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VARIOUS PROPERTIES OF A CERTAIN CLASS OF MULTIVALENT ANALYTIC FUNCTIONS

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Dedicated to Professor Grigore Ştefan Sălăgean on his 60^{th} *birthday*

Abstract. By using the techniques of Briot-Bouquet differential subordination, we study various properties and characteristics of the subclass $V_{p,q,s}^{\lambda}(\alpha_1;\beta_1;A,B)$ of multivalent analytic functions.

1. **Introduction**

Let $A(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad (p \in \mathbb{N} = \{1, 2, 3, ...\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Let Ω denote the class of bounded analytic functions satisfying $\omega(0) = 0$ and $|\omega(z)| \leq |z|$ for $z \in U$. For functions $f(z) \in A(p)$ given by (1.1) and $g(z) \in A(p)$ defined by $g(z) = z^p + \sum\limits_{k=1}^{\infty} b_{k+p} z^{k+p}$ $(p \in \mathbb{N})$, the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f * g)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} b_{k+p} z^{k+p} = (g * f)(z).$$

For given arbitrary numbers A, B ($-1 \leq B < A \leq 1$), we denote by $P(A, B)$ the class of functions of the form:

$$\varphi(z) = 1 + b_1 z + b_2 z^2 + ..., \tag{1.2}$$

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which are analytic in U and satisfy the following condition:

$$\varphi(z) \prec \frac{1+Az}{1+Bz} \quad (z \in U).$$

(Here the symbol $\prec$ stands for subordination.) The class $P(A, B)$ was investigated by Janowski [11].

For a function $f(z) \in A(p)$ given by (1.1), the generalized Bernardi-Libera-Livingston integral operator $F_{\delta,p}$ is defined by (see [5])

$$\begin{aligned} F_{\delta,p}(f)(z) &= \frac{\delta+p}{z^{\delta}} \int_0^z t^{\delta-1} f(t)dt \\ &= z^p + \sum_{k=1}^{\infty} \left(\frac{\delta+p}{\delta+p+k}\right) a_{k+p} z^{k+p} \quad (\delta > -p; z \in U). \end{aligned} \tag{1.3}$$

It readily follows from (1.3) that $f \in A(p) \Longleftrightarrow F_{\delta,p} \in A(p)$. Furthermore, we have

$$\theta_m(z) = F_{\delta_m,p}(F_{\delta_{m-1},p}...(F_{\delta_1,p}(z)))$$

$$= z^p + \sum_{k=1}^{\infty} \left(\prod_{j=1}^{m} \frac{\delta_j+p}{\delta_j+p+k} \right) a_{k+p} z^{k+p} \quad (\delta_j > -p;\ j = 1, ..., m). \tag{1.4}$$

For complex parameters $\alpha_1, ..., \alpha_q$ and $\beta_1, ..., \beta_s$ ($\beta_j \notin \mathbb{Z}_0^-$; $\mathbb{Z}_0^- = \{0, -1, -2, ...\}$; $j = 1, ..., s$,) the generalized hypergeometric function ${}_qF_s(z)$ is defined (cf., e.g., [28]) as follows:

$${}_qF_s(z) \equiv {}_q F_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k...(\alpha_q)_k}{(\beta_1)_k...(\beta_s)_k} \frac{z^k}{(1)_k} \tag{1.5}$$

$$(q \leq s+1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in U),$$

where $(x)_k$ is the Pochhammer symbol defined (in terms of the Gamma function) by

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = \begin{cases} x(x+1)...(x+k-1) & (k \in \mathbb{N} \text{ and } x \in \mathbb{C}) \\ 1 & (k = 0 \text{ and } x \in \mathbb{C}\backslash\{0\}). \end{cases}$$

We note that the series (1.5) converges absolutely for $z \in U$ and hence represents an analytic function in the open unit disk U (see [29]). Corresponding to a function $\mathcal{F}_p(\alpha_1, ..., \alpha_q, \beta_1, ..., \beta_s; z)$ defined by

$$\mathcal{F}_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = z^p \ {}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z),$$

Dziok and Srivastava [6] defined a linear operator $H_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s) : A(p) \rightarrow A(p)$ by the following Hadamard product:

$$H_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s)f(z) = \mathcal{F}_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) * f(z),$$

$$(q \leq s + 1; q, s \in \mathbb{N}_0; z \in U).$$

If $f \in A(p)$ is given by (1.1), then we have

$$H_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s)f(z) = z^p + \sum_{k=1}^{\infty} \Gamma_k a_{k+p} z^{k+p} \ , \tag{1.6}$$

where

$$\Gamma_k = \frac{(\alpha_1)_k ... (\alpha_q)_k}{(\beta_1)_k ... (\beta_s)_k (1)_k} \qquad (k \in \mathbb{N}).$$

For convenience, we write

$$H_{p,q,s}(\alpha_1; \beta_1) = H_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s).$$

It follows from (1.6) that

$$H_{p,2,1}(p, 1; p)f(z) = f(z), \ H_{p,2,1}(p + 1, 1; p)f(z) = \frac{zf^{'}(z)}{p}$$

and

$$\begin{aligned} z(H_{p,q,s}(\alpha_1; \beta_1)f(z))^{'} \ &= \ (\beta_1 - 1)H_{p,q,s}(\alpha_1; \beta_1 - 1)f(z) \\ &+ \ (p + 1 - \beta_1)H_{p,q,s}(\alpha_1; \beta_1)f(z). \end{aligned} \tag{1.7}$$

The linear operator $H_{p,q,s}(\alpha_1; \beta_1)$ includes various other linear operators which were considered in earlier works. In particular, for $f \in A(p)$ we have the following observations:

(i) $H_{1,2,1}(a, b; c)f(z) = I_c^{a,b} f(z)$ $(a, b \in \mathbb{C}; c \notin \mathbb{Z}_0^-)$, where $I_c^{a,b}$ is the linear operator investigated by Hohlov [10];

(ii) $H_{p,2,1}(n + p, 1; 1)f(z) = D^{n+p-1} f(z)$ $(n > -p; p \in \mathbb{N})$, where D^{n+p-1} is the linear operator studied by Goel and Sohi [8]. In the case when $p = 1$, $D^n f(z)$ is the $n - th$ Ruscheweyh derivative of $f(z)$ (see [22]);

(iii) $H_{p,2,1}(\delta+p,1;\delta+p+1)f(z)= F_{\delta,p}(f)(z)$ $(\delta > -p)$, where $F_{\delta,p}$ is the generalized Bernardi–Libera–Livingston integral operator ([5]);

(iv) $H_{p,2,1}(p+1,1;p+1-\mu)f(z)= \Omega_z^{(\mu,p)}f(z)$ $(-\infty < \mu < p+1)$, where $\Omega_z^{(\mu,p)}$ $(-\infty < \mu < p+1)$ is the extended fractional differintegral operator (see [20]), defined by

$$\begin{aligned}\Omega_z^{(\mu,p)}f(z) &= z^p + \sum_{k=1}^{\infty} \frac{\Gamma(k+p+1)\Gamma(p+1-\mu)}{\Gamma(p+1)\Gamma(k+p+1-\mu)} a_{k+p} z^{k+p} \\ &= \frac{\Gamma(p+1-\mu)}{\Gamma(p+1)} z^{\mu} D_z^{\mu} f(z) \quad (-\infty < \mu < p+1),\end{aligned}$$

where $D_z^{\mu}f(z)$ is, respectively, the fractional integral of $f(z)$ of order $-\mu$ when $-\infty < \mu < 0$ and the fractional derivative of $f(z)$ of order μ when $0 < \mu < p+1$ (see, for details [18], [19] and [20]). The fractional differential operator $\Omega_z^{(\mu,p)}$ with $0 \le \mu < 1$ was investigated by Srivastava and Aouf [27].

(v) $H_{p,2,1}(a,1;c)f(z)= L_p(a;c)f(z)$ $(a \in \mathbb{R}; c \in \mathbb{R}\backslash\mathbb{Z}_0^-)$, where $L_p(a;c)$ is the linear operator studied by Saitoh [24] which yields the operator $L(a;c)f(z)$ introduced by Carlson and Shaffer [3] for $p = 1$;

(vi) $H_{1,2,1}(\mu,1;\lambda+1)f(z)= I_{\lambda,\mu}f(z)$ $(\lambda > -1; \mu > 0)$, where $I_{\lambda,\mu}$ is the Choi–Saigo–Srivastava operator [5];

(vii) $H_{p,2,1}(p+1,1;n+p)f(z)= I_{n,p}f(z)$ $(n > -p; p \in \mathbb{N})$, where $I_{n,p}$ is the Noor integral operator of $(n+p-1)-th$ order , studied by Liu and Noor [15];

(viii) $H_{p,2,1}(\lambda+p,c;a)f(z)= I_p^{\lambda}(a;c)f(z)$ $(a,c \in \mathbb{R}\backslash\mathbb{Z}_0^-; \lambda > -p)$, where $I_p^{\lambda}(a;c)$ is the Cho–Kwon–Srivastava operator [4].

Now, by making use of the Dziok–Srivastava operator $H_{p,q,s}(\alpha_1;\beta_1)$, we introduce a subclass of functions in $A(p)$ as follows.

Definition 1.1. *A function $f(z) \in A(p)$ is said to be in the class $V_p^{\lambda}(\alpha_1, ..., \alpha_q;$ $\beta_1, ..., \beta_s; A, B)$ $((\alpha_j > 0;\ j = 1, ..., q),\ (\beta_j \notin \mathbb{Z}_0^-;\ j = 1, ..., s),\ \beta_1 > 1,\ \lambda \geq 0$ and $-1 \leq B < A \leq 1)$, if and only if it satisfies*

$$(1-\lambda)\frac{H_{p,q,s}(\alpha_1;\beta_1)f(z)}{z^p} + \lambda\frac{H_{p,q,s}(\alpha_1;\beta_1-1)f(z)}{z^p} \prec \frac{1+Az}{1+Bz}\ . \tag{1.8}$$

For convenience, we write $V_{p,q,s}^{\lambda}(\alpha_1;\beta_1;A,B) = V_p^{\lambda}(\alpha_1,..,\alpha_q;\beta_1,..,\beta_s;A,B)$.

We note that

(i) $V_{1,2,1}^{1}(2,1;2;1-2\alpha,-1) = R(\alpha) \quad (0 \leq \alpha < 1)$ [7];

(ii) $V_{p,2,1}^{1}(p+1,1;p+1;1,\frac{1}{M}-1) = S_p(M) \quad (M > \frac{1}{2})$ [26];

(iii) $V_{1,2,1}^{1}(2,1;2;2\alpha\beta-1,2\beta-1) = R_1(\alpha,\beta) \quad (0 \leq \alpha < 1,\ 0 < \beta \leq 1)$ [16];

(iv) $V_{1,2,1}^{1}(2,1;2;(2\alpha-1)\beta,\beta) = R(\alpha,\beta) \quad (0 \leq \alpha < 1,\ 0 < \beta \leq 1)$ [12];

(v) $V_{1,2,1}^{1}(n+2,1;2;A,B) = V_n(A,B) \quad (n > -1)$ [14];

(vi) $V_{1,2,1}^{1}(n+2,1;2;B+(A-B)(1-\alpha),B) = V_n(A,B,\alpha) \quad (n > -1; 0 \leq \alpha < 1)$ [2];

(vii) $V_{p,2,1}^{\lambda}(p+1,1,p+1-\mu;\beta(1-(2\alpha/p)),-\beta) = V_p^{\lambda}(\mu,\alpha,\beta)$; where $V_p^{\lambda}(\mu,\alpha,\beta)$ denotes the class of functions $f\ (z) \in A(p)$ satisfying the condition:

$$\left|\frac{(1-\lambda)\Omega_z^{(\mu,p)}f(z) + \lambda\Omega_z^{(1+\mu,p)}f(z) - z^p}{(1-\lambda)\Omega_z^{(\mu,p)}f(z) + \lambda\Omega_z^{(1+\mu,p)}f(z) + (1-(2\alpha/p))z^p}\right| < \beta \quad (z \in U),$$

where $0 \leq \mu < 1$, $0 \leq \alpha < p$, $p \in \mathbb{N}$ and $0 < \beta \leq 1$;

(viii) $V_{p,q,s}^{\lambda}(\alpha_1;\beta_1;1,\frac{1}{M}-1) = V_{p,q,s}^{\lambda}(\alpha_1;\beta_1;M) \quad (M > \frac{1}{2})$, where $V_{p,q,s}^{\lambda}(\alpha_1;\beta_1;$ $M)$ denotes the class of functions $f\ (z) \in A(p)$ satisfying the condition:

$$\left|\left[(1-\lambda)\frac{H_{p,q,s}(\alpha_1;\beta_1)f(z)}{z^p} + \lambda\frac{H_{p,q,s}(\alpha_1;\beta_1-1)f(z)}{z^p}\right] - M\right| < M$$

$$(M > \frac{1}{2}; z \in U);$$

(ix) $V_{p,2,1}^{1}(p+1,1;p+2-\mu;1,\frac{1}{M}-1) = V_p(\mu,M) \quad (M > \frac{1}{2}; -\infty < \mu < p+1)$, where $V_p(\mu,M)$ denotes the class of functions $f\ (z) \in A(p)$ satisfying the condition:

$$\left|\frac{\Omega_z^{(\mu,p)}f(z)}{z^p} - M\right| < M \quad (M > \frac{1}{2}; -\infty < \mu < p+1; z \in U).$$

2. **Preliminaries**

To prove our main results, we need the following lemmas.

Lemma 2.1. [9] *Let the function $h(z)$ be analytic and convex (univalent) in U with $h(0)=1$ and let the function $\phi(z)$ given by (1.2) be analytic in U. If*

$$\phi(z)+\frac{z\phi^{'}(z)}{\gamma}\prec h(z)\quad (\mathrm{Re}(\gamma)\geq 0;\ \gamma\neq 0),$$

then

$$\phi(z)\prec\psi(z)=\frac{\gamma}{z^{\gamma}}\int_{0}^{z}t^{\gamma-1}h(t)dt\prec h(z)\ ,$$

and $\psi(z)$ is the best dominant.

Lemma 2.2. [25] *Let $\Phi(z)$ be analytic in U with*

$$\Phi(0)=1\qquad \textit{and}\quad \mathrm{Re}(\Phi(z))>\frac{1}{2}\ (z\in U).$$

Then, for any function $F(z)$ analytic in U, $(\Phi*F)(U)$ is contained in the convex hull of $F(U)$.

Lemma 2.3. [29] *For real or complex numbers a,b and c $(c\neq 0,-1,-2,...)$, we have*

$$\int_{0}^{1}t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt=\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\ {}_2F_1(a,b;c;z)\ (\mathrm{Re}\,(c)>\mathrm{Re}\,(b)>0); \tag{2.1}$$

and

$${}_2F_1(a,b;c;z)=(1-z)^{-a}\ {}_2F_1(a,c-b;c;\frac{z}{z-1}). \tag{2.2}$$

Lemma 2.4. [13] *Let $\omega(z)=\sum\limits_{k=1}^{\infty}d_kz^k\in\Omega$, if ν is a complex number, then*

$$\left|d_2-\nu d_1^2\right|\leq\max\{1,|\upsilon|\}. \tag{2.3}$$

Equation (2.3) may be attend with the functions $\omega(z)=z$ and $\omega(z)=z^2$, respectively, for $|\upsilon|\geq 1$ and $|\upsilon|<1$.

3. Main results

Otherwise unless mention throughout this paper, we assume that $-1 \le B < A \le 1$, $\lambda > 0$, $p \in \mathbb{N}$, $\beta_1 > 1$ and $z \in U$.

Theorem 3.1. *Let the function f defined by (1.1) be in the class $V^{\lambda}_{p,q,s}(\alpha_1;\beta_1;A,B)$. Then*

$$\frac{H_{p,q,s}(\alpha_1;\beta_1)f(z)}{z^p} \prec Q(z) \prec \frac{1+Az}{1+Bz}\ , \tag{3.1}$$

where

$$Q(z) = \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1+Bz)^{-1}\ {}_2F_1\left(1,1;\frac{\beta_1-1}{\lambda}+1;\frac{Bz}{Bz+1}\right) & (B \neq 0) \\ 1+\frac{\beta_1-1}{\beta_1-1+\lambda}Az & (B=0), \end{cases}$$

is the best dominant of (3.1). Furthermore,

$$\mathrm{Re}\left\{\frac{H_{p,q,s}(\alpha_1;\beta_1)f(z)}{z^p}\right\} > \eta(\lambda,\beta_1,A,B), \tag{3.2}$$

where

$$\eta(\lambda,\beta_1,A,B) = \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1-B)^{-1}\ {}_2F_1\left(1,1;\frac{\beta_1-1}{\lambda}+1;\frac{B}{B-1}\right) & (B \neq 0) \\ 1-\frac{\beta_1-1}{\beta_1-1+\lambda}A & (B=0). \end{cases}$$

The estimate in (3.2) is best possible.

Proof. Setting

$$\phi(z) = \frac{H_{p,q,s}(\alpha_1;\beta_1)f(z)}{z^p}\ . \tag{3.3}$$

Then $\phi(z)$ is of the form (1.2) and is analytic in U. Differentiating (3.3), and using identity (1.7) in the resulting equation, we have

$$\begin{aligned}(1-\lambda)\frac{H_{p,q,s}(\alpha_1;\beta_1)f(z)}{z^p} + \lambda\frac{H_{p,q,s}(\alpha_1;\beta_1-1)f(z)}{z^p} &= \phi(z) + \frac{\lambda z\phi'(z)}{\beta_1-1} \\ &\prec \frac{1+Az}{1+Bz}\ .\end{aligned}$$

Now, by using Lemma 2.1 for $\gamma = \frac{\beta_1 - 1}{\lambda}$, we deduce that

$$\begin{aligned} \phi(z) \prec Q(z) &= \frac{\beta_1 - 1}{\lambda} z^{-\frac{\beta_1 - 1}{\lambda}} \int_0^z t^{\frac{\beta_1 - 1}{\lambda} - 1} \left(\frac{1 + At}{1 + Bt}\right) dt \\ &= \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1 + Bz)^{-1} \, {}_2F_1\left(1, 1; \frac{\beta_1 - 1}{\lambda} + 1; \frac{Bz}{Bz + 1}\right) & (B \neq 0) \\ 1 + \frac{\beta_1 - 1}{\beta_1 - 1 + \lambda} Az & (B = 0), \end{cases} \end{aligned}$$

by change of variables followed by using the identities (2.1) and (2.2) (with $a = 1$, $b = \frac{\beta_1 - 1}{\lambda}$ and $c = b + 1$). This proves the assertion (3.1) of Theorem 3.1. Next, to prove (3.2), it suffices to show that

$$\inf_{|z| < 1} \{\operatorname{Re}(Q(z))\} = Q(-1). \tag{3.4}$$

For $|z| \leq r < 1$, we have

$$\operatorname{Re}\left\{\frac{1 + Az}{1 + Bz}\right\} \geq \frac{1 - Ar}{1 - Br}.$$

Setting

$$g(s, z) = \frac{1 + Asz}{1 + Bsz} \text{ and } d\mu(s) = \frac{\beta_1 - 1}{\lambda} s^{\frac{\beta_1 - 1}{\lambda} - 1} ds \quad (0 \leq s \leq 1),$$

we get

$$Q(z) = \int_0^1 g(s, z) d\mu(s),$$

so that

$$\operatorname{Re}\{Q(z)\} \geq \int_0^1 \frac{1 - Asr}{1 - Bsr} d\mu(s) = Q(-r) \quad (|z| \leq r < 1).$$

Letting $r \to 1^-$ in the above inequality, we obtain the assertion (3.4). The result in (3.2) is best possible as the function $Q(z)$ is the best dominant of (3.1). □

Corollary 3.2. *For* $0 < \lambda_2 < \lambda_1$, *we have*

$$V_{p,q,s}^{\lambda_1}(\alpha_1; \beta_1; A, B) \subset V_{p,q,s}^{\lambda_2}(\alpha_1; \beta_1; A, B).$$

Proof. Let $f \in V_{p,q,s}^{\lambda_1}(\alpha_1; \beta_1; A, B)$.

Then by Theorem 3.1, we have $f \in V^{0}_{p,q,s}(\alpha_1;\beta_1;A,B)$. Since

$$\begin{aligned}
& (1-\lambda_2)\frac{H_{p,q,s}(\alpha_1;\beta_1)f(z)}{z^p} + \lambda_2\frac{H_{p,q,s}(\alpha_1;\beta_1-1)f(z)}{z^p} \\
= & \left(1-\frac{\lambda_2}{\lambda_1}\right)\frac{H_{p,q,s}(\alpha_1;\beta_1)f(z)}{z^p} \\
& +\frac{\lambda_2}{\lambda_1}\left\{(1-\lambda_1)\frac{H_{p,q,s}(\alpha_1;\beta_1)f(z)}{z^p} + \lambda_1\frac{H_{p,q,s}(\alpha_1;\beta_1-1)f(z)}{z^p}\right\} \\
\prec & \frac{1+Az}{1+Bz}\ ,
\end{aligned}$$

we see that $f \in V^{\lambda_2}_{p,q,s}(\alpha_1;\beta_1;A,B)$. □

Taking $\lambda = s = \alpha_2 = 1,\ q = 2,\ \alpha_1 = p+1,\ \beta_1 = n+p,\ A = 1-\frac{2\alpha}{p}$ and $B = -1$ in Theorem 3.1, we get the following corollary.

Corollary 3.3. *Let the function f given by (1.1) satisfy*

$$\mathrm{Re}\left\{\frac{I_{n-1,p}f(z)}{z^p}\right\} > \frac{\alpha}{p} \quad (0 \le \alpha < p;\ n > -p).$$

Then

$$\mathrm{Re}\left\{\frac{I_{n,p}f(z)}{z^p}\right\} > \frac{\alpha}{p} + \left(1-\frac{\alpha}{p}\right)\left\{{}_2F_1\left(1,1;p+n;\frac{1}{2}\right)-1\right\}.$$

The result is best possible.

Putting $n = 1$ in Corollary 3.3, we have the following corollary.

Corollary 3.4. *If $f \in A(p)$ satisfies*

$$\mathrm{Re}\left\{\frac{f'(z)}{z^{p-1}}\right\} > \alpha \quad (0 \le \alpha < p),$$

then

$$\mathrm{Re}\left\{\frac{f(z)}{z^p}\right\} > \frac{\alpha}{p} + \left(1-\frac{\alpha}{p}\right)\left\{{}_2F_1\left(1,1;p+1;\frac{1}{2}\right)-1\right\}.$$

The result is best possible.

Remark 3.5. The above result improves the corresponding result of Saitoh [23, Corollary 2].

Theorem 3.6. *Let $f(z) \in V^{0}_{p,q,s}(\alpha_1;\beta_1;A,B)$, then the function $F_{\delta,p}$ defined by (1.3) satisfies*

$$\frac{H_{p,q,s}(\alpha_1;\beta_1)F_{\delta,p}(z)}{z^p} \prec q(z) \prec \frac{1+Az}{1+Bz}\ , \tag{3.5}$$

where

$$q(z)=\begin{cases} \frac{A}{B}+\left(1-\frac{A}{B}\right)(1+Bz)^{-1}\ {}_2F_1\left(1,1;p+\delta+1;\frac{Bz}{Bz+1}\right) & (B\neq 0)\\ 1+\frac{p+\delta}{p+\delta+1}Az & (B=0), \end{cases}$$

and $q(z)$ is the best dominant of (3.5). Furthermore,

$$\operatorname{Re}\left\{\frac{H_{p,q,s}(\alpha_1,\beta_1)F_{\delta,p}(z)}{z^p}\right\}>\xi(\delta,p,A,B)\ , \tag{3.6}$$

where

$$\xi(\delta,p,A,B)=\begin{cases} \frac{A}{B}+\left(1-\frac{A}{B}\right)(1-B)^{-1}\ {}_2F_1\left(1,1;p+\delta+1;\frac{B}{B-1}\right) & (B\neq 0)\\ 1-\frac{p+\delta}{p+\delta+1}A & (B=0). \end{cases}$$

The estimate in (3.6) is best possible.

Proof. Let

$$\phi(z)=\frac{H_{p,q,s}(\alpha_1,\beta_1)F_{\delta,p}(z)}{z^p}\ . \tag{3.7}$$

Then $\phi(z)$ is analytic in U with $\phi(0)=1$. Differentiating (3.7) and using the identity

$$z(H_{p,q,s}(\alpha_1;\beta_1)F_{\delta,p}(z))^{'}=(\delta+p)H_{p,q,s}(\alpha_1;\beta_1)f(z)-\delta H_{p,q,s}(\alpha_1;\beta_1)F_{\delta,p}(z) \tag{3.8}$$

in the resulting equation, we obtain

$$\phi(z)+\frac{z\phi^{'}(z)}{\delta+p}=\frac{H_{p,q,s}(\alpha_1;\beta_1)f(z)}{z^p}\prec\frac{1+Az}{1+Bz}\quad(z\in U).$$

Now, by using Lemma 2.1 for $\gamma=\delta+p$, we deduce that

$$\phi(z)\prec q(z)=(\delta+p)z^{-(\delta+p)}\int_0^z t^{\delta+p-1}\left(\frac{1+At}{1+Bt}\right)dt.$$

The assertions (3.5) and (3.6) can now be deduced on the same lines that used in Theorem 3.1. This completes the proof of Theorem 3.6. □

Taking $A=1-\frac{2\alpha}{p}$ $(0\leq\alpha<p)$ and $B=-1$ in Theorem 3.6, we get the following corollary.

Corollary 3.7. *If $f\in A(p)$ satisfies*

$$\operatorname{Re}\left\{\frac{H_{p,q,s}(\alpha_1;\beta_1)f(z)}{z^p}\right\}>\frac{\alpha}{p}\quad(0\leq\alpha<p),$$

then

$$\operatorname{Re}\left\{\frac{H_{p,q,s}(\alpha_1;\beta_1)F_{\delta,p}(z)}{z^p}\right\} > \frac{\alpha}{p} + \left(1-\frac{\alpha}{p}\right)\left\{{}_2F_1\left(1,1;p+\delta+1;\frac{1}{2}\right)-1\right\}.$$

The result is best possible.

Taking $s=\alpha_2=1$, $q=2$, $\alpha_1=p+1$ and $\beta_1=p+1-\mu$ $(-\infty<\mu<p+1)$ in Corollary 3.7, we get the following corollary.

Corollary 3.8. *If $f\in A(p)$ satisfies*

$$\operatorname{Re}\left\{\frac{\Omega_z^{(\mu,p)}f(z)}{z^p}\right\} > \frac{\alpha}{p} \quad (0\le\alpha<p;-\infty<\mu<p+1),$$

then

$$\operatorname{Re}\left\{\frac{\Omega_z^{(\mu,p)}F_{\delta,p}(z)}{z^p}\right\} > \frac{\alpha}{p} + \left(1-\frac{\alpha}{p}\right)\left\{{}_2F_1\left(1,1;p+\delta+1;\frac{1}{2}\right)-1\right\}.$$

The result is best possible.

Corollary 3.9. *Under the hypothesis of Corollary 3.7, the function $\theta_m(z)$ defined by (1.4)satisfies*

$$\operatorname{Re}\left\{\frac{H_{p,q,s}(\alpha_1;\beta_1)\theta_m(z)}{z^p}\right\} > \frac{\rho_m}{p},$$

where $\rho_0=\alpha$ and

$$\rho_j=\rho_{j-1}+(p-\rho_{j-1})\left\{{}_2F_1\left(1,1;p+\delta+1;\frac{1}{2}\right)-1\right\} \quad (j=1,2,...,m).$$

The result is best possible.

Taking $s=\alpha_2=1, q=2$, $\alpha_1=p+1$ and $\beta_1=n+p$ $(n>-p)$ in Corollary 3.7, we have the following corollary.

Corollary 3.10. *If $f\in A(p)$ satisfies*

$$\operatorname{Re}\left\{\frac{I_{n,p}f(z)}{z^p}\right\} > \frac{\alpha}{p} \quad (0\le\alpha<p),$$

then

$$\operatorname{Re}\left\{\frac{I_{n,p}F_{\delta,p}(z)}{z^p}\right\} > \frac{\alpha}{p} + (1-\frac{\alpha}{p})\left\{{}_2F_1\left(1,1;p+\delta+1;\frac{1}{2}\right)-1\right\}.$$

The result is the best possible.

Putting $n = 0$ in Corollary 3.10, we have the following corollary which in turn improves the corresponding result of Fukui et al. [7] for $p = 1$.

Corollary 3.11. *If $f \in A(p)$ satisfies*

$$\operatorname{Re}\left\{\frac{f'(z)}{z^{p-1}}\right\} > \alpha \quad (0 \leq \alpha < p),$$

then

$$\operatorname{Re}\left\{\frac{F'_{\delta,p}(z)}{z^{p-1}}\right\} > \alpha + (p-\alpha)\left\{{}_2F_1\left(1,1;p+\delta+1;\frac{1}{2}\right)-1\right\}.$$

The result is best possible.

Theorem 3.12. *For $f \in A(p)$, we have*

$$f \in V^0_{p,q,s}(\alpha_1;\beta_1;A,B) \Leftrightarrow F_{\beta_1-p-1,p} \in V^1_{p,q,s}(\alpha_1;\beta_1;A,B).$$

Proof. Using identity (3.8) and

$$z(H_{p,q,s}(\alpha_1;\beta_1)F_{\delta,p}(z))' = (\beta_1-1)H_{p,q,s}(\alpha_1;\beta_1-1)F_{\delta,p}(z) + (p+1-\beta_1)H_{p,q,s}(\alpha_1;\beta_1)F_{\delta,p}(z),$$

for $\delta = \beta_1 - p - 1$, we deduce that

$$H_{p,q,s}(\alpha_1;\beta_1)f(z) = H_{p,q,s}(\alpha_1;\beta_1-1)F_{\beta_1-p-1,p}(z)$$

and the assertion of Theorem 3.12 follows by using the definition of the class $V^\lambda_{p,q,s}(\alpha_1;\beta_1;A,B)$. □

Theorem 3.13. *If the function $f(z)$ given by (1.1) belongs to the class $V^\lambda_{p,q,s}(\alpha_1;\beta_1;A,B)$, then*

$$|a_{k+p}| \leq \frac{(A-B)(\beta_1-1)_{k+1}(\beta_2)_k...(\beta_s)_k(1)_k}{(\beta_1-1+\lambda k)(\alpha_1)_k...(\alpha_q)_k} \quad (k \geq 1). \tag{3.9}$$

The estimate is sharp.

Proof. Since$f(z) \in V^\lambda_{p,q,s}(\alpha_1;\beta_1;A,B)$, then

$$(1-\lambda)\frac{H_{p,q,s}(\alpha_1;\beta_1)f(z)}{z^p} + \lambda\frac{H_{p,q,s}(\alpha_1;\beta_1-1)f(z)}{z^p} = p(z), \tag{3.10}$$

where $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k \in P(A, B)$. Substituting the power series expansion of $H_{p,q,s}(\alpha_1;\beta_1)f(z)$, $H_{p,q,s}(\alpha_1;\beta_1-1)f(z)$ and $p(z)$ in (3.10) and equating the coefficients of z^k on the both sides of the resulting equation, we obtain

$$\frac{(\beta_1-1+\lambda k)(\alpha_1)_k...(\alpha_q)_k}{(\beta_1-1)_{k+1}(\beta_2)_k...(\beta_s)_k k!}a_{k+p} = p_k \quad (k \geq 1). \tag{3.11}$$

Using the well-known [1] coefficient estimates

$$|p_k| \leq A - B \qquad (k \geq 1),$$

in (3.11), we get the required result (3.9). The estimate in (3.9) is sharp for the functions $f_k(z)$ defined by

$$(1-\lambda)\frac{H_{p,q,s}(\alpha_1;\beta_1)f_k(z)}{z^p} + \lambda\frac{H_{p,q,s}(\alpha_1;\beta_1-1)f_k(z)}{z^p} = \frac{1+Az^k}{1+Bz^k} \qquad (k \geq 1).$$

Clearly, $f_k(z) \in V^{\lambda}_{p,q,s}(\alpha_1;\beta_1;A,B)$ for each $k \geq 1$. It is easy to see that the functions $f_k(z)$ have the series expansion

$$f_k(z) = z^p + \frac{(A-B)(\beta_1-1)_{k+1}(\beta_2)_k...(\beta_s)_k\ (1)_k}{(\beta_1-1+\lambda k)(\alpha_1)_k...(\alpha_q)_k}z^{k+p} + ... ,$$

show that the estimates in (3.9) are sharp. □

Taking $A = \lambda = s = \alpha_2 = 1, q = 2$, $\alpha_1 = p+1$ and $\beta_1 = p+2-\mu$ $(-\infty < \mu < p+1)$, $B = \frac{1}{M} - 1$ $(M > \frac{1}{2})$ in Theorem 3.13, we have the following corollary.

Corollary 3.14. *If the function $f(z)$ given by (1.1) belongs to the class $V_p(\mu, M)$, then*

$$|a_{k+p}| \leq \frac{(2M-1)(p+1-\mu)_k}{M(p+1)_k} \quad (k \geq 1).$$

The estimate is sharp.

Theorem 3.15. *Let f given by (1.1) belongs to the class $V^{\lambda}_{p,q,s}(\alpha_1;\beta_1;A,B)$ and ζ be any complex number. Then*

$$\left|a_{p+2} - \zeta a_{p+1}^2\right| \leq \frac{2(A-B)(\beta_1-1)_3(\beta_2)_2...(\beta_s)_2}{(\beta_1-1+2\lambda)(\alpha_1)_2...(\alpha_q)_2}.$$

$$.\max\left\{1, \left|B + \zeta\frac{(\beta_1-1)_2\ \beta_2...\beta_s(A-B)(\beta_1-1+2\lambda)(\alpha_1+1)...(\alpha_q+1)}{2\alpha_1...\alpha_q\ (\beta_1+1)...(\beta_s+1)(\beta_1-1+\lambda)^2}\right|\right\}. \tag{3.12}$$

The estimate in (3.12) is sharp.

Proof. From (1.8), we deduce that

$$(1-\lambda)\frac{H_{p,q,s}(\alpha_1;\beta_1)f(z)}{z^p}+\lambda\frac{H_{p,q,s}(\alpha_1;\beta_1-1)f(z)}{z^p}-1$$

$$=\left[A-B\left\{(1-\lambda)\frac{H_{p,q,s}(\alpha_1;\beta_1)f(z)}{z^p}+\lambda\frac{H_{p,q,s}(\alpha_1;\beta_1-1)f(z)}{z^p}\right\}\right]\omega(z), \quad (3.13)$$

where $\omega(z)=\sum\limits_{k=1}^{\infty}\omega_k z^k$ is analytic in U and satisfies $|\omega(z)|\le|z|$ for $z\in U$. Substituting the power series expansion of $H_{p,q,s}(\alpha_1;\beta_1)f(z)$, $H_{p,q,s}(\alpha_1;\beta_1-1)f(z)$ and $\omega(z)$ in (3.13), and equating the coefficients of z and z^2, we get

$$a_{p+1}=\frac{(A-B)(\beta_1-1)_2\ \beta_2...\beta_s}{(\beta_1-1+\lambda)\ \alpha_1...\alpha_q}\omega_1, \quad (3.14)$$

$$a_{p+2}=\frac{2(A-B)(\beta_1-1)_3\ (\beta_2)_2...(\beta_s)_2}{(\beta_1-1+2\lambda)\ (\alpha_1)_2...(\alpha_q)_2}(\omega_2-B\omega_1^2). \quad (3.15)$$

From (3.14) and (3.15), we have

$$\left|a_{p+2}-\zeta a_{p+1}^2\right|=\frac{2(A-B)(\beta_1-1)_3(\beta_2)_2...(\beta_s)_2}{(\beta_1-1+2\lambda)(\alpha_1)_2...(\alpha_q)_2}\left|\omega_2-\upsilon\omega_1^2\right|, \quad (3.16)$$

where

$$\upsilon=B+\zeta\frac{(\beta_1-1)_2\ \beta_2...\beta_s(A-B)(\beta_1-1+2\lambda)(\alpha_1+1)...(\alpha_q+1)}{2\alpha_1...\alpha_q\ (\beta_1+1)...(\beta_s+1)(\beta_1-1+\lambda)^2}.$$

Now, by using (2.3) in (3.16), we get the required result. The result (3.12) is sharp as the estimate (2.3) is sharp. □

Taking $A=\lambda=s=\alpha_2=1, q=2$, $\alpha_1=p+1$ and $\beta_1=p+2-\mu$ $(-\infty<\mu<p+1)$, $B=\frac{1}{M}-1$ $(M>\frac{1}{2})$ in Theorem 3.15, we have the following corollary.

Corollary 3.16. *Let f, given by (1.1), belongs to the class $V_p(\mu,M)$, and ζ be any complex number. Then*

$$\left|a_{p+2}-\zeta a_{p+1}^2\right|\le\frac{(2M-1)(p+1-\mu)_2}{M(p+1)_2}.$$

$$.\max\left\{1,\left|\frac{1-M}{M}+\zeta\frac{(2M-1)(p+2)(p+1-\mu)}{M(p+1)(p+2-\mu)}\right|\right\}.$$

The estimate is sharp.

Theorem 3.17. *Let $f \in V^{\lambda}_{p,q,s}(\alpha_1;\beta_1;A,B)$ and $g \in A(p)$ with $\mathrm{Re}\{\frac{g(z)}{z^p}\} > \frac{1}{2}$ for $z \in U$. Then the function $h = f * g$ belongs to the class $V^{\lambda}_{p,q,s}(\alpha_1;\beta_1;A,B)$.*

Proof. We can write

$$(1-\lambda)\frac{H_{p,q,s}(\alpha_1;\beta_1)h(z)}{z^p} + \lambda\frac{H_{p,q,s}(\alpha_1;\beta_1-1)h(z)}{z^p}$$

$$= \left\{(1-\lambda)\frac{H_{p,q,s}(\alpha_1;\beta_1)f(z)}{z^p} + \lambda\frac{H_{p,q,s}(\alpha_1;\beta_1-1)f(z)}{z^p}\right\} * \frac{g(z)}{z^p}. \tag{3.17}$$

Since $\mathrm{Re}\{\frac{g(z)}{z^p}\} > \frac{1}{2}$ in U and $f \in V^{\lambda}_{p,q,s}(\alpha_1;\beta_1;A,B)$, it follows from (3.17) and Lemma 2.2 that $h \in V^{\lambda}_{p,q,s}(\alpha_1,\beta_1;A,B)$. The proof is completed. □

Corollary 3.18. *Let $f \in V^{\lambda}_{p,q,s}(\alpha_1,\beta_1;A,B)$ and $g \in A(p)$ satisfy*

$$\mathrm{Re}\left\{(1-\mu)\frac{g(z)}{z^p} + \mu\frac{g'(z)}{pz^{p-1}}\right\} > \frac{3-2\,{}_2F_1(1,1;\frac{p}{\mu}+1;1/2)}{2\left\{2-{}_2F_1(1,1;\frac{p}{\mu}+1;1/2)\right\}} \quad (\mu > 0; z \in U). \tag{3.18}$$

Then $f * g \in V^{\lambda}_{p,q,s}(\alpha_1;\beta_1;A,B)$.

Proof. From Theorem 3.1 (for $q = 2$, $s = 1$, $\alpha_1 = \beta_1 = p+1$, $\alpha_2 = 1$, $\lambda = \mu > 0$, $A = \frac{{}_2F_1(1,1;\frac{p}{\mu}+1;1/2)-1}{2-{}_2F_1(1,1;\frac{p}{\mu}+1;1/2)}$, and $B = -1$), condition (3.18) implies

$$\mathrm{Re}\left\{\frac{g(z)}{z^p}\right\} > \frac{1}{2}.$$

Using this, it follows from Theorem 3.17 that $f * g \in V^{\lambda}_{p,q,s}(\alpha_1;\beta_1;A,B)$. □

Theorem 3.19. *If each of the functions $f(z)$ given by (1.1) and $g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p}z^{k+p}$ belongs to the class $V^{\lambda}_{p,q,s}(\alpha_1;\beta_1;A,B)$, then so does the function $h(z) = (1-\lambda)H_{p,q,s}(\alpha_1;\beta_1)(f*g)(z) + \lambda H_{p,q,s}(\alpha_1;\beta_1-1)(f*g)(z)$.*

Proof. Since $f \in V^{\lambda}_{p,q,s}(\alpha_1;\beta_1;A,B)$, it follow by (3.13) that

$$\left|(1-\lambda)\frac{H_{p,q,s}(\alpha_1;\beta_1)f(z)}{z^p} + \lambda\frac{H_{p,q,s}(\alpha_1;\beta_1-1)f(z)}{z^p} - 1\right|$$

$$< \left|A - B\left\{(1-\lambda)\frac{H_{p,q,s}(\alpha_1;\beta_1)f(z)}{z^p} + \lambda\frac{H_{p,q,s}(\alpha_1;\beta_1-1)f(z)}{z^p}\right\}\right|,$$

which is equivalent to

$$\left|(1-\lambda)\frac{H_{p,q,s}(\alpha_1;\beta_1)f(z)}{z^p} + \lambda\frac{H_{p,q,s}(\alpha_1;\beta_1-1)f(z)}{z^p} - \xi\right| < \eta \quad (z \in U), \tag{3.19}$$

where

$$\xi = \frac{1-AB}{1-B^2} \quad \text{and} \quad \eta = \frac{A-B}{1-B^2}.$$

It is known [17] that if $G(z) = \sum\limits_{k=0}^{\infty} g_k z^k$ is analytic in U and $|G(z)| \leq E$, then

$$\sum_{k=0}^{\infty} |g_k|^2 \leq E^2. \tag{3.20}$$

Applying (3.20) to (3.19), we get

$$(1-\xi)^2 + \sum_{k=1}^{\infty} \left\{ \frac{(\beta_1 - 1 + \lambda k)(\alpha_1)_k ... (\alpha_q)_k}{(\beta_1 - 1)_{k+1}(\beta_2)_k ... (\beta_s)_k (1)_k} \right\}^2 |a_{k+p}|^2 < \eta^2,$$

that is, that

$$\sum_{k=1}^{\infty} \left\{ \frac{(\beta_1 - 1 + \lambda k)(\alpha_1)_k ... (\alpha_q)_k}{(\beta_1 - 1)_{k+1}(\beta_2)_k ... (\beta_s)_k (1)_k} \right\}^2 |a_{k+p}|^2 < \frac{(A-B)^2}{1-B^2}. \tag{3.21}$$

Similarly,

$$\sum_{k=1}^{\infty} \left\{ \frac{(\beta_1 - 1 + \lambda k)(\alpha_1)_k ... (\alpha_q)_k}{(\beta_1 - 1)_{k+1}(\beta_2)_k ... (\beta_s)_k (1)_k} \right\}^2 |b_{k+p}|^2 < \frac{(A-B)^2}{1-B^2}. \tag{3.22}$$

Now, for $|z| = r < 1$, by applying Cauchy-Schwarz inequality, we find that

$$\left| (1-\lambda) \frac{H_{p,q,s}(\alpha_1; \beta_1) h(z)}{z^p} + \lambda \frac{H_{p,q,s}(\alpha_1; \beta_1 - 1) h(z)}{z^p} - \xi \right|^2$$

$$= \left| (1-\xi) + \sum_{k=1}^{\infty} \left\{ \frac{(\beta_1 - 1 + \lambda k)(\alpha_1)_k ... (\alpha_q)_k}{(\beta_1 - 1)_{k+1}(\beta_2)_k ... (\beta_s)_k (1)_k} \right\}^2 a_{k+p} b_{k+p} z^k \right|^2$$

$$\leq (1-\xi)^2 + 2(1-\xi) \sum_{k=1}^{\infty} \left\{ \frac{(\beta_1 - 1 + \lambda k)(\alpha_1)_k ... (\alpha_q)_k}{(\beta_1 - 1)_{k+1}(\beta_2)_k ... (\beta_s)_k (1)_k} \right\}^2 |a_{k+p}| \, |b_{k+p}| \, r^k$$

$$+ \left| \sum_{k=1}^{\infty} \left\{ \frac{(\beta_1 - 1 + \lambda k)(\alpha_1)_k ... (\alpha_q)_k}{(\beta_1 - 1)_{k+1}(\beta_2)_k ... (\beta_s)_k (1)_k} \right\}^2 a_{k+p} b_{k+p} z^k \right|^2$$

$$\leq (1-\xi)^2 + 2(1-\xi) \left[\sum_{k=1}^{\infty} \left\{ \frac{(\beta_1 - 1 + \lambda k)(\alpha_1)_k ... (\alpha_q)_k}{(\beta_1 - 1)_{k+1}(\beta_2)_k ... (\beta_s)_k (1)_k} \right\}^2 |a_{k+p}|^2 \, r^k \right]^{1/2} .$$

$$\cdot \left[\sum_{k=1}^{\infty} \left\{ \frac{(\beta_1 - 1 + \lambda k)(\alpha_1)_k ... (\alpha_q)_k}{(\beta_1 - 1)_{k+1}(\beta_2)_k ... (\beta_s)_k (1)_k} \right\}^2 |b_{k+p}|^2 \, r^k \right]^{1/2}$$

$$+ \left[\sum_{k=1}^{\infty} \left\{ \frac{(\beta_1 - 1 + \lambda k)(\alpha_1)_k ... (\alpha_q)_k}{(\beta_1 - 1)_{k+1}(\beta_2)_k ... (\beta_s)_k (1)_k} \right\}^2 |a_{k+p}|^2 \, r^k \right].$$

$$\cdot\left[\sum_{k=1}^{\infty}\left\{\frac{(\beta_1-1+\lambda k)(\alpha_1)_k...(\alpha_q)_k}{(\beta_1-1)_{k+1}(\beta_2)_k...(\beta_s)_k(1)_k}\right\}^2|b_{k+p}|^2\,r^k\right]$$

$$\leq(1-\xi)^2+2(1-\xi)\left[\sum_{k=1}^{\infty}\left\{\frac{(\beta_1-1+\lambda k)(\alpha_1)_k...(\alpha_q)_k}{(\beta_1-1)_{k+1}(\beta_2)_k...(\beta_s)_k(1)_k}\right\}^2|a_{k+p}|^2\right]^{1/2}\cdot$$

$$\cdot\left[\sum_{k=1}^{\infty}\left\{\frac{(\beta_1-1+\lambda k)(\alpha_1)_k...(\alpha_q)_k}{(\beta_1-1)_{k+1}(\beta_2)_k...(\beta_s)_k(1)_k}\right\}^2|b_{k+p}|^2\right]^{1/2}$$

$$+\left[\sum_{k=1}^{\infty}\left\{\frac{(\beta_1-1+\lambda k)(\alpha_1)_k...(\alpha_q)_k}{(\beta_1-1)_{k+1}(\beta_2)_k...(\beta_s)_k(1)_k}\right\}^2|a_{k+p}|^2\right]\cdot$$

$$\cdot\left[\sum_{k=1}^{\infty}\left\{\frac{(\beta_1-1+\lambda k)(\alpha_1)_k...(\alpha_q)_k}{(\beta_1-1)_{k+1}(\beta_2)_k...(\beta_s)_k(1)_k}\right\}^2|b_{k+p}|^2\right]$$

$$\leq(1-\xi)^2+2(1-\xi)\frac{(A-B)^2}{1-B^2}+\frac{(A-B)^4}{(1-B^2)^2}$$

$$=\left\{\frac{B(A-B)}{1-B^2}\right\}^2+2\frac{B(A-B)^3}{(1-B^2)^2}+\frac{(A-B)^4}{(1-B^2)^2}<\eta^2,$$

by using (3.22) and (3.23).

Thus, again with the aid of (3.20), we have $h\in V^{\lambda}_{p,q,s}(\alpha_1,\beta_1;\,A,B)$. □

Theorem 3.20. *Let* $f\in V^{\lambda}_{p,q,s}(\alpha_1;\beta_1;A,B)$ *and* $S_n(z)=z^p+\sum\limits_{k=1}^{n-1}a_{k+p}z^{k+p}$ $(n\geq 2)$. *Then for* $z\in U$, *we have*

$$\operatorname{Re}\left[\frac{\int_0^z t^{-p}(H_{p,q,s}(\alpha_1;\beta_1)S_n(t))dt}{z}\right]>\eta(\lambda,\beta_1,A,B),$$

where $\eta(\lambda,\beta_1,A,B)$ *is defined as in Theorem 3.1.*

Proof. Singh and Singh [25] proved that

$$\operatorname{Re}\left\{1+\sum_{k=1}^{n-1}\frac{z^k}{k+1}\right\}>\frac{1}{2}\quad(z\in U).\tag{3.23}$$

Writing

$$\frac{\int_0^z t^{-p}(H_{p,q,s}(\alpha_1;\beta_1)S_n(t))dt}{z}=\frac{H_{p,q,s}(\alpha_1;\beta_1)f(z)}{z^p}*\left\{1+\sum_{k=1}^{n-1}\frac{z^k}{k+1}\right\}$$

and making use of (3.23), Theorem 3.1 and Lemma 2.2, the assertion of the theorem follows at once. □

Remark 3.21. *By taking* $q = 2$, $s = 1$, $\alpha_1 = a$ $(a > 0)$, $\alpha_2 = 1$ *and* $\beta_1 = c$ $(c > 1;$ $c \notin \mathbb{Z}_0^-)$ *in our results, we obtain the results obtained by Patel and Sahoo [21].*

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