STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LV**, Number 3, September 2010

SOME PROPERTIES OF A NEW CLASS OF CERTAIN ANALYTIC FUNCTIONS OF COMPLEX ORDER

LUMINIŢA-IOANA COTÎRLĂ

Dedicated to Professor Grigore Ştefan Sălăgean on his 60^{th} *birthday*

Abstract. In this paper we introduce a new class, $\mathcal{F}_n(b, M)$ of certain analytic functions. For this class we determine sufficient condition in terms of coefficients, coefficient estimate, and maximization theorem concerning the coefficients.

1. **Introduction and preliminaries**

Let A be the class of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic and univalent in the open unit disk

$$U = \{z \in \mathbb{C} : |z| < 1\}.$$

For n a positive integer and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H} : \ f(z) = a + a_n z^n + \dots\}.$$

We use Ω to denote the class of functions $w(z)$ in U satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ for $z \in U$.

For a function $f(z)$ in A, we define

$$I^0 f(z) = f(z); \tag{1.2}$$

Received by the editors: 16.12.2009.
2000 *Mathematics Subject Classification.* 30C45.
Key words and phrases. Integral operator, analytic function, complex order.

$$I^1 f(z) = If(z) = \int_0^z f(t)t^{-1}dt; \tag{1.3}$$

and

$$I^n f(z) = I(I^{n-1}f(z)), \quad (z \in U \quad and \quad n \in \mathbb{N} = \{1, 2, 3, ...\}). \tag{1.4}$$

The integral operator I^n was introduced by Sălăgean in [8]. We note that, for a function $f \in A$ of the form (1.1)

$$I^n f(z) = z + \sum_{k=2}^{\infty} k^{-n} a_k z^k, \quad (z \in U, n \in \mathbb{N}).$$

In [1], [2], [3], [4], [7] and others papers, are introduced and studied certain subclasses of analitic functions defined by Sălăgean operator defined in [8]. Recently,in[5], [6] are studied some class of analytic functions defined by the integral operator defined in [8].

With the help of the integral operator I^n, we say that a function $f(z)$ belonging to A is in the class $\mathcal{F}_n(b, M)$ if and only if

$$\left| \frac{1}{b}(\frac{I^n f(z)}{I^{n+1}f(z)} - 1) + 1 - M \right| < M, \tag{1.5}$$

where $M > \frac{1}{2}$, $z \in U$ and $b \neq 0$ is complex number.

We shall need in this paper the following lemma:

Lemma 1.1. [4] *Let* $w(z) = \sum_{k=1}^{\infty} c_k z^k \in \Omega$ *if* μ *is any complex number, then*

$$|c_2 - \mu c_1^2| \leq max\{1, |\mu|\} \tag{1.6}$$

for any complex μ. *Equality in (1.6) may be attained for the functions* $w(z) = z^2$ *and* $w(z) = z$ *for* $|\mu| < 1$ *and* $|\mu| \geq 1$, *respectively.*

We know from [3] that $f(z) \in H_n(b, M)$ if and only if for $z \in U$

$$\frac{I^n f(z)}{I^{n+1}f(z)} = \frac{1 + [b(1+m) - m]w(z)}{1 - mw(z)},$$

where $m = 1 - \frac{1}{M}$, $(M > \frac{1}{2})$ and $w(z) \in \Omega$.

The purpose of the present paper is to determine sufficient condition in terms of coefficients for function belong to $\mathcal{F}_n(b, M)$, coefficient estimate, and maximization of $|a_3 - \mu a_2^2|$ on the class $\mathcal{F}_n(b, M)$ for complex value of μ.

2. Main results

Theorem 2.1. *Let the function $f(z)$ be defined by (1.1). If*

$$\sum_{k=2}^{\infty}\left\{\left(1-\frac{1}{k}\right)+\left|\frac{b(1+m)}{k}+m(1-\frac{1}{k})\right|\right\}\frac{|a_k|}{k^n}\leq |b(1+m)|, \tag{2.1}$$

holds, then $f(z)$ belongs to $\mathcal{F}_n(b,M)$, where $m=1-\frac{1}{M}$ $\quad(M>\frac{1}{2})$.

Proof. Suppose that the inequality (2.1) holds. Then we have for $z\in U$

$$|I^n f(z)-I^{n+1}f(z)|-|b(1+m)I^{n+1}f(z)+m(I^n f(z)-I^{n+1}f(z)|$$

$$=\left|\sum_{k=2}^{\infty}\frac{1}{k^n}(1-\frac{1}{k})a_k z^k\right|-\left|b(1+m)\left\{z+\sum_{k=2}^{\infty}\frac{a_k}{k^{n+1}}z^k\right\}+m\sum_{k=2}^{\infty}\frac{1}{k^n}(1-\frac{1}{k})a_k z^k\right|$$

$$\leq\sum_{k=2}^{\infty}\frac{1}{k^n}\left(1-\frac{1}{k}\right)|a_k|r^k-\left\{|b(1+m)|r-\sum_{k=2}^{\infty}\left|\frac{b(1+m)}{k}+m\left(1-\frac{1}{k}\right)\right|\frac{|a_k|}{k^n}r^k\right\}$$

$$=\sum_{k=2}^{\infty}\frac{1}{k^n}|a_k|r^k\left\{\left(1-\frac{1}{k}\right)+\left|\frac{b(1+m)}{k}+m\left(1-\frac{1}{k}\right)\right|\right\}-|b(1+m)|r.$$

Letting $r\to -1$, then we have

$$|I^n f(z)-I^{n+1}f(z)|-|b(1+m)I^{n+1}f(z)+m(I^n f(z)-I^{n+1}f(z)|$$

$$=\sum_{k=2}^{\infty}\left\{\left(1-\frac{1}{k}\right)+\left|\frac{b(1+m)}{k}+m\left(1-\frac{1}{k}\right)\right|\right\}\frac{1}{k^n}|a_k|r^k-|b(1+m)|\leq 0,$$

by (2.1). Hence, it follows that

$$\left|\frac{\frac{I^n f(z)}{I^{n+1}f(z)}-1}{b(1+m)+m\{\frac{I^n f(z)}{I^{n+1}f(z)}-1\}}\right|<1,\quad z\in U.$$

Letting

$$w(z)=\frac{\frac{I^n f(z)}{I^{n+1}f(z)}-1}{b(1+m)+m\{\frac{I^n f(z)}{I^{n+1}f(z)}-1\}},$$

then $w(0)=0$, $w(z)$ is analytic in $|z|<1$ and $|w(z)|<1$. Hence, we have

$$\frac{I^n f(z)}{I^{n+1}f(z)}=\frac{1+[b(1+m)-m]w(z)}{1-mw(z)},\quad m=1-\frac{1}{M},\quad M>\frac{1}{2},w(z)\in\Omega,$$

and this shows that $f(z)$ belongs to $\mathcal{F}_n(b,M)$.

Theorem 2.2. *Let the function $f(z)$defined by (1.1)be in the class $\mathcal{F}_n(b, M)$, $z \in U$.*

a) For

$$2m\left(1-\frac{1}{k}\right)Re\{b\} > \left(1-\frac{1}{k}\right)^2(1-m)-|b|^2(1+m),$$

let

$$N=\left[\frac{2m\left(1-\frac{1}{k}\right)Re\{b\}}{\left(1-\frac{1}{k}\right)^2(1-m)-|b|^2(1+m)}\right], \quad k=1,3,...,j-1.$$

Then

$$|a_j| \le \frac{1}{\frac{1}{j^n}(1-\frac{1}{j})!}\prod_{k=2}^{j}\left|\frac{b(1+m)}{k}+\left(\frac{k-2}{k}\right)m\right|, \tag{2.2}$$

for $j = 2, 3, ..., N+2$; and

$$|a_j| \le \frac{1}{\frac{1}{j^n}(1-\frac{1}{j})(N+1)!}\prod_{k=2}^{N+3}\left|\frac{b(1+m)}{k}+\left(\frac{k-2}{k}\right)m\right|, \tag{2.3}$$

for $j > N+2$.

b) If

$$2m\left(1-\frac{1}{k}\right)Re\{b\} \le \left(1-\frac{1}{k}\right)^2(1-m)-|b|^2(1+m),$$

then

$$|a_j| \le \frac{(1+m)|b|}{\frac{1}{j^n}(1-\frac{1}{j})}, \quad for j \ge 2, \tag{2.4}$$

where $m = 1-\frac{1}{M}$ $\quad (M > \frac{1}{2})$ and $b \neq 0$ complex.

Proof. Since $f(z) \in \mathcal{F}_n(b, M)$, from

$$\frac{I^n f(z)}{I^{n+1} f(z)} = \frac{1+[b(1+m)-m]w(z)}{1-mw(z)},$$

where $m = 1-\frac{1}{M}$ $\quad (M > \frac{1}{2})$ and $w(z) \in \Omega$, we have that

$$\sum_{k=2}^{\infty}\frac{1}{k^n}\left(1-\frac{1}{k}\right)a_k z^k = w(z)\left\{z(1+m)b+\sum_{k=2}^{\infty}\frac{1}{k^n}\left[\frac{b(1+m)}{k}+m\left(1-\frac{1}{k}\right)\right]a_k z^k\right\}. \tag{2.5}$$

The equality (2.5) can be written in the form

$$\sum_{k=2}^{j}\frac{1}{k^n}\left(1-\frac{1}{k}\right)a_k z^k + \sum_{k=2}^{\infty} d_k z^k =$$

$$= \left\{ b(1+m)z + \sum_{k=2}^{j-1} \frac{1}{k^n} \left[\frac{b(1+m)}{k} + m\left(1-\frac{1}{k}\right) \right] a_k z^k \right\} w(z),$$

where d_j's are some appropriates complex numbers. Then since $|w(z)| < 1$, we have

$$\left| \sum_{k=2}^{j} \frac{1}{k^n} \left(1-\frac{1}{k}\right) a_k z^k + \sum_{k=j+1}^{\infty} d_k z^k \right| \leq \tag{2.6}$$

$$\left| b(1+m)z + \sum_{k=2}^{j-1} \frac{1}{k^n} \left[\frac{b(1+m)}{k} + m\left(1-\frac{1}{k}\right) \right] a_k z^k \right|.$$

Squaring both sides of (2.6) and integrating round $|z| = r < 1$,we get, after taking the limit with $r \to 1$

$$\frac{1}{j^{2n}}(1-\frac{1}{j})^2|a_j|^2 \leq (1+m)^2|b|^2 + \tag{2.7}$$

$$+\sum_{k=2}^{j-1} \frac{1}{k^{2n}} \left\{ \left| \frac{b(1+m)}{k} + m\left(1-\frac{1}{k}\right) \right|^2 - \left(1-\frac{1}{k}\right)^2 \right\} |a_k|^2.$$

Now there may be following two cases:

(a) Let

$$\frac{2m(k-1)Re\{b\}}{k^2} > \frac{(k-1)^2(1-m)}{k^2} - \frac{(1+m)|b|^2}{k^2}.$$

Suppose that $j \leq n+2$.Then for $j = 2$, (2.7) gives

$$|a_2| \leq (1+m)|b|2^{n+1}$$

which gives (2.2) for $j = 2$. We establish (2.2), by mathematical induction. Suppose (2.2)is valid for $k = 2, 3, ..., j-1$. Then it follows from (2.7)

$$\frac{1}{j^{2n}} \left(1-\frac{1}{j}\right)^2 |a_j|^2 \leq$$

$$(1+m)^2|b|^2 + \sum_{k=2}^{j-1} \frac{1}{k^{2n}} \left\{ \left| \frac{b(1+m)}{k} + m\left(1-\frac{1}{k}\right) \right|^2 - \left(1-\frac{1}{k}\right)^2 \right\} \times$$

$$\times \frac{1}{\frac{1}{k^{2n}}((1-\frac{1}{k})!)^2} \prod_{p=2}^{k} \left| \frac{b(1+m)}{k} + m\left(\frac{p+2}{p}\right) \right|^2$$

$$= \frac{1}{((1-\frac{1}{j})!)^2} \prod_{k=2}^{j} \left| \frac{b(1+m)}{k} + \left(\frac{k-2}{k}\right) m \right|^2.$$

Thus, we get

$$|a_j| \le \frac{1}{\frac{1}{j^n}(1-\frac{1}{j})!}\prod_{k=2}^{j}\left|\frac{b(1+m)}{k}+\left(\frac{k-2}{k}\right)m\right|,$$

which completes the proof of (2.2). Next, we suppose $j > N+2$. Then (2.7) gives

$$\frac{1}{j^{2n}}(1-\frac{1}{j})^2|a_j|^2 \le$$

$$\le (1+m)^2|b|^2+\sum_{k=2}^{N+2}\frac{1}{k^{2n}}\left\{\left|\frac{b(1+m)}{k}+m\left(1-\frac{1}{k}\right)\right|^2-\left(1-\frac{1}{k}\right)^2\right\}|a_k|^2+$$

$$+\sum_{k=N+3}^{j-1}\frac{1}{k^{2n}}\left\{\left|\frac{b(1+m)}{k}+m\left(1-\frac{1}{k}\right)\right|^2-\left(1-\frac{1}{k}\right)^2\right\}|a_k|^2 \le$$

$$\le (1+m)^2|b|^2+\sum_{k=2}^{N+2}\frac{1}{k^{2n}}\left\{\left|\frac{b(1+m)}{k}+m\left(1-\frac{1}{k}\right)\right|^2-\left(1-\frac{1}{k}\right)^2\right\}|a_k|^2.$$

On substituting upper estimates for $a_2, a_3, ..., a_{N+2}$ obtained above, and simplifying, we obtain (2.3).

(b) Let

$$2m\left(1-\frac{1}{k}\right)Re\{b\} \le \left(1-\frac{1}{k}\right)^2(1-m)-(1+m)|b|^2,$$

then it follows from (2.7)

$$\frac{1}{j^{2n}}\left(1-\frac{1}{j}\right)^2|a_j|^2 \le (1+m)^2|b|^2, \quad (j \ge 2)$$

which prove (2.4).

Theorem 2.3. *If a function $f(z)$ defined by (1.1) is in the class $\mathcal{F}_n(b,M)$ and μ is any complex number, then*

$$|a_3-\mu a_2^2| \le \frac{3^{n+1}}{2}|b(1+m)|max\{1,|d|\} \tag{2.8}$$

where

$$d = \frac{b(1+m)}{2\cdot 3^{n+1}}[2^{2n+4}\mu-3^{n+1}]-\frac{m}{2}. \tag{2.9}$$

The result is sharp.

Proof. Since $f(z) \in \mathcal{F}_n(b,M)$, we have

$$w(z) = \frac{I^nf(z)-I^{n+1}f(z)}{[b(1+m)-m]I^{n+1}f(z)+mI^nf(z)} =$$

$$= \frac{\sum_{k=2}^{\infty} \frac{a_k}{k^n} z^{k-1} \left(1 - \frac{1}{k}\right)}{b(1+m) + \sum_{k=2}^{\infty} \frac{a_k}{k^n} z^{k-1} \left[\frac{b(1+m)}{k} + m\left(1 - \frac{1}{k}\right)\right]} =$$

$$= \frac{\sum_{k=2}^{\infty} \frac{a_k}{k^n} z^{k-1} \left(1 - \frac{1}{k}\right)}{b(1+m)} \times \left\{ 1 + \frac{\sum_{k=2}^{\infty} \frac{a_k}{k^n} z^{k-1} \left[\frac{b(1+m)}{k} + m\left(1 - \frac{1}{k}\right)\right]}{b(1+m)} \right\}. \quad (2.10)$$

Now compare the coefficients of z and z^2 on both sides of (2.10). Thus we obtain

$$a_2 = 2^{n+1} b(1+m) c_1 \quad (2.11)$$

and

$$a_3 = \frac{3^{n+1} b(1+m)}{2} \left\{ c_2 + \left[\frac{b(1+m)}{2} + \frac{m}{2}\right] c_1^2 \right\}. \quad (2.12)$$

Hence

$$a_3 - \mu a_2^2 = \frac{3^{n+1}}{2} b(1+m) \{c_2 - c_1^2 d\}, \quad (2.13)$$

where

$$d = \frac{b(1+m)}{2 \cdot 3^{n+1}} [2^{2n+4}\mu - 3^{n+1}] - \frac{m}{2}.$$

Taking modulus both sides in (2.13), we have

$$|a_3 - \mu a_2^2| \leq \frac{3^{n+1}}{2} |b(1+m)| \cdot |c_2 - d c_1^2|. \quad (2.14)$$

Using Lemma 1.1.in (2.14), we have

$$|a_3 - \mu a_2^2| \leq \frac{3^{n+1}}{2} |b(1+m)| max\{1, |d|\}.$$

Finally, the assertion (2.8) of Theorem 2.3. is sharp in view of the fact that the assertion (1.6) of Lemma 1.1 is sharp.

References

[1] Alintas, O., Ozkan, O., Srivastava, H. M., *Majorization by starlike functions of complex order*, Complex Variables Theory Appl., **46** (2001), 207-218.

[2] Al-Oboudi, F. M., *On univalent functions defined by a generalized Sălăgean operator*, Int. J. Math. Sci., **2** (2004), 1429-1436.

[3] Aouf, M. K., Al-Amiri, A. M., *On certain fractional operators for certain subclass of prestarlike functions defined by Sălăgean operator*, J. Fract. Calc., **22** (2002), 47-56.

[4] Aouf, M. K., Darwish, H. E., Attiya, A. A., *On a class of certain analytic functions of complex order*, Indian J.Pure Appl. Math., **32** (2001), 1443-1452.

[5] Cotîrlă, L. I., *Harmonic multivalent functions defined by integral operator*, Studia Univ. Babeş-Bolyai, **1** (2009), 65-74.

[6] Cotîrlă, L. I., *A differential sandwich theorem for analytic functions defined by the integral operator*, Studia Univ. Babeş-Bolyai, **2** (2009), 13-22.

[7] G. Murugusundaramoorthy, H. M. Srivastava, *Neighorhoods of certain classes of analytic functions of complex order*, J. Pure Appl. Math., **52**, Article **24** (2004), 1-8 (electronic).

[8] Sălăgean, G. S. *Subclasses of univalent functions*, Complex Analysis, Fifth Romanian-Finnish Seminar, Part 1 (Bucharest, 1981), Lecture Notes in Math., vol. 1013, Springer, Berlin, 1983, 362-372.

Babeş-Bolyai University
Faculty of Mathematics and Computer Science
400084 Cluj-Napoca, Romania
E-mail address: luminita.cotirla@yahoo.com