, such that

$$\omega(0) = 0, \ |\omega(z)| < 1 \quad (z \in \mathcal{U}) .$$

For complex parameters β, γ and functions $h \in \mathcal{A}$, $\omega \in \Omega$, we consider the first-order differential equation of the form

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = (h \circ \omega)(z), \quad q(0) = h(0) = 1. \tag{1.1}$$

If there exist a function $\omega \in \Omega$, such that the function $q \in \mathcal{A}$ is a solution of the Cauchy problem (1.1) then we write

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} \prec h(z). \tag{1.2}$$

The expression (1.2) is a first-order differential subordination and it is called the Briot-Bouquet differential subordination.

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More general, we say that a function $f \in \mathcal{A}$ is *subordinate* to a function $F \in \mathcal{A}$, and write $f(z) \prec F(z)$, if and only if there exists a function $\omega \in \Omega$, such that

$$f(z) = (F \circ \omega)(z) \quad (z \in \mathcal{U}).$$

Moreover, we say that f is subordinate to F in $\mathcal{U}(R)$, if $f(Rz) \prec F(Rz)$. We shall write

$$f(z) \prec_R F(z)$$

in this case. In particular, if F is univalent in $\mathcal{U}$ we have the following equivalence (cf. [5]):

$$f(z) \prec F(z) \iff f(0) = F(0) \text{ and } f(\mathcal{U}) \subset F(\mathcal{U}).$$

Let $\mathcal{M}$ denote the class of functions f of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \tag{1.3}$$

which are analytic in $\mathcal{D} = \mathcal{U} \setminus \{0\}$. By $f * g$ we denote the *Hadamard product* (*or convolution*) of f, $g \in \mathcal{M}$, defined by

$$(f * g)(z) = \left(\sum_{n=-1}^{\infty} a_n z^n\right) * \left(\sum_{n=-1}^{\infty} b_n z^n\right) := \sum_{n=-1}^{\infty} a_n b_n z^n.$$

Let λ, σ be positive real numbers. Motivated by the Sălăgean operator [6] we consider the linear operator $D_\sigma^\lambda : \mathcal{M} \to \mathcal{M}$ defined by

$$D_\sigma^\lambda f(z) = \left(f * h_{\lambda,\sigma}\right)(z),$$

where

$$h_{\lambda,\sigma}(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{n+\sigma+1}{\sigma}\right)^\lambda z^n \quad (z \in \mathcal{D}).$$

It is closely related to Cho and Srivastava operator [1] (see also [7]) and the multiplier transformations studied by Flett [3].

For a function $f \in \mathcal{M}$ we have

$$z\left[D_\sigma^\lambda f(z)\right]' = \sigma D_\sigma^{\lambda+1} f(z) - (1+\sigma) D_\sigma^\lambda f(z). \tag{1.4}$$

A function $f \in \mathcal{A}$ of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (z \in \mathcal{U}(r))$$

is said to be p *-valently starlike* in $\mathcal{U}(r)$ if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad (z \in \mathcal{U}(r);\ 0 < r \leq 1).$$

Note that all functions p-valently starlike in $\mathcal{U}(r)$ are p-valent in $\mathcal{U}(r)$. In particular we have

$$f(z) \neq 0 \quad (z \in \mathcal{U}(r) \setminus \{0\}).$$

Let h be a function convex in $\mathcal{U}$ with

$$h(0) = 1,\ \mathrm{Re}\, h(z) > 0 \quad (z \in \mathcal{U}) \tag{1.5}$$

and let t be a complex number. We denote by $\mathcal{V}(t, \lambda, \sigma; h)$ the class of functions $f \in \mathcal{M}$ satisfying the following condition:

$$z\left[(1-t)\, D_\sigma^\lambda f(z) + tD_\sigma^{\lambda+1} f(z)\right] \prec h(z), \tag{1.6}$$

in terms of subordination.

Moreover we define the class $\mathcal{W}(t, \lambda, \sigma; h)$ of functions $f \in \mathcal{M}$ satisfying the following condition:

$$\frac{(1-t)\, D_\sigma^{\lambda+1} f(z) + tD_\sigma^{\lambda+2} f(z)}{(1-t)\, D_\sigma^\lambda f(z) + tD_\sigma^{\lambda+1} f(z)} \prec h(z). \tag{1.7}$$

In particular for real constants $A, B,\ -1 \leq A < B \leq 1$, we denote

$$\begin{aligned} \mathcal{V}\,(t, \lambda, \sigma; A, B) &= \mathcal{V}\left(t, \lambda, \sigma; \frac{1+Az}{1+Bz}\right), \\ \mathcal{W}\,(t, \lambda, \sigma; A, B) &= \mathcal{W}\left(t, \lambda, \sigma; \frac{1+Az}{1+Bz}\right). \end{aligned}$$

In the paper we present some inclusion relations for the defined classes.

2. **Main results**

We shall need some lemmas.

Lemma 2.1. [4] *Let* w *be a nonconstant function analytic in* $\mathcal{U}(r)$ *with* $w(0) = 0$*. If*

$$|w(z_0)| = \max\{|w(z)|\,;\ |z| \leq |z_0|\} \quad (z_0 \in \mathcal{U}(r))\,,$$

then there exists a real number k $(k \geq 1)$*, such that*

$$z_0 w'(z_0) = k w(z_0).$$

We shall need also a modified result of Eenigenburg, Miller, Mocanu and Reade [2] (see also [5]).

Lemma 2.2. *Let* h *be a convex function in* U, *with*

$$\mathrm{Re}[\beta h(z) + \gamma] > 0 \quad (z \in \mathcal{U})$$

If a function q *satisfies the Briot-Bouquet differential subordination (1.2) in* $\mathcal{U}(R)$, *i.e*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} \prec_R h(z),$$

then

$$q(z) \prec_R h(z).$$

Making use of above lemmas, we get the following two theorem.

Theorem 2.3.

$$\mathcal{V}(t, \lambda + m, \sigma; h) \subset \mathcal{V}(t, \lambda, \sigma; h) \quad (m \in \mathbf{N}).$$

Proof. It is clear that it is sufficient to prove the theorem for $m = 1$. Let a function f belong to the class $\mathcal{V}(t, \lambda + 1, \sigma; h)$ or equivalently

$$z\left[(1-t)\, D_\sigma^{\lambda+1} f(z) + t D_\sigma^{\lambda+2} f(z)\right] \prec h(z). \tag{2.1}$$

It is sufficient to verify the condition (1.6). The function

$$q(z) = z\left[(1-t)\, D_\sigma^{\lambda} f(z) + t D_\sigma^{\lambda+1} f(z)\right] \tag{2.2}$$

is analytic in $\mathcal{U}$ and $q(0) = 1$. Taking the derivative of (2.2) we get

$$z\left[(1-t)\, D_\sigma^{\lambda+1} f(z) + t D_\sigma^{\lambda+2} f(z)\right] = q(z) + \frac{zq'(z)}{\sigma} \quad (z \in \mathcal{U})\,. \tag{2.3}$$

Thus by (2.1) we have

$$q(z)+\frac{zq'(z)}{\sigma}\prec h(z).$$

Lemma 2.2 now yields

$$q(z)\prec h(z).$$

Thus by (2.2) $f\in\mathcal{V}(t,\lambda,\sigma;h)$ and this proves Theorem 2.3. □

Theorem 2.4.

$$\mathcal{W}(t,\lambda{+}m,\sigma;h)\subset\mathcal{W}(t,\lambda,\sigma;h)\quad(m\in\mathbf{N}).$$

Proof. It is clear that it is sufficient to prove the theorem for $m=1$. Let a function f belong to the class $\mathcal{W}(t,\lambda{+}1,\sigma;h)$ or equivalently

$$\frac{(1-t)\,D_\sigma^{\lambda+2}f(z)+tD_\sigma^{\lambda+3}f(z)}{(1-t)\,D_\sigma^{\lambda+1}f(z)+tD_\sigma^{\lambda+2}f(z)}\prec h(z). \tag{2.4}$$

It is sufficient to verify the condition (1.7). If we put

$$R=\sup\left\{r:\ (1-t)\,D_\sigma^{\lambda}f(z)+tD_\sigma^{\lambda+1}f(z)\neq 0,\ 0<|z|<r\right\}, \tag{2.5}$$

then the function

$$q(z)=\frac{(1-t)\,D_\sigma^{\lambda+1}f(z)+tD_\sigma^{\lambda+2}f(z)}{(1-t)\,D_\sigma^{\lambda}f(z)+tD_\sigma^{\lambda+1}f(z)} \tag{2.6}$$

is analytic in $\mathcal{U}(R)$ and $q(0)=1$. Taking the logarithmic derivative of (2.6) and applying (1.4) we get

$$\frac{(1-t)\,D_\sigma^{\lambda+2}f(z)+tD_\sigma^{\lambda+3}f(z)}{(1-t)\,D_\sigma^{\lambda+1}f(z)+tD_\sigma^{\lambda+2}f(z)}=q(z)+\frac{zq'(z)}{\sigma q(z)}\quad(z\in\mathcal{U}(R)). \tag{2.7}$$

Thus by (2.4) we have

$$q(z)+\frac{zq'(z)}{\sigma q(z)}\prec_R h(z).$$

Lemma 2.2 now yields

$$q(z)\prec_R h(z). \tag{2.8}$$

By (2.6) it suffices to verify that $R=1$. Let p be the positive integer such that $p>\sigma$ and let

$$F(z)=z^{p+1}\left[(1-t)\,D_\sigma^{\lambda}f(z)+tD_\sigma^{\lambda+1}f(z)\right]\quad(z\in\mathcal{U}).$$

Then by (1.4), (2.6) and (2.8) we have

$$\frac{zF'(z)}{F(z)}=\sigma\frac{(1-t)\,D_\sigma^{\lambda+1}f(z)+tD_\sigma^{\lambda+2}f(z)}{(1-t)\,D_\sigma^{\lambda}f(z)+tD_\sigma^{\lambda+1}f(z)}+p-\sigma\prec_R\sigma h(z)+p-\sigma.$$

Thus by (1.5) we obtain

$$\mathrm{Re}\frac{zF'(z)}{F(z)} > 0 \quad (z \in \mathcal{U}(R)).$$

It means, that F is p-valently starlike in $\mathcal{U}(R)$ and consequently it is p-valent in $\mathcal{U}(R)$. Thus we see that F can not vanish on $|z| = R$ if $R < 1$. Hence by (2.5) we have $R = 1$ and the proof of Theorem 2.4 is complete. □

Putting $h(z) = \frac{1+Az}{1+Bz}$ in Theorems 2.2 and 2.3 we obtain the following two corollaries:

Corollary 2.5.

$$\mathcal{V}(t, \lambda + m, \sigma; A, B) \subset \mathcal{V}(t, \lambda, \sigma; A, B) \quad (m \in \mathbf{N}).$$

Corollary 2.6.

$$\mathcal{W}(t, \lambda+m, \sigma; A, B) \subset \mathcal{W}(t, \lambda, \sigma; A, B) \quad (m \in \mathbf{N}).$$

Using Lemma 2.1 we show the following sufficient conditions for functions to belong to the class $\mathcal{W}(t, \lambda, \sigma; A, B)$.

Theorem 2.7. *Let σ, λ, A, B be real numbers, and let*

$$\sigma > 0,\ \lambda > 0,\ -1 \leq A < B \leq 1,\ B - A \geq 2AB. \tag{2.9}$$

If a function $f \in M$ satisfies the inequality

$$\left|\frac{(1-t)\,D_\sigma^{\lambda+2}f(z) + tD_\sigma^{\lambda+3}f(z)}{(1-t)\,D_\sigma^{\lambda+1}f(z) + tD_\sigma^{\lambda+2}f(z)} - 1\right| < \frac{(B-A)(1+\sigma-\sigma A) - 2AB}{\sigma(1+B)(1-A)} \quad (z \in \mathcal{U}), \tag{2.10}$$

then f belongs to the class $W(t, \lambda, \sigma; A, B)$.

Proof. Let a function f belong to the class $\mathcal{M}$. Putting

$$q(z) = \frac{1 + Aw(z)}{1 + Bw(z)} \quad (z \in \mathcal{U}(R)) \tag{2.11}$$

in (2.7), we obtain

$$\frac{(1-t)\,D_\sigma^{\lambda+2}f(z) + tD_\sigma^{\lambda+3}f(z)}{(1-t)\,D_\sigma^{\lambda+1}f(z) + tD_\sigma^{\lambda+2}f(z)} = \frac{1+Aw(z)}{1+Bw(z)} + \frac{1}{\sigma}\left(\frac{Azw'(z)}{1+Aw(z)} - \frac{Bzw'(z)}{1+Bw(z)}\right).$$

Consequently, we have

$$F(z) = w(z)\left\{\frac{zw'(z)}{\sigma w(z)}\left(\frac{A}{1+Aw(z)} - \frac{B}{1+Bw(z)}\right) - \frac{B-A}{1+Bw(z)}\right\}, \tag{2.12}$$

where
$$F(z) = \frac{(1-t)\,D_\sigma^{\lambda+2} f(z) + tD_\sigma^{\lambda+3} f(z)}{(1-t)\,D_\sigma^{\lambda+1} f(z) + tD_\sigma^{\lambda+2} f(z)} - 1.$$
By (1.7), (2.6) and (2.11) it is sufficient to verify that w is analytic in $\mathcal{U}$ and
$$|w(z)| < 1 \quad (z \in \mathcal{U}).$$
Now, suppose that there exists a point $z_0 \in \mathcal{U}(R)$, such that
$$|w(z_0)| = 1,\ |w(z)| < 1 \ \ (|z| < |z_0|).$$
Then, applying Lemma 2.1, we can write
$$z_0 w'(z_0) = kw(z_0),\ w(z_0) = e^{i\theta} \quad (k \geq 1).$$
Combining these with (2.12), we obtain
$$\begin{aligned} |F(z_0)| &= \left|\frac{k}{\sigma}\left(\frac{-A}{1+Ae^{i\theta}} + \frac{B}{1+Be^{i\theta}}\right) + \frac{B-A}{1+Be^{i\theta}}\right| \\ &\geq \frac{k}{\sigma}\mathrm{Re}\left(\frac{-A}{1+Ae^{i\theta}} + \frac{B}{1+Be^{i\theta}}\right) + \frac{B-A}{1+B}. \end{aligned}$$
Thus, by (2.9) we have
$$\begin{aligned} |F(z_0)| &\geq \frac{k}{\sigma}\left(\frac{-A}{1-A} + \frac{B}{1+B}\right) + \frac{B-A}{1+B} \\ &\geq \frac{(B-A)\,(1+\sigma-\sigma A) - 2AB}{\sigma\,(1+B)\,(1-A)}. \end{aligned}$$
Since this result contradicts (2.10) we conclude that w is the analytic function in $\mathcal{U}(R)$ and $|w(z)| < 1$ $(z \in \mathcal{U}\,(R))$. Applying the same methods as in the proof of Theorem 2.4 we obtain $R = 1$, which completes the proof of Theorem 2.7. □

Putting $t = 0,\ A = 2\alpha - 1$ and $B = 1$ in Corollaries 2.5 and 2.6 and Theorem 2.7 we obtain following relationships for the operator D_σ^λ .

Corollary 2.8. *Let* $0 \leq \alpha < 1$ *and* $m \in N$. *If a function* $f \in M$ *satisfies the inequality*
$$\mathrm{Re}\left(zD_\sigma^{\lambda+m} f(z)\right) > \alpha \quad (z \in \mathcal{D})\,,$$
then
$$\mathrm{Re}\left(zD_\sigma^{\lambda} f(z)\right) > \alpha \quad (z \in \mathcal{D})\,.$$

Corollary 2.9. *Let* $0 \leq \alpha < 1$ *and* $m \in N$. *If a function* $f \in M$ *satisfies the inequality*

$$\operatorname{Re}\left\{\frac{D_\sigma^{\lambda+m+1}f(z)}{D_\sigma^{\lambda+m}f(z)}\right\} > \alpha \quad (z \in \mathcal{D}),$$

then

$$\operatorname{Re}\left\{\frac{D_\sigma^{\lambda+1}f(z)}{D_\sigma^{\lambda}f(z)}\right\} > \alpha \quad (z \in \mathcal{D}).$$

Corollary 2.10. *Let* $0 \leq \alpha \leq 2/3$. *If a function* $f \in M$ *satisfies the inequality*

$$\left|\frac{D_\sigma^{\lambda+2}f(z)}{D_\sigma^{\lambda+1}f(z)} - 1\right| < 1 - \alpha + \frac{2-3\alpha}{2\sigma(1-\alpha)} \quad (z \in \mathcal{D}),$$

then

$$\operatorname{Re}\left\{\frac{D_\sigma^{\lambda+1}f(z)}{D_\sigma^{\lambda}f(z)}\right\} > a \quad (z \in \mathcal{D}).$$

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