STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LV**, Number 3, September 2010

ON ORDER OF CONVOLUTION CONSISTENCE OF THE ANALYTIC FUNCTIONS

URSZULA BEDNARZ AND JANUSZ SOKÓŁ

Dedicated to Professor Grigore Ştefan Sălăgean on his 60^{th} *birthday*

Abstract. In this paper we consider the convolution of certain classes of analytic functions. We discuss when it is in a given class. By means of the Sălăgean integral operator we define a constant S which describes a measure of convolution consistence of three classes. We shall examine some special families for which we can determine the order of convolution consistence.

1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the unit disc $\mathcal{U} = \{z : |z| < 1\}$ on the complex plane $\mathbb{C}$. Let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ consisting of functions normalized by $f(0) = 0$, $f'(0) = 1$ and let $\mathcal{S} \subset \mathcal{A}$ denote the class of functions univalent in $\mathcal{U}$. Everywhere in this paper $z \in \mathcal{U}$ unless we make a note. A function f maps $\mathcal{U}$ onto a starlike domain with respect to $w_0 = 0$ if and only if

$$\mathfrak{Re}\left[\frac{zf'(z)}{f(z)}\right] > 0 \quad (z \in \mathcal{U}). \tag{1.1}$$

It is well known that if an analytic function f satisfies (1.1) and $f(0) = 0$, $f'(0) \neq 0$, then f is univalent and starlike in $\mathcal{U}$.

A set E is said to be convex if and only if it is starlike with respect to each of its points, that is if and only if the linear segment joining any two points of E lies

Received by the editors: 25.04.2010.

2000 *Mathematics Subject Classification.* 30C45, 30C50, 30C55.

Key words and phrases. Hadamard product, integral convolution, k-starlike functions, k-uniformly convex functions, order of convolution consistence, Sălăgean integral operator, starlike functions, convex functions, functions with positive real part.

entirely in E. Let f be analytic and univalent in $\mathcal{U}$. Then f maps $\mathcal{U}$ onto a convex domain E if and only if

$$\mathfrak{Re}\left[1+\frac{zf''(z)}{f'(z)}\right]>0 \quad (z\in\mathcal{U}). \tag{1.2}$$

Such a function f is said to be convex in $\mathcal{U}$ (or briefly convex). The set of all functions $f\in\mathcal{A}$ that are starlike univalent in $\mathcal{U}$ will be denoted by $\mathcal{ST}$. The set of all functions $f\in\mathcal{A}$ that are convex univalent in $\mathcal{U}$ by $\mathcal{CV}$. Recall that the Hadamard product or convolution of two power series

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \quad\text{and}\quad g(z)=z+\sum_{n=2}^{\infty}b_nz^n,$$

is defined as

$$(f*g)(z)=z+\sum_{n=2}^{\infty}a_nb_nz^n$$

and the integral convolution is defined by

$$(f\otimes g)(z)=z+\sum_{n=2}^{\infty}\frac{a_nb_n}{n}z^n.$$

It is well known [10] that if $f,g\in\mathcal{CV}$, then $f*g\in\mathcal{CV}$ while if $f,g\in\mathcal{ST}$, then $f*g$ may not be in $\mathcal{ST}$ and even may fail to be univalent. To examine deeply this problem let us consider the Sălăgean integral operator (see [12]) $\mathcal{I}^s:\mathcal{A}\to\mathcal{A}$, $s\in\mathbb{R}$, such that

$$\mathcal{I}^sf(z)=\mathcal{I}^s\left(\sum_{n=1}^{\infty}a_nz^n\right)=\sum_{n=1}^{\infty}\frac{a_n}{n^s}z^n.$$

Now, one can ask if exists there a number $s\in\mathbb{R}$ such that

$$\mathcal{I}^s(f*g)\in\mathcal{ST}\quad \forall f,g\in\mathcal{ST}.$$

The answer there is in Theorem 2.1 below. This problem may be consider more generally for other classes of functions when the Sălăgean integral operator is defined on $\mathcal{H}$ as follows

$$\mathcal{I}^s\left(a_0+\sum_{n=1}^{\infty}a_nz^n\right)=a_0+\sum_{n=1}^{\infty}\frac{a_n}{n^s}z^n.$$

Definition 1.1. Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be subsets of $\mathcal{H}$. We say that the three $(\mathcal{X},\mathcal{Y},\mathcal{Z})$ is S–closed under convolution if there exists a number $S = S(\mathcal{X},\mathcal{Y},\mathcal{Z})$ such that

$$\begin{aligned} S(\mathcal{X},\mathcal{Y},\mathcal{Z}) &= \min\{s\in\mathbb{R} : \mathcal{I}^s(f*g)\in\mathcal{Z} \quad \forall f\in\mathcal{X} \ \ \forall g\in\mathcal{Y}\} \qquad (1.3) \\ &= \min\{s\in\mathbb{R} : \mathcal{I}^s(\mathcal{X}*\mathcal{Y})\subseteq\mathcal{Z}\}, \end{aligned}$$

where $\mathcal{I}^s$ denote the Sălăgean integral operator. The number $S(\mathcal{X},\mathcal{Y},\mathcal{Z})$ is called the order of convolution consistence the three $(\mathcal{X},\mathcal{Y},\mathcal{Z})$. It would be called the Sălăgean number.

2. Main results

We shall examine some special families for which we can determine the order of convolution consistence. First we shall restrict our attention to the classes of starlike and convex functions.

Theorem 2.1. *The order of convolution consistence of the class $\mathcal{ST}$ is equal to 1:*

$$S(\mathcal{ST},\mathcal{ST},\mathcal{ST}) = 1. \qquad (2.1)$$

Proof. It is well known [10] that $\mathcal{ST}\otimes\mathcal{ST} = \mathcal{ST}$ and $\mathcal{I}^1(f*g) = f\otimes g$. Thus if $f,g\in\mathcal{ST}$, then $\mathcal{I}^1(f*g)\in\mathcal{ST}$. This means that $S(\mathcal{ST},\mathcal{ST},\mathcal{ST})\le 1$. If we consider the functions $f,g\in\mathcal{ST}$ such that

$$f(z) = g(z) = \frac{z}{(1-z)^2} \quad (z\in\mathcal{U}),$$

then

$$\mathcal{I}^s(f*g) = \sum_{n=1}^{\infty} n^{2-s}z^n.$$

The coefficients of the functions in the class $\mathcal{ST}$ cannot be greater than n. If we want that $n^{2-s}\le n$, then $s\ge 1$. Therefore we deduce that $S(\mathcal{ST},\mathcal{ST},\mathcal{ST}) = 1$. □

Theorem 2.2. *We have the following orders of convolution consistence*

(i) $S(\mathcal{CV},\mathcal{CV},\mathcal{ST}) = -1$,
(ii) $S(\mathcal{CV},\mathcal{ST},\mathcal{ST}) = 0$,
(iii) $S(\mathcal{ST},\mathcal{ST},\mathcal{CV}) = 2$,
(iv) $S(\mathcal{CV},\mathcal{CV},\mathcal{CV}) = 0$,
(v) $S(\mathcal{CV},\mathcal{ST},\mathcal{CV}) = 1$.

Proof. (i) It is well known [10] that $\mathcal{CV} * \mathcal{ST} = \mathcal{ST}$. Let $f, g \in \mathcal{CV}$. Then $zg' \in \mathcal{ST}$ and $\mathcal{I}^{-1}(f * g)(z) = f(z) * (zg'(z)) \in \mathcal{ST}$, so $S(\mathcal{CV}, \mathcal{CV}, \mathcal{ST}) \le -1$. If

$$f(z) = g(z) = \frac{z}{1-z} \in \mathcal{CV},$$

then

$$\mathcal{I}^s(f * g) = \sum_{n=1}^{\infty} n^{-s} z^n.$$

Because the coefficients of the functions in the class $\mathcal{ST}$ cannot be greater than n we obtain the condition $n^{-s} \le n$. Therefore we deduce that $s \ge -1$ and then $S(\mathcal{CV}, \mathcal{CV}, \mathcal{ST}) = -1$.

The proofs of (ii) – (v) run as the proof of (i). □

To find the order of convolution consistence of other classes let us recall the classes of k-uniformly convex and of k-starlike functions:

$$k\text{-}\mathcal{UCV} := \left\{ f \in \mathcal{S} : \mathfrak{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] > k\left|\frac{zf''(z)}{f'(z)}\right|, \quad (z \in \mathcal{U};\ 0 \le k < \infty) \right\},$$

$$k\text{-}\mathcal{ST} := \left\{ f \in \mathcal{S} : \mathfrak{Re}\left[\frac{zf'(z)}{f(z)}\right] > k\left|\frac{zf'(z)}{f(z)} - 1\right|, \quad (z \in \mathcal{U};\ 0 \le k < \infty) \right\}.$$

The class k-$\mathcal{UCV}$ was introduced by Kanas and Wiśniowska [5], where its geometric definition and connections with the conic domains were considered. The class k-$\mathcal{UCV}$ was defined pure geometrically as a subclass of univalent functions, that map each circular arc contained in the unit disk $\mathcal{U}$ with a center $\xi, |\xi| \le k \ (0 \le k < \infty)$, onto a convex arc. The notion of k-uniformly convex function is a natural extension of the classical convexity. Observe that, if $k = 0$ then the center ξ is the origin and the class k-$\mathcal{UCV}$ reduces to the class of convex univalent functions $\mathcal{CV}$. Moreover for $k = 1$ corresponds to the class of uniformly convex functions $\mathcal{UCV}$ introduced by Goodman [2] and studied extensively by Rønning [9] and independently by Ma and Minda [8]. The class k-$\mathcal{ST}$ is related to the class k-$\mathcal{UCV}$ by means of the well-known Alexander equivalence between the usual classes of convex $\mathcal{CV}$ and starlike $\mathcal{ST}$ functions (see also the works [4, 6, 7, 8, 9] for further developments involving each of the classes k-$\mathcal{UCV}$ and k-$\mathcal{ST}$). Moreover, in [1] the authors studied the properties of the integral convolution of the neighborhoods of these classes. To start examine the order of convolution consistence connected with the classes k-$\mathcal{UCV}$ and k-$\mathcal{ST}$ we need recall some basic results about these classes. Let us denote (see [4])

$$P_1(k) = \begin{cases} \frac{8(\arccos k)^2}{\pi^2(1-k^2)} & \text{for} \quad 0 \le k < 1 \\ \frac{8}{\pi^2} & \text{for} \quad k = 1 \\ \frac{\pi^2}{4\sqrt{t}(1+t)(k^2-1)\mathcal{K}^2(t)} & \text{for} \quad k > 1 \end{cases}, \tag{2.2}$$

where $t \in (0,1)$ is determined by $k = \cosh(\pi\mathcal{K}'(t)/[4\mathcal{K}(t)])$, $\mathcal{K}$ is the Legendre's complete Elliptic integral of the first kind

$$\mathcal{K}(t) = \int_0^1 \frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-t^2x^2)}}$$

and $\mathcal{K}'(t) = \mathcal{K}(\sqrt{1-t^2})$ is the complementary integral of $\mathcal{K}(t)$. Let Ω_k be a domain such that $1 \in \Omega_k$ and

$$\partial\Omega_k = \left\{w = u + iv : \quad u^2 = k^2(u-1)^2 + k^2v^2\right\}, \quad 0 \le k < \infty.$$

The domain Ω_k is elliptic for $k > 1$, hyperbolic when $0 < k < 1$, parabolic when $k = 1$, and a right half-plane when $k = 0$. If $\widetilde{p}_\alpha$ is an analytic function with $\widetilde{p}_\alpha(0) = 1$ which maps the unit disc $\mathcal{U}$ conformally onto the region Ω_k, then $P_1(k) = \widetilde{p}_\alpha'(0)$. $P_1(k)$ is strictly decreasing function of the variable k and its values are included in the interval $(0, 2]$.

Lemma 2.3. (see [4]) *Let $0 \le k < \infty$ and let $f \in k$-$\mathcal{ST}$ be of the form*

$$f(z) = z + \sum_{n=2}^{\infty} a_n(k)z^n \quad (|z| < 1),$$

then

$$|a_n(k)| \le \frac{(P_1(k))_{(n-1)}}{(n-1)!}, \quad n = 2, 3, \dots ,$$

where $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n = \begin{cases} 1 & (n = 0) \\ \lambda(\lambda+1) \cdot \ldots \cdot (\lambda+n-1) & (n \in \mathbf{N}). \end{cases}$$

For $k = 0$ the estimates are sharp; otherwise only the bound on $|a_2(k)|$ is sharp.

Lemma 2.4. (see [4]) *Let* $0 \le k < \infty$ *and let* $f \in k\text{-}\mathcal{UCV}$ *be of the form*

$$f(z) = z + \sum_{n=2}^{\infty} a_n(k) z^n \quad (|z| < 1),$$

then

$$|a_n(k)| \le \frac{(P_1(k))_{(n-1)}}{n!}, \quad n = 2, 3, \dots ,$$

where $P_1(k)$ *is given in (2.2). For* $k = 0$ *the estimates are sharp; otherwise only the bound on* $|a_2(k)|$ *is sharp.*

Theorem 2.5. *The following inequalities hold true*

(i) $\log_2 P_1(k) \le S(k\text{-}\mathcal{ST}, k\text{-}\mathcal{ST}, k\text{-}\mathcal{ST}) \le 1$,

(ii) $1 + \log_2 P_1(k) \le S(k\text{-}\mathcal{ST}, k\text{-}\mathcal{ST}, k\text{-}\mathcal{UCV}) \le 2$,

(iii) $S(k\text{-}\mathcal{ST}, \mathcal{CV}, k\text{-}\mathcal{UCV}) = 1$,

(iv) $S(k\text{-}\mathcal{ST}, \mathcal{CV}, k\text{-}\mathcal{ST}) = 0$,

(v) $S(k\text{-}\mathcal{UCV}, \mathcal{CV}, k\text{-}\mathcal{UCV}) = 0$,

whenever there exist the above orders of convolution consistence.

Proof. (i) In [4] it was proved that if $f, g \in k\text{-}\mathcal{ST}$ then $f \otimes g \in k\text{-}\mathcal{ST}$ so $\mathcal{I}^1(f * g) = f \otimes g \in k\text{-}\mathcal{ST}$. Therefore $S(k\text{-}\mathcal{ST}, k\text{-}\mathcal{ST}, k\text{-}\mathcal{ST}) \le 1$, whenever it there exist. Suppose that

$$f(z) = g(z) = z \exp \int_0^z \frac{\widetilde{p}_\alpha(t) - 1}{t} \, \mathrm{d}t = z + P_1(k) z^2 + \cdots , \tag{2.3}$$

where $P_1(k)$ is given in (2.2). Then $f, g \in k\text{-}\mathcal{ST}$ and by Lemma 2.3 for the second coefficient we have

$$\mathcal{I}^s(f * g) \in k\text{-}\mathcal{ST} \Rightarrow \frac{P_1(k) P_1(k)}{2^s} \le P_1(k) \Leftrightarrow P_1(k) \le 2^s.$$

Therefore we deduce that $S(k\text{-}\mathcal{ST}, k\text{-}\mathcal{ST}, k\text{-}\mathcal{ST}) \ge \log_2 P_1(k)$. Notice that $P_1(k)$ is strictly decreasing function of the variable k and its values are included in the interval $(0, 2]$.

(ii) This proof runs as the previous proof.

(iii) Let $f \in k\text{-}\mathcal{ST}$ and $g \in \mathcal{CV}$. Then [4] $f \otimes g \in k\text{-}\mathcal{UCV}$ so $\mathcal{I}^1(f * g) \in k\text{-}\mathcal{UCV}$, hence $S(\mathcal{CV}, \mathcal{CV}, \mathcal{ST}) \le 1$. If f is given as in (2.3) and $g(z) = z/(1 - z) \in \mathcal{CV}$, then by Lemma 2.4

$$\mathcal{I}^s(f * g) \in k\text{-}\mathcal{UCV} \Rightarrow \frac{P_1(k)}{2^s} \le \frac{P_1(k)}{2} \Leftrightarrow s \ge 1.$$

Therefore $S(k\text{-}\mathcal{ST}, \mathcal{CV}, k\text{-}\mathcal{UCV}) = 1$

(iv), (v) Those proofs run as the previous proof. ☐

Lemma 2.6. (see [11]) *Let F and G be in $\mathcal{CV}$. Then*

$$f \prec F \quad and \quad g \prec G \quad \Rightarrow \quad f * g \prec F * G. \tag{2.4}$$

Let us consider for $\alpha < 1$ the class of functions:

$$\mathcal{P}(\alpha) = \{p : zp(z) \in \mathcal{A} \text{ and } \mathfrak{Re}\,[p(z)] > \alpha \text{ for } z \in \mathcal{U}\}.$$

Lemma 2.7. *If $h \in \mathcal{P}(\alpha)$ and $h(z) = 1 + a_1 z + a_2 z^2 + \cdots$, then the function*

$$H(z) = 1 + \sum_{n=1}^{\infty} \frac{a_n}{n} z^n \quad (z \in \mathcal{U}) \tag{2.5}$$

satisfies

$$H(z) \prec 1 - 2(1-\alpha)\log(1-z) \quad (z \in \mathcal{U}) \tag{2.6}$$

and belongs to the class $\mathcal{P}(1 + 2(\alpha - 1)\log 2)$.

Proof. It is well known that the function

$$g(z) = -\log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n} \quad (z \in \mathcal{U})$$

belongs to the class $\mathcal{CV}$ of convex univalent functions so $g(z) + 1$ is convex univalent too. Thus as in (2.4) we have

$$\begin{cases} h(z) \prec \frac{1+(1-2\alpha)z}{1-z} \\ g(z) + 1 \prec g(z) + 1 \end{cases} \Rightarrow \quad h(z) * (g(z)+1) \prec \frac{1+(1-2\alpha)z}{1-z} * (g(z)+1)\,,$$

Therefore we can write

$$\begin{aligned} h(z) * (g(z)+1) &= 1 + \sum_{n=1}^{\infty} \frac{a_n}{n} z^n \\ &\prec \frac{1+(1-2\alpha)z}{1-z} * (1 - \log(1-z)) \\ &= \left[1 + 2(1-\alpha)(z + z^2 + \cdots)\right] * (1 - \log(1-z)) \\ &= 1 - 2(1-\alpha)\log(1-z). \end{aligned} \tag{2.7}$$

The function

$$H(z) = \frac{1+(1-2\alpha)z}{1-z} * (1 - \log(1-z)) = 1 - 2(1-\alpha)\log(1-z) \quad (z \in \mathcal{U})$$

is convex univalent as a convolution of convex univalent functions and is typically–real so the geometric properties of the image of $H(\mathcal{U})$ show that

$$\min\{\mathfrak{Re} H(z) : |z| < 1\} = H(-1) = 1 + 2(\alpha - 1)\log 2.$$

Therefore from (2.7) we obtain that $H \in \mathcal{P}(1 + 2(\alpha - 1)\log 2)$. □

Lemma 2.8. [13] *If $a \le 1$, $b \le 1$, and $f \in \mathcal{P}(a)$, $g \in \mathcal{P}(b)$ for $z \in \mathcal{U}$, then*

$$\mathfrak{Re}[(f * g)(z)] > c \quad for\ z \in \mathcal{U},$$

were $c = 1 - 2(1 - a)(1 - b)$.

Theorem 2.9. *If $S(\mathcal{P}(\alpha), \mathcal{P}(\alpha), \mathcal{P}(\delta))$ there exists, then*

(i) $\mu \le S(\mathcal{P}(\alpha), \mathcal{P}(\alpha), \mathcal{P}(\delta)) \le 1$,
where $\delta = 1 - 4(1 - \alpha)^2 \log 2$, $\mu = -\frac{\log(2\log 2)}{\log 2} = -0.732\ldots$,

(ii) $S(\mathcal{P}(\alpha), \mathcal{P}(\beta), \mathcal{P}(\gamma)) = 0$,
where $\gamma = 1 - 2(1 - \alpha)(1 - \beta)$,

(iii) $1 + \log_2 \frac{(1-\alpha)(1-\beta)}{1-\gamma} \le S(\mathcal{P}(\alpha), \mathcal{P}(\beta), \mathcal{P}(\gamma)) \le 0$,
where $\gamma < 1 - 2(1 - \alpha)(1 - \beta)$.

Proof. (i) Let $g \in \mathcal{P}(\alpha)$ and let

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n.$$

Let h, H be given as in Lemma 2.7. Therefore we have $H \in \mathcal{P}(\gamma)$, where

$$\gamma = 1 + 2(\alpha - 1)\log 2.$$

Further, by Lemma 2.8 we have

$$\begin{aligned} \mathcal{I}^1(g * h)(z) &= 1 + \sum_{n=1}^{\infty} \frac{a_n b_n}{n} z^n = g(z) * H(z) \\ &\in \mathcal{P}(1 - 2(1 - \alpha)(1 - \gamma)) \\ &= \mathcal{P}(1 - 4(1 - \alpha)^2 \log 2), \end{aligned} \tag{2.8}$$

so $S(\mathcal{P}(\alpha), \mathcal{P}(\alpha), \mathcal{P}(\delta)) \le 1$. Suppose that

$$h(z) = g(z) = 1 + 2(1 - \alpha)\sum_{n=1}^{\infty} z^n \in \mathcal{P}(\alpha).$$

It is known that if $1 + a_1 z + \cdots \in \mathcal{P}(\delta)$, then $|a_n| \le 2(1-\delta)$. Therefore, examining the second coefficients we get

$$\mathcal{I}^s(g*h) \in \mathcal{P}(\delta) \Rightarrow \frac{4(1-\alpha)^2}{2^s} \le 2(4(1-\alpha)^2 \log 2) \Leftrightarrow \frac{1}{2\log 2} \le 2^s \Leftrightarrow s > \log_2 \frac{1}{2\log 2}$$

and we can see that $S(\mathcal{P}(\alpha), \mathcal{P}(\alpha), \mathcal{P}(\delta)) \ge \mu$, where $\mu = -\frac{\log(2\log 2)}{\log 2} = -0.732\ldots$.

For the proof of (ii) notice that by Lemma 2.8 if $f \in \mathcal{P}(\alpha)$ and $g \in \mathcal{P}(\beta)$, then $\mathcal{I}^0(f*g) \in \mathcal{P}(\gamma))$. This means that $S(\mathcal{P}(\alpha), \mathcal{P}(\beta), \mathcal{P}(\gamma)) \le 0$. If

$$\begin{aligned} f(z) &= 1 + 2(1-\alpha)\sum_{n=1}^{\infty} z^n \in \mathcal{P}(\alpha) \\ g(z) &= 1 + 2(1-\beta)\sum_{n=1}^{\infty} z^n \in \mathcal{P}(\beta), \end{aligned} \tag{2.9}$$

then for the second coefficient we have

$$\mathcal{I}^s(f*g) \in \mathcal{P}(\gamma) \Rightarrow \frac{4(1-\alpha)(1-\beta)}{2^s} \le 2(1-\gamma) \Leftrightarrow 2^s \ge 1.$$

Therefore we deduce that $S(\mathcal{P}(\alpha), \mathcal{P}(\beta), \mathcal{P}(\gamma)) = 0$.

In order to prove (iii) notice that by Lemma 2.8 if $f \in \mathcal{P}(\alpha)$ and $g \in \mathcal{P}(\beta)$, then

$$\mathcal{I}^0(f*g) \in \mathcal{P}(1-2(1-\alpha)(1-\beta)) \subseteq \mathcal{P}(\gamma).$$

This means that $S(\mathcal{P}(\alpha), \mathcal{P}(\beta), \mathcal{P}(\gamma)) \le 0$. If $f \in \mathcal{P}(\alpha)$ and $g \in \mathcal{P}(\beta)$ are given as in (2.9), then for the second coefficient we have

$$\mathcal{I}^s(f*g) \in \mathcal{P}(\gamma) \Rightarrow \frac{4(1-\alpha)(1-\beta)}{2^s} \le 2(1-\gamma) \Leftrightarrow 2^{s-1} \ge \frac{(1-\alpha)(1-\beta)}{1-\gamma}.$$

Thus we see that

$$1 + \log_2 \frac{(1-\alpha)(1-\beta)}{1-\gamma} \le S(\mathcal{P}(\alpha), \mathcal{P}(\beta), \mathcal{P}(\gamma)).$$

Note that if $\gamma < 1 - 2(1-\alpha)(1-\beta)$, then

$$1 + \log_2 \frac{(1-\alpha)(1-\beta)}{1-\gamma} < 0.$$

□

References

[1] Bednarz, U., Kanas, S., *Stability of the integral convolution of k-uniformly convex and k-starlike functions*, Journal of Appl. Analisis, **10** (2004), no. ,1 105-115.

[2] Goodman, A. W., *On uniformly convex functions, Ann. Polon. Math.*, **56** (1991), 87-92.

[3] Kanas, S., *Stability of convolution and dual sets for the class of k-uniformly convex and k-starlike functions*, Folia Sci. Univ. Tech. Resov., **22** (1998), 51-64.

[4] Kanas, S., Wiśniowska, A., *Conic regions and k-uniform convexity II*, Folia Sci. Univ. Tech. Resov., **22**(1998), 65-78.

[5] Kanas, S., Wiśniowska, A., *Conic regions and k-uniform convexity*, J. Comput. Appl. Math., **105** (1999), 327-336.

[6] Kanas, S., Wiśniowska, A., *Conic regions and k-starlike functions*, Rev. Roumaine Math. Pures Appl., **45** (2000), 647-657.

[7] Kanas, S., Srivastava, H. M., *Linear operators associated with k-uniformly convex functions*, Integral Transform. Spec. Funct., **9** (2000), no. 2, 121-132.

[8] Ma, W., Minda, D., *Uniformly convex functions*, Ann. Polon. Math., **57** (1992), no. 2, 165-175.

[9] Rønning, F., *Uniformly convex functions and a corresponding class of starlike functions*, Proc. Amer. Math. Soc., **118** (1993), 189-196.

[10] Ruscheweyh, S., Sheil-Small, T., *Hadamard product of schlicht functions and the Poyla-Schoenberg conjecture*, Comm. Math. Helv., **48** (1973), 119-135.

[11] Ruscheweyh, S., Stankiewicz, J., *Subordination under convex univalent function*, Bull. Pol. Acad. Sci. Math., **33** (1985), 499-502.

[12] Sălăgean, G. S., *Subclasses of univalent functions.* Complex Analysis, Fifth Romanian Finnish Seminar, Part 1 (Bucharest, 1981), Lecture Notes in Math., 1013, Springer, Berlin, 1983, 362-372.

[13] Stankiewicz, J., Stankiewicz, Z., *Some Applications of the Hadamard Convolution in the Thory of Functions*, Ann. Univ. Mariae Curie-Skłodowska, **40** (1986), 251-265.

Department of Mathematics
Rzeszów University of Technology
ul. W. Pola 2, 35-959 Rzeszów, Poland
E-mail address: `ubednarz@prz.edu.pl`

Department of Mathematics
Rzeszów University of Technology
ul. W. Pola 2, 35-959 Rzeszów, Poland
E-mail address: `jsokol@prz.edu.pl`