STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LV**, Number 3, September 2010

ONE-SIDED CLEAN RINGS

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Dedicated to Professor Grigore Ştefan Sălăgean on his 60^{th} *birthday*

Abstract. Replacing units by one-sided units in the definition of clean rings (and modules), new classes of rings (and modules) are defined and studied, generalizing most of the properties known in the clean case.

1. Introduction

For a ring with identity, we denote by $U(R)$ the units, $U_l(R)$ and $U_r(R)$ the left respectively right invertible elements of R (shortly, right-units or left-units), and by $N(R)$ the nilpotent elements.

An element in a ring R is *right (or left) clean* if it is a sum of an idempotent and a right (respectively left) unit. A ring R is *right clean* if all its elements are right clean and it is *left clean* if R^{op} is right clean. Moreover, it is *one-sided clean* if each element is left or right clean. These classes are included in the class of *almost clean* rings considered by McGovern ([8]: every element is a sum of a non-zero divisor and an idempotent) and studied further (in the commutative case) by Ahn and D. D. Anderson ([1]).

Further, a ring R is *weakly right exchange* if for every element $a \in R$ there are two orthogonal idempotents f, f' with $f \in aR$, $f' \in (1-a)R$, such that $f + f' \cong 1$.

In this paper the main results are the following

- *Let* $e^2 = e \in R$ *be such that* eRe *and* $(1-e)R(1-e)$ *are both right clean rings. Then* R *is a right clean ring.*

Received by the editors: 01.03.2010.

2000 *Mathematics Subject Classification.* 16L30, 16U60, 16D60.

Key words and phrases. Right clean, weakly right exchange rings and modules.

- *Any ring* $R = U_l(R) \cup U_r(R) \cup N(R)$ *is both right and left clean.*
- *Any right clean ring is weakly right exchange.*

and,

- *A ring* R *is weakly right exchange if and only if for every* $a \in R$ *there are elements* $b, c \in R$ *such that* $bab = b$, $c(1-a)c = c$, $ab(1-a)c = 0 = (1-a)cab$.

Finally results on strongly respectively weakly one-sided clean rings are given.

2. **Right clean rings**

In the sequel we will merely state our results for right clean rings, but most of them have a left or one-sided analogue.

Obviously Dedekind finite (and in particular abelian or commutative) one-sided clean rings are (strongly) clean.

The following is immediate from definitions

Lemma 2.1. *(i) Every homomorphic image of a right clean ring is right clean.*

(ii) A direct product of rings $\prod R_i$ *is right clean if and only if each* R_i *is right clean.*

The next result is elementary. We supply a proof for later reference.

Proposition 2.2. *Let* A, B *be rings,* ${}_AC_B$ *a bimodule and* $R = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$. *Then* R *is right clean if and only if* A *and* B *are right clean.*

Proof. If R is right clean, the maps $f : R \longrightarrow A$, $f\left(\begin{bmatrix} a & c \\ 0 & b \end{bmatrix}\right) = a$ and $g : R \longrightarrow B$, $g\left(\begin{bmatrix} a & c \\ 0 & b \end{bmatrix}\right) = b$ are ring epimorphisms, and so A, B are right clean by (i), previous Lemma.

Conversely, let $\begin{bmatrix} a & c \\ 0 & b \end{bmatrix} \in R$. Then there are $u_a \in U_l(A)$, $e_a = e_a^2 \in A$ with $a = u_a + e_a$ and a similar decomposition for b. Suppose $v_a u_a = 1 = v_b u_b$. Clearly

$\begin{bmatrix} a & c \\ 0 & b \end{bmatrix} = \begin{bmatrix} u_a & c \\ 0 & u_b \end{bmatrix} + \begin{bmatrix} e_a & 0 \\ 0 & e_b \end{bmatrix}$ where $\begin{bmatrix} e_a & 0 \\ 0 & e_b \end{bmatrix}^2 = \begin{bmatrix} e_a & 0 \\ 0 & e_b \end{bmatrix}$ and $\begin{bmatrix} u_a & c \\ 0 & u_b \end{bmatrix} \in U_l(R)$. Indeed, $\begin{bmatrix} v_a & -v_a c v_b \\ 0 & v_b \end{bmatrix}$ is a left inverse for $\begin{bmatrix} u_a & c \\ 0 & u_b \end{bmatrix}$. □

Remark 2.3. This property fails for one-sided clean rings A and B.

Proposition 2.4. *Let $e^2 = e \in R$ be such that eRe and $(1-e)R(1-e)$ are both right clean rings. Then R is a right clean ring.*

Proof. Using the Pierce decomposition of the ring R, let $\begin{bmatrix} a & x \\ y & b \end{bmatrix} \in R = \begin{bmatrix} eRe & eR(1-e) \\ (1-e)Re & (1-e)R(1-e) \end{bmatrix}$. For $u_1 u = e$ and $a = f + u$ in eRe, $v_1 v = 1 - e$ and $b - yu_1x = g + v$ in $(1-e)R(1-e)$, $\begin{bmatrix} a & x \\ y & b \end{bmatrix}$ decomposes into $\begin{bmatrix} f & 0 \\ 0 & g \end{bmatrix} + \begin{bmatrix} u & x \\ y & v + yu_1x \end{bmatrix}$ and all we need is a left inverse for the latter. But this is $\begin{bmatrix} e & -u_1x \\ 0 & 1-e \end{bmatrix} \begin{bmatrix} u_1 & 0 \\ 0 & v_1 \end{bmatrix} \begin{bmatrix} e & 0 \\ -yu_1 & 1-e \end{bmatrix} = \begin{bmatrix} u_1 + u_1xv_1yu_1 & -u_1xv_1 \\ -v_1yu_1 & v_1 \end{bmatrix}$. □

By induction, we have

Theorem 2.5. *If $1 = e_1 + e_2 + ... + e_n$ in a ring R where e_i are orthogonal idempotents and each e_iRe_i is right clean, then R is right clean.*

Hence

Corollary 2.6. *If R is right clean then so is the matrix ring $\mathcal{M}_n(R)$.*

As in the clean case, we were not able to prove that corner rings (even full) of right (or left or one-sided) clean rings have the same property.

Only recently, classes of rings defined by equalities like: $R = U(R) \cup \mathrm{Id}(R)$ or, $R = U(R) \cup \mathrm{Id}(R) \cup -\mathrm{Id}(R)$ (here $\mathrm{Id}(R)$ denotes the idempotent elements of R), have received a great deal of attention (see [2] and [1] for the commutative case). In a similar vein, examples of right clean rings are provided by the next Proposition.

Proposition 2.7. *Any ring $R = U_l(R) \cup U_r(R) \cup N(R)$ is both right and left clean.*

Proof. We first show that *every right unit is right clean.* Let $a \in U_l(R)$ and $ba = 1$. Then $e = ab$ is an idempotent, so is $1 - e$, and using the decomposition $a = (1 - e) + (a + (e - 1))$ we have to find a left inverse for $a + (e - 1)$. But this is $ebe + (e-1)$ since $(ebe + (e-1))(a + (e-1)) = ebea + ea - a + 0 + 1 - e = 1$ (because $ebea = abbaba = ab = e$).

Coming back to the proof of the Proposition, if $a \in N(R)$ it is well-known that $1 - a = u \in U(R)$ and so $a = 1 - u$ is even strongly clean. If $a \in U_l(R) \cup U_r(R)$ we just use the previous result and its left analogue. □

Remark 2.8. 1) In general $(a + (e - 1))(ebe + (e - 1)) = 1$ fails (equivalently $(e-1)(b+1) = 0$).

2) A slightly larger class is suggested by the following example which can be found in David Arnold's 1982 book ([3]): " In the endomorphism ring of a torsion-free strongly indecomposable Abelian group of finite rank, every element is a monomorphism (i.e., a non-zero divisor) or nilpotent".

3) Recently, H. Chen (see [5]) has proved that regular one-sided unit-regular rings are (though he does not consider this notion) exactly one-sided clean. So these are also examples for the notion we deal with.

3. **Right clean modules**

For the sake of completeness we first restate some results given in [4]: let $f, e \in S = \mathrm{End}(M_R)$ with $e^2 = e$, $A = \ker e$ and $B = \mathrm{im} e$.

Proposition 3.1. *$f - e$ is a monomorphism if and only if the restrictions $f|_A$, $(1 - f)|_B$ are monomorphisms and $fA \cap (1 - f)B = 0$.*

$f - e$ is an epimorphism if and only if and $fA + (1 - f)B = M$.

Lemma 3.2. *$f - e$ is a unit in S if and only if the restrictions $f|_A$, $(1 - f)|_B$ are monomorphisms and $fA \oplus (1 - f)B = M$.*

Observe that the (double) restriction (for the domain - we use | and for the codomain - we use $\widetilde{\ }$) $\widetilde{f|_A} : A \longrightarrow fA$ and $\widetilde{(1-f)|_B} : B \longrightarrow (1 - f)B$ are always onto, so $f|_A$, $(1 - f)|_B$ are monomorphisms if and only if $\widetilde{f|_A}$ and $\widetilde{(1-f)|_B}$ are isomorphisms. If $fA\cap(1-f)B=0$, then $u=\widetilde{f|_A}\oplus\widetilde{(1-f)|_B}:A\oplus B\longrightarrow fA\oplus(1-f)B$ is an isomorphism too (the codomain sum is direct, but not necessarily equal to M).

Therefore, our analogues are

Lemma 3.3. *Let $f,e\in S=\mathrm{End}(M_R)$ with $e^2=e$, $A=\ker e$ and $B=\mathrm{im}e$. Then $f-e\in U_l(S)$ if and only if the restrictions $f|_A$, $(1-f)|_B$ are monomorphisms, $fA\cap(1-f)B=0$ and the monomorphism $\widetilde{f|_A}\oplus\widetilde{(1-f)|_B}\in S$ has a left inverse in S.*

Proposition 3.4. *An element $f\in\mathrm{End}(M_R)$ is right clean if and only if there is a R-module decomposition $M=A\oplus B$ such that the restrictions $f|_A$, $(1-f)|_B$ are monomorphisms, $fA\cap(1-f)B=0$ and the monomorphism $\widetilde{f|_A}\oplus\widetilde{(1-f)|_B}:M\longrightarrow M$ has a left inverse in* $\mathrm{End}(M_R)$.

Remark 3.5. 1) Due to Theorem 2.5, finite direct sums of right clean modules are right clean.

2) Using Lemma 2.1, if $M_R=A\oplus B$ and $\mathrm{Hom}_R(A,B)=0$, then M is right clean if and only if A, B are right clean.

4. **Weakly exchange rings**

A ring is called (right) *exchange* (or *suitable* in [10]) if for every equation $a+a'=1$ there are idempotents $e\in aR$ and $e'\in a'R$ such that $e+e'=1$.

Since these idempotents are complementary, they must be orthogonal (and commute).

Recall that an idempotent $e\in R$ is *isomorphic* to 1 if and only if there are elements $u,v\in R$ with $vu=1$ and $e=uv$ (equivalently, $eR\cong R$ as right R-modules). If $e\neq 1$, such a ring is not Dedekind finite.

We define *weakly right exchange* rings R by the conditions: for every equation $a+a'=1$ there are two orthogonal idempotents f,f' with $f\in aR$, $f'\in a'R$, such that $f+f'\cong 1$ (obviously, since the idempotents f,f' are orthogonal, their sum is also an idempotent).

According to the above definition, there are elements $u, v \in R$ with $vu = 1$ and $f + f' = uv$.

Remark 4.1. We must require these two idempotents to be orthogonal. Indeed, if we require only $vu = 1$ and $f + f' = uv$ (i.e., $f + f' \cong 1$), then $f + f'$ is an idempotent ($uvuv = uv$) and this implies $f + f' = (f + f')^2 = ff' + f'f + f + f'$ and so only $ff' + f'f = 0$ (so not orthogonal nor commuting).

We can naturally associate with these (orthogonal but not necessarily complementary) idempotents two complementary idempotents, two by two isomorphic, namely vfu and $vf'u$.

1) $vfu + vf'u = v(f + f')u = vuvu = vu = 1$

2) $(vfu)^2 = vfuvfu = vf(f + f')fu = vfu$ (and so is $vf'u$)

3) $vfu \cong f$ and $vf'u \cong f'$: indeed, $vfu = (vfu)^2 = vf.uvfu \cong uvfu.vf = (f + f')f(f + f')f = f$, and similarly, $vf'u \cong f'$.

Remark 4.2. Related to lifting idempotents, since $f \in aR$ and $f' \in (1 - a)R$, all we can check is

$$f - a(f + f') = (1 - a)f - af' \in (a - a^2)R.$$

Obviously, if u is a unit, $f + f' = 1$ and $f - a \in (a - a^2)R$ shows that idempotents can be lifted.

Theorem 4.3. *Any right clean ring is weakly right exchange.*

Proof. If $a = u + e$ with $e^2 = e$ and $vu = 1$ (but not necessarily $uv = 1$), since $(uev)^2 = uevuev = uev$, we consider the idempotent

$$f' = uev.$$

Similarly, $(u(1 - e)v)^2 = u(1 - e)vu(1 - e)v = u(1 - e)v$ and we denote

$$f = u(1 - e)v = uv - uev.$$

Take $b = uv + (1 - a)v = (1 - e)v$ and $c = uv - av = -ev$. Then $ab = f$, $(1 - a)c = f'$ and so $f \in aR$ and $f' \in (1 - a)R$.

Thus $ff' = f'f = 0$ (these idempotents are orthogonal) and the sum $f + f' = uv$ (is an idempotent) isomorphic with 1.

Moreover $vfu = 1 - e$ is idempotent (and f, f' are isomorphic to complementary idempotents: $f \cong 1 - e$, and $f' \cong e$). □

Remark 4.4. In a right clean ring the following is also true:

(a) We have $bf = b$ (i.e., $bab = b$) and $bf' = 0$ and similarly $cf' = c$ (i.e., $c(1-a)c = c$) and $cf = 0$. We also have $f'u = (1-f)u$ and $vf' = v(1-f)$.

(b) As in the clean initial case, $c = b + v$, and $a^2 - a = (a - 1 + f)u = (a - f')u$, and since this relation cannot be solved for $f - 1 + a$ or for $f' - a$ (in order to obtain $f - 1 + a$ or $f' - a$ in $(a - a^2)R$), idempotents cannot be lifted modulo any right (or left) ideal.

Actually, since $f \in aR$ and $f' \in (1-a)R$, all we can check is

$$f - a(f + f') = (1-a)f - af' \in (a - a^2)R.$$

(c) Obviously, if u is a unit, $f + f' = 1$ and $f - a \in (a - a^2)R$ shows that idempotents can be lifted.

It is well-known that exchange rings were ring theoretic described by Monk (see [9]). Here is the characterization for weakly right exchange rings.

Theorem 4.5. *A ring R is weakly right exchange if and only if for every $a \in R$ there are elements $b, c \in R$ such that $bab = b$, $c(1-a)c = c$, $ab(1-a)c = 0 = (1-a)cab$.*

Proof. If R is weakly right exchange, take orthogonal idempotents $f = at \in aR$ and $f' = (1-a)s \in (1-a)R$. Then $b = tat$ satisfies $bab = b$, $ab = f$ and $c = s(1-a)s$ satisfies $c(1-a)c = c$ and $f' = (1-a)c$. Since f, f' are orthogonal, we also have $ab(1-a)c = 0 = (1-a)ca$ and $(1-ab)(1-a)c + ab = (1-f)f' + f = f + f'$ is (an idempotent) isomorphic to 1.

Conversely, $f = ab$ and $f' = (1-a)c$ are readily checked to be orthogonal idempotents and $f + f' = (1-ab)(1-a)c + ab$ is (an idempotent) isomorphic to 1. □

Similarly (right exchange and left exchange properties are equivalent), an **open problem** remains: are weakly right exchange rings also weakly left exchange?

5. **Strongly one-sided clean rings**

All the above one-sided clean notions have corresponding strongly versions.

Unlike the strongly clean version, here $ue = eu$ does not imply $u^{-1}e = eu^{-1}$. Therefore R is *strongly right clean* if it is right clean, $ue = eu$ and $ve = ev$.

Proposition 5.1. *Let $e^2 = e \in R$. An element $a \in eRe$ is strongly right clean in R if and only if a is strongly right clean in eRe.*

Proof. First notice that if $a \in eRe$ then $a(1-e) = (1-e)a = 0$ and so $a = ae = ea = eae$.

If $a = g + u$ is strongly right clean in R, then $(g+u)(1-e) = 0$ implies $1-e = -vg(1-e) = -gv(1-e)$ and so (by left multiplication with g) $g(1-e) = 1-e$. Thus (using also $(1-e)a = 0$) $eg = ge$. Therefore $eg = ege = ge$ is an idempotent in eRe. Since a and g commute with e, so is $u = a - g$. Hence $eu = eue = ue$ has eve as left inverse in eRe. Finally, $a = eae = e(g+u)e = ege + eue$ is strongly right clean in eRe.

Conversely, if $a = f + v$ is strongly right clean in eRe with $fv = vf$, $f^2 = f \in eRe$ and $w \in eRe$, $wv = e$ then $a = (a-u) + u$ is strongly right clean in R as $w + (1-e)$ is a left inverse for $u = v + (1-e)$ and $a - u = f + (1-e)$ is idempotent (sum of two orthogonal idempotents). ☐

Remark 5.2. The converse does not use $ev = ve$ from our definition.

Corollary 5.3. *Corner rings of strongly right clean rings are strongly right clean.*

Further, strongly right clean is not a Morita invariant property. The example given in [11], i.e. the localization $\mathbf{Z}_{(2)}$ can be used in order to disprove: R strongly right clean implies $\mathcal{M}_n(R)$ strongly right clean.

6. **Weakly left-clean rings**

We can get even closer to almost clean rings by weakening our right clean elements as follows: an element $a \in R$ is *weakly left-clean* if it is the sum of an idempotent e and a left nonzero-divisor (or left cancellable element) u of R, and a ring is *weakly left-clean* if all its elements share this property.

Remark 6.1. For regular rings, right clean and weakly left-clean coincide (Ex. 1.4, [7]).

In this setting, the *weak left-clean modules* are characterized by Proposition 4.4 in [4].

However, since images of non-zero divisors may not be non-zero divisors, properties for such rings are worse, compared with the right clean rings.

Direct products of weakly left-clean rings are weakly left-clean.

Homomorphic images of weakly left-clean rings may not be weakly left-clean.

Thus, (see Lemma 2.1) *if A, B are rings, ${}_AC_B$ a bimodule and* $R = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$*, then R weakly left-clean generally does not imply A and B weakly left-clean.*

Nevertheless, the converse is true:

Proposition 6.2. *If A, B are weakly left-clean rings and ${}_AC_B$ is a bimodule then* $R = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ *is also weakly left-clean.*

Proof. With the notations in the proof of Lemma 2.1, if u_a, u_b are left non-zero divisors, so is $\begin{bmatrix} u_a & c \\ 0 & u_b \end{bmatrix}$ in R.

Indeed, it is readily checked that matrices of the type $\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$ with left non-zero divisors x and z, are left non-zero divisors in R. □

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