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ON A FRACTIONAL DIFFERENTIAL INCLUSION WITH BOUNDARY CONDITIONS

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Abstract. We prove a Filippov type existence theorem for solutions of a fractional differential inclusion defined by a nonconvex set-valued map with Dirichlet boundary conditions. The method consists in application of the contraction principle in the space of selections of the set-valued map instead of the space of solutions.

1. Introduction

In this note we study the following problem

$$-D^{\alpha}x(t) \in F(t, x(t)) \quad a.e.\ ([0,1]), \tag{1.1}$$

$$x(0) = x(1) = 0, \tag{1.2}$$

where $\alpha \in (1,2]$, D^{α} is the standard Riemann-Liouville fractional derivative and $F : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map.

Differential equations with fractional order have recently proved to be strong tools in the modelling of many physical phenomena; for a complete bibliography on this topic we refer to [23]. As a consequence there was an intensive development of the theory of differential equations of fractional order ([2, 15, 20, 22, 24] etc.).

The study of fractional differential inclusions was initiated by El-Sayed and Ibrahim ([16]). Very recently several qualitative results for fractional differential inclusions were obtained in [3, 18].

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The present note is motivated by a recent paper of Ouahab ([23]) where several existence results concerning problem (1.1)-(1.2) are obtained. The aim of our paper is to provide an additional existence result for problem (1.1)-(1.2). More exactly, we prove a Filippov type result concerning the existence of solutions to the boundary value problem (1.1)-(1.2). We recall that for a differential inclusion defined by a lipschitzian set-valued map with nonconvex values, Filippov's theorem ([17]) consists in proving the existence of a solution starting from a given "quasi" or "almost" solution. Moreover, the result provides an estimation between the "quasi" solution and the solution obtained.

Our approach is different from the ones in [23] and consists in the application of the set-valued contraction principle in the space of selections of the set-valued map instead of the space of solutions. We note that the idea of applying the set-valued contraction principle due to Covitz and Nadler ([14]) in the space of derivatives of the solutions belongs to Tallos ([19], [25]) and it was already used for similar results obtained for other classes of differential inclusions ([5-13]).

The paper is organized as follows: in Section 2 we present the notations, definitions and the preliminary results to be used in the sequel and in Section 3 we prove the main result.

2. **Preliminaries**

In this short section we sum up some basic facts that we are going to use later.

Let (X,d) be a metric space and consider a set valued map T on X with nonempty closed values in X. T is said to be a λ -contraction if there exists $0<\lambda<1$ such that:

$$d_H(T(x),T(y))\leq \lambda d(x,y) \quad \forall x,y\in X,$$

where $d_H(.,.)$ denotes the Pompeiu-Hausdorff distance. Recall that the Pompeiu-Hausdorff distance of the closed subsets $A,B\subset X$ is defined by

$$d_H(A,B)=\max\{d^*(A,B),d^*(B,A)\},\quad d^*(A,B)=\sup\{d(a,B);a\in A\},$$

where $d(x,B)=\inf_{y\in B} d(x,y)$.

If X is complete, then every set valued contraction has a fixed point, i.e. a point $z\in X$ such that $z\in T(z)$ ([14]).

We denote by $Fix(T)$ the set of all fixed points of the set-valued map T. Obviously, $Fix(T)$ is closed.

Proposition 2.1. ([21]) *Let X be a complete metric space and suppose that T_1, T_2 are λ-contractions with closed values in X. Then*

$$d_H(Fix(T_1), Fix(T_2)) \leq \frac{1}{1-\lambda} \sup_{z\in X} d(T_1(z), T_2(z)).$$

Let $I:=[0,1]$, denote by $C(I,\mathbf{R})$ the Banach space of all continuous functions from I to $\mathbf{R}$ and by $L^1(I,\mathbf{R})$ we denote the Banach space of Lebegue integrable functions $u(.): I\to\mathbf{R}$ endowed with the norm $||u||_1=\int_0^1 |u(t)|dt$.

Definition 2.2. a) *The fractional integral of order $\alpha>0$* of a Lebesgue integrable function $f(.):(0,\infty)\to\mathbf{R}$ is defined by

$$I_0^\alpha f(t)=\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds,$$

provided the right-hand side is pointwise defined on $(0,\infty)$ and $\Gamma(.)$ is the (Euler's) Gamma function.

b) *The fractional derivative of order $\alpha>0$* of a continuous function $f(.):(0,\infty)\to\mathbf{R}$ is defined by

$$\frac{d^\alpha f(t)}{dt^\alpha}=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{-\alpha+n-1} f(s)ds,$$

where $n=[\alpha]+1$, provided the right-hand side is pointwise defined on $(0,\infty)$.

Definition 2.3. A function $x(.)\in C(I,\mathbf{R})$ is called a solution of problem (1.1)-(1.2) if there exists a function $v(.)\in L^1(I,\mathbf{R})$ with $v(t)\in F(t,x(t))$, *a.e.* (I) such that $-D^\alpha x(t)=v(t)$, *a.e.* (I) and conditions (1.2) are satisfied.

We need the following result ([1]).

Lemma 2.4. ([1]) *Let* $f(.) : [0,1] \to \mathbf{R}$ *be continuous. Then* $x(.)$ *is the unique solution of the boundary value problem*

$$D^\alpha x(t) + f(t) = 0 \quad t \in I, \tag{2.1}$$

$$x(0) = x(1) = 0, \tag{2.2}$$

if and only if

$$x(t) = \int_0^1 G(t,s)f(s)ds, \tag{2.3}$$

where

$$G(t,s) := \frac{1}{\Gamma(\alpha)} \begin{cases} [t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}, & \text{if} \quad 0 \le s < t \le 1, \\ [t(1-s)]^{\alpha-1}, & \text{if} \quad 0 \le t < s \le 1. \end{cases}$$

Note that $|G(t,s)| \le \frac{2}{\Gamma(\alpha)}\ \forall t,s \in I$.

In the sequel we assume the following conditions on F.

Hypothesis 2.5. i) $F(.,.) : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ *has nonempty closed values and for every* $x \in \mathbf{R}$ $F(.,x)$ *is measurable.*

ii) *There exists* $L(.) \in L^1(I,\mathbf{R})$ *such that for almost all* $t \in I, F(t,\cdot)$ *is L(t)-Lipschitz in the sense that*

$$d_H(F(t,x),F(t,y)) \le L(t)|x-y| \quad \forall\ x,y \in \mathbf{R}$$

and $d(0,F(t,0)) \le L(t)$ *a.e.* (I).

3. **The main result**

We are able now to prove our main result.

Theorem 3.1. *Assume that Hypothesis 2.5 is satisfied and* $\frac{2}{\Gamma(\alpha)}||L||_1 < 1$. *Let* $y(.) \in C(I,\mathbf{R})$ *be such that there exists* $q(.) \in L^1(I,\mathbf{R})$ *with* $d(-D^\alpha y(t),$
$F(t,y(t))) \le q(t)$, *a.e.* (I), $y(0) = y(1) = 0$.

Then for every $\varepsilon > 0$ *there exists* $x(.)$ *a solution of (1.1)-(1.2) satisfying for all* $t \in I$

$$|x(t) - y(t)| \le \frac{2}{\Gamma(\alpha) - 2||L||_1}\int_0^1 q(t)dt + \varepsilon. \tag{3.1}$$

Proof. For $u(.) \in L^1(I, \mathbf{R})$ define the following set valued maps:

$$M_u(t) = F(t, \int_0^1 G(t,s)u(s)ds), \quad t \in I,$$

$$T(u) = \{\phi(.) \in L^1(I, \mathbf{R}); \quad \phi(t) \in M_u(t) \quad a.e.\ (I)\}.$$

It follows from the definition and Lemma 2.4 that $x(.)$ is a solution of (1.1)-(1.2) if and only if $-D^\alpha x(.)$ is a fixed point of $T(.)$.

We shall prove first that $T(u)$ is nonempty and closed for every $u \in L^1(I, \mathbf{R})$. The fact that the set valued map $M_u(.)$ is measurable is well known. For example the map $t \to \int_0^1 G(t,s)u(s)ds$ can be approximated by step functions and we can apply Theorem III. 40 in [4]. Since the values of F are closed with the measurable selection theorem (Theorem III.6 in [4]) we infer that $M_u(.)$ admits a measurable selection ϕ. One has

$$|\phi(t)| \leq d(0, F(t,0)) + d_H(F(t,0), F(t, \int_0^1 G(t,s)u(s)ds) \leq$$

$$\leq L(t)\left(1 + \frac{2}{\Gamma(\alpha)}\int_0^1 |u(s)|ds\right),$$

which shows that $\phi \in L^1(I, \mathbf{R})$ and $T(u)$ is nonempty.

On the other hand, the set $T(u)$ is also closed. Indeed, if $\phi_n \in T(u)$ and $||\phi_n - \phi||_1 \to 0$ then we can pass to a subsequence ϕ_{n_k} such that $\phi_{n_k}(t) \to \phi(t)$ for a.e. $t \in I$, and we find that $\phi \in T(u)$.

We show next that $T(.)$ is a contraction on $L^1(I, \mathbf{R})$.

Let $u, v \in L^1(I, \mathbf{R})$ be given, $\phi \in T(u)$ and let $\delta > 0$. Consider the following set-valued map:

$$H(t) = M_v(t) \cap \left\{x \in \mathbf{R};\ |\phi(t) - x| \leq L(t)\left|\int_0^1 G(t,s)(u(s) - v(s))ds\right| + \delta\right\}.$$

From Proposition III.4 in [4], $H(.)$ is measurable and from Hypothesis 2.5 ii) $H(.)$ has nonempty closed values. Therefore, there exists $\psi(.)$ a measurable selection of $H(.)$. It follows that $\psi \in T(v)$ and according with the definition of the norm we have

$$||\phi - \psi||_1 = \int_0^1 |\phi(t) - \psi(t)|dt \leq \int_0^1 L(t)\left(\int_0^1 |G(t,s)|.|u(s) - v(s)|ds\right)dt +$$

$$\int_0^1 \delta dt = \int_0^1 \left(\int_0^1 L(t)|G(t,s)|dt \right) |u(s) - v(s)|ds + \delta \leq \frac{2}{\Gamma(\alpha)} ||L||_1 ||u - v||_1 + \delta.$$

Since $\delta > 0$ was chosen arbitrary, we deduce that

$$d(\phi, T(v)) \leq \frac{2}{\Gamma(\alpha)} ||L||_1 ||u - v||_1.$$

Replacing u by v we obtain

$$d_H(T(u), T(v)) \leq \frac{2}{\Gamma(\alpha)} ||L||_1 ||u - v||_1,$$

thus $T(.)$ is a contraction on $L^1(I, \mathbf{R})$.

We consider next the following set-valued maps

$$F_1(t, x) = F(t, x) + q(t)[-1, 1], \quad (t, x) \in I \times \mathbf{R},$$

$$M_u^1(t) = F_1(t, \int_0^1 G(t, s)u(s)ds), \quad t \in I, \quad u(.) \in L^1(I, \mathbf{R}),$$

$$T_1(u) = \{\psi(.) \in L^1(I, \mathbf{R}); \quad \psi(t) \in M_u^1(t) \quad a.e.\ (I)\}.$$

Obviously, $F_1(., .)$ satisfies Hypothesis 2.5.

Repeating the previous step of the proof we obtain that T_1 is also a $\frac{2}{\Gamma(\alpha)} ||L||_1$-contraction on $L^1(I, \mathbf{R})$ with closed nonempty values.

We prove next the following estimate

$$d_H(T(u), T_1(u)) \leq \int_0^1 q(t)dt. \tag{3.2}$$

Let $\phi \in T(u), \delta > 0$ and define

$$H_1(t) = M_u^1(t) \cap \{z \in \mathbf{R}; \quad |\phi(t) - z| \leq q(t) + \delta\}.$$

With the same arguments used for the set valued map $H(.)$, we deduce that $H_1(.)$ is measurable with nonempty closed values. Hence let $\psi(.)$ be a measurable selection of $H_1(.)$. It follows that $\psi \in T_1(u)$ and one has

$$||\phi - \psi||_1 = \int_0^1 |\phi(t) - \psi(t)|dt \leq \int_0^1 [q(t) + \delta]dt \leq \int_0^1 q(t) + \delta.$$

Since δ is arbitrary, as above we obtain (3.2).

We apply Proposition 2.1 and we infer that

$$d_H(Fix(T), Fix(T_1)) \leq \frac{1}{1-\frac{2}{\Gamma(\alpha)}||L||_1}\int_0^1 q(t)dt.$$

Since $-D^\alpha y(.) \in Fix(T_1)$ it follows that there exists $u(.) \in Fix(T)$ such that for any $\varepsilon > 0$

$$||-D^\alpha y - u||_1 \leq \frac{1}{1-\frac{2}{\Gamma(\alpha)}||L||_1}\int_0^1 q(t)dt + \frac{\Gamma(\alpha)\varepsilon}{2}.$$

We define $x(t) = \int_0^1 G(t,s)u(s)ds$, $t \in I$ and we have

$$|x(t) - y(t)| \leq \int_0^1 |G(t,s)|.|u(s) + D^\alpha y(s)|ds \leq$$

$$\leq \frac{2}{\Gamma(\alpha)}||u + D^\alpha y||_1 \leq \frac{2}{\Gamma(\alpha) - 2||L||_1}||q||_1 + \varepsilon,$$

which completes the proof. □

Remark 3.2. The assumption in Theorem 3.1 is satisfied, in particular, for $y(.) = 0$ and therefore, via Hypothesis 2.5, with $q(.) = L(.)$. In this case, Theorem 3.1 provides an existence result for problem (1.1)-(1.2) together with a priori bounds for the solution. More precisely, the estimate (3.1) becomes in this case

$$|x(t)| \leq \frac{2||L||_1}{\Gamma(\alpha) - 2||L||_1} + \varepsilon, \quad \forall t \in I \tag{3.3}$$

In [23] among other existence results for problem (1.1)-(1.2) it is obtained in Theorem 4.9 the existence of solutions by applying, as usual in the study of the existence of solutions using fixed points, the contraction principle in the space of solutions. This approach does not allows to obtain an estimate as in (3.3).

On the other hand, in [23], Theorem 6.2, another Filippov type result for problem (1.1)-(1.2) is provided. Its proof follows Filippov's ideas and uses Kuratowsky and Ryll-Nardjewski selection theorem (e.g., [4]). More exactly, if the assumptions in Theorem 3.1 are satisfied then there exists $x(.) \in C(I, \mathbf{R})$ a solution of problem (1.1)-(1.2) such that, for all $t \in I$

$$|x(t) - y(t)| \leq \frac{2}{\Gamma(\alpha)}||q||_1 + \frac{16||q||_1^3}{\Gamma^2(\alpha)(\Gamma(\alpha) - 2||L||_1)}||L||_1. \tag{3.4}$$

We note that in our approach we obtain a "pointwise" estimate from a norm estimate and in general the estimates in (3.1) and (3.4) are not comparable. However, in particular cases the estimate in (3.1) is better that the one in (3.4). If the function $q(.) \in L^1(I, \mathbf{R})$ satisfies $\int_0^1 q(t)dt > \frac{\sqrt{\Gamma(\alpha)}}{2}$, than if we take in (3.1)

$$\varepsilon = \frac{4||q||_1||L||_1(4||q||_1^2 - \Gamma(\alpha))}{\Gamma^2(\alpha)(\Gamma(\alpha) - 2||L||_1)}$$

we obtain (3.4).

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