STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LV**, Number 3, September 2010

A NOTE ON DIFFERENTIAL SUPERORDINATIONS USING A MULTIPLIER TRANSFORMATION AND RUSCHEWEYH DERIVATIVE

ALINA ALB LUPAŞ

Dedicated to Professor Grigore Ştefan Sălăgean on his 60^{th} *birthday*

Abstract. In the present paper we define a new operator, by means of convolution product between Ruscheweyh operator and the multiplier transformation $I(m, \lambda, l)$. For functions f belonging to the class $\mathcal{A}_n$ we define the differential operator $IR^m_{\lambda,l} : \mathcal{A}_n \to \mathcal{A}_n$,

$$IR^m_{\lambda,l} f(z) := (I(m, \lambda, l) * R^m) f(z),$$

where $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \ldots, z \in U\}$ is the class of normalized analytic functions. We study some differential superordinations regarding the operator $IR^m_{\lambda,l}$.

1. Introduction

Denote by U the unit disc of the complex plane $U = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U.

Let

$$\mathcal{A}(p, n) = \{f \in \mathcal{H}(U) : f(z) = z^p + \sum_{j=p+n}^{\infty} a_j z^j, \quad z \in U\},$$

with $\mathcal{A}(1, n) = \mathcal{A}_n$ and

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(U), f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \ldots, z \in U\}$$

for $a \in \mathbb{C}$ and $p, n \in \mathbb{N}$.

Received by the editors: 26.04.2010.

2000 *Mathematics Subject Classification.* 30C45, 30A20, 34A40.

Key words and phrases. Differential superordination, convex function, best subordinant, differential operator.

If f and g are analytic functions in U, we say that f is superordinate to g, written $g \prec f$, if there is a function w analytic in U, with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $g(z) = f(w(z))$ for all $z \in U$. If f is univalent, then $g \prec f$ if and only if $f(0) = g(0)$ and $g(U) \subseteq f(U)$.

Let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ and h analytic in U. If p and $\psi\left(p\left(z\right), zp'\left(z\right); z\right)$ are univalent in U and satisfies the (first-order) differential superordination

$$h(z) \prec \psi(p(z), zp'(z); z), \qquad z \in U, \tag{1.1}$$

then p is called a solution of the differential superordination. The analytic function q is called a subordinant of the solutions of the differential superordination, or more simply a subordinant, if $q \prec p$ for all p satisfying (1.1).

An univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (1.1) is said to be the best subordinant of (1.1). The best subordinant is unique up to a rotation of U.

Definition 1.1. [7] For $f \in \mathcal{A}(p,n)$, $p, n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, $\lambda, l \geq 0$, the operator $I_p\left(m, \lambda, l\right) f(z)$ is defined by the following infinite series

$$I_p\left(m, \lambda, l\right) f(z) := z^p + \sum_{j=p+n}^{\infty} \left(\frac{p + \lambda\left(j-1\right) + l}{p+l}\right)^m a_j z^j.$$

Remark 1.2. It follows from the above definition that

$$I_p\left(0, \lambda, l\right) f(z) = f(z),$$

$$\left(p+l\right) I_p\left(m+1, \lambda, l\right) f(z) = \left[p(1-\lambda) + l\right] I_p\left(m, \lambda, l\right) f(z) + \lambda z \left(I_p\left(m, \lambda, l\right) f(z)\right)',$$

$z \in U$.

Remark 1.3. If $p = 1$, we have $\mathcal{A}(1, n) = \mathcal{A}_n$, $I_1\left(m, \lambda, l\right) f(z) = I\left(m, \lambda, l\right)$ and

$$\left(l+1\right) I\left(m+1, \lambda, l\right) f(z) = \left[l + 1 - \lambda\right] I\left(m, \lambda, l\right) f(z) + \lambda z \left(I\left(m, \lambda, l\right) f(z)\right)',$$

$z \in U$.

Remark 1.4. If $f \in \mathcal{A}_n$, $f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j$, then

$$I\left(m, \lambda, l\right) f\left(z\right) = z + \sum_{j=n+1}^{\infty} \left(\frac{1 + \lambda\left(j-1\right) + l}{l+1}\right)^m a_j z^j, \ z \in U.$$

Remark 1.5. For $l = 0$, $\lambda \geq 0$, the operator $D_\lambda^m = I(m, \lambda, 0)$ was introduced and studied by Al-Oboudi [6], which is reduced to the Sălăgean differential operator $S^m = I(m, 1, 0)$ [10] for $\lambda = 1$.

Definition 1.6. (Ruscheweyh [9]) For $f \in \mathcal{A}_n$, $m, n \in \mathbb{N}$, the operator R^m is defined by $R^m : \mathcal{A}_n \to \mathcal{A}_n$,

$$\begin{aligned} R^0 f(z) &= f(z) \\ R^1 f(z) &= z f'(z) \\ &\ldots \\ (m+1) R^{m+1} f(z) &= z\left(R^m f(z)\right)' + m R^m f(z), \quad z \in U. \end{aligned}$$

Remark 1.7. If $f \in \mathcal{A}_n$, $f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j$, then

$$R^m f(z) = z + \sum_{j=n+1}^{\infty} C_{m+j-1}^m a_j z^j, \; z \in U.$$

Definition 1.8. [8] We denote by Q the set of functions that are analytic and injective on $\overline{U} \backslash E(f)$, where $E(f) = \{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\}$, and $f'(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(f)$. The subclass of Q for which $f(0) = a$ is denoted by $Q(a)$.

We will use the following lemmas.

Lemma 1.9. *(Miller and Mocanu* [8]*) Let h be a convex function with $h(0) = a$, and let $\gamma \in \mathbb{C} \backslash \{0\}$ be a complex number with Re $\gamma \geq 0$. If $p \in \mathcal{H}[a, n] \cap Q$, $p(z) + \frac{1}{\gamma} z p'(z)$ is univalent in U and*

$$h(z) \prec p(z) + \frac{1}{\gamma} z p'(z), \qquad z \in U,$$

then

$$q(z) \prec p(z), \qquad z \in U,$$

where

$$q(z) = \frac{\gamma}{n z^{\gamma/n}} \int_0^z h(t) t^{\gamma/n - 1} dt, \; z \in U.$$

The function q is convex and is the best subordinant.

Lemma 1.10. *(Miller and Mocanu [8]) Let q be a convex function in U and let*

$$h(z) = q(z) + \frac{1}{\gamma} z q'(z), \ z \in U,$$

where $Re\ \gamma \geq 0$.

If $p \in \mathcal{H}[a,n] \cap Q$, $p(z) + \frac{1}{\gamma} z p'(z)$ is univalent in U and

$$q(z) + \frac{1}{\gamma} z q'(z) \prec p(z) + \frac{1}{\gamma} z p'(z), \qquad z \in U,$$

then

$$q(z) \prec p(z), \qquad z \in U,$$

where

$$q(z) = \frac{\gamma}{n z^{\gamma/n}} \int_0^z h(t) t^{\gamma/n - 1} dt, \ z \in U.$$

The function q is the best subordinant.

2. **Main results**

Definition 2.1. ([4]) Let $m, n, \lambda, l \in \mathbb{N}$. Denote by $IR^m_{\lambda,l}$ the operator given by the Hadamard product (the convolution product) of the operator $I(m, \lambda, l)$ and the Ruscheweyh operator R^m, $IR^m_{\lambda,l} : \mathcal{A}_n \to \mathcal{A}_n$,

$$IR^m_{\lambda,l} f(z) = (I(m, \lambda, l) * R^m) f(z).$$

Remark 2.2. If $f \in \mathcal{A}_n$, $f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j$, then

$$IR^m_{\lambda,l} f(z) = z + \sum_{j=n+1}^{\infty} \left(\frac{1 + \lambda(j-1) + l}{l+1} \right)^m C^m_{m+j-1} a_j^2 z^j, \ z \in U.$$

Remark 2.3. For $l = 0$, $\lambda \geq 0$, we obtain the Hadamard product DR^m_λ [1] of the generalized Sălăgean operator D^m_λ and Ruscheweyh operator R^m.

For $l = 0$ and $\lambda = 1$, we obtain the Hadamard product SR^m [5] of the Sălăgean operator S^m and Ruscheweyh operator R^m.

Theorem 2.4. *Let h be a convex function, $h(0) = 1$. Let $m, n, \lambda, l \in \mathbb{N}$, $f \in \mathcal{A}_n$ and suppose that*

$$\frac{l+1}{[\lambda(l-m+2) - (l+1)] z} \cdot \left[(m+1) IR^{m+1}_{\lambda,l} f(z) - (m-2) IR^m_{\lambda,l} f(z) \right]$$

$$+\left(1-\frac{l+1}{\lambda\left(l-m+2\right)-\left(l+1\right)}\right)-\frac{2\left(l+1\right)\left(m-1\right)-2\lambda m}{\lambda\left(l-m+2\right)-\left(l+1\right)}\int_0^z\frac{IR^m_{\lambda,l}f\left(t\right)-t}{t^2}dt$$

is univalent and $\left(IR^m_{\lambda,l}f\left(z\right)\right)'\in\mathcal{H}\left[1,n\right]\cap Q$. *If*

$$h(z)\prec\frac{l+1}{\left[\lambda\left(l-m+2\right)-\left(l+1\right)\right]z}\left[\left(m+1\right)IR^{m+1}_{\lambda,l}f\left(z\right)-\left(m-2\right)IR^m_{\lambda,l}f\left(z\right)\right] \quad (2.1)$$

$$+\left(1-\frac{l+1}{\lambda\left(l-m+2\right)-\left(l+1\right)}\right)-\frac{2\left(l+1\right)\left(m-1\right)-2\lambda m}{\lambda\left(l-m+2\right)-\left(l+1\right)}\int_0^z\frac{IR^m_{\lambda,l}f\left(t\right)-t}{t^2}dt,$$

$z\in U$, *then*

$$q(z)\prec\left(IR^m_{\lambda,l}f\left(z\right)\right)',\qquad z\in U,$$

where

$$q(z)=\frac{\lambda\left(l-m+2\right)-\left(l+1\right)}{\lambda\left(l+1\right)nz^{\frac{\lambda(l-m+2)-(l+1)}{\lambda(l+1)n}}}\int_0^z h\left(t\right)t^{\frac{\lambda(l-m-nl-n+2)-(l+1)}{\lambda(l+1)n}}dt.$$

The function q is convex and it is the best subordinant.

Proof. With notation

$$p\left(z\right)=\left(IR^m_{\lambda,l}f\left(z\right)\right)'=1+\sum_{j=n+1}^{\infty}\left(\frac{1+\lambda\left(j-1\right)+l}{l+1}\right)^m C^m_{m+j-1}ja_j^2z^{j-1}\text{ and }p\left(0\right)=1,$$

we obtain for $f(z)=z+\sum_{j=n+1}^{\infty}a_jz^j$,

$$p\left(z\right)+zp'\left(z\right)=1+\sum_{j=n+1}^{\infty}\left(\frac{1+\lambda\left(j-1\right)+l}{l+1}\right)^m C^m_{m+j-1}ja_j^2z^{j-1}$$

$$+\sum_{j=n+1}^{\infty}\left(\frac{1+\lambda\left(j-1\right)+l}{l+1}\right)^m C^m_{m+j-1}j\left(j-1\right)a_j^2z^{j-1}$$

$$=\frac{1}{z}\left(\frac{m+1}{\lambda}IR^{m+1}_{\lambda,l}f\left(z\right)-\frac{m-2}{\lambda}IR^m_{\lambda,l}f\left(z\right)\right)+\frac{\lambda\left(m-1\right)-\left(l+1\right)}{\lambda\left(l+1\right)}\left(IR^m_{\lambda,l}f\left(z\right)\right)'$$

$$+\left(1-\frac{m-1}{l+1}-\frac{2}{\lambda}\right)-\frac{2\left(l+1\right)\left(m-1\right)-2\lambda m}{\lambda\left(l+1\right)}\int_0^z\frac{IR^m_{\lambda,l}f\left(t\right)-t}{t^2}dt.$$

Therefore

$$p\left(z\right)+\frac{\lambda\left(l+1\right)}{\lambda\left(l-m+2\right)-\left(l+1\right)}zp'\left(z\right)$$

$$=\frac{l+1}{\left[\lambda\left(l-m+2\right)-\left(l+1\right)\right]z}\left[\left(m+1\right)IR^{m+1}_{\lambda,l}f\left(z\right)-\left(m-2\right)IR^m_{\lambda,l}f\left(z\right)\right]$$

$$+\left(1-\frac{l+1}{\lambda\left(l-m+2\right)-\left(l+1\right)}\right)-\frac{2\left(l+1\right)\left(m-1\right)-2\lambda m}{\lambda\left(l-m+2\right)-\left(l+1\right)}\int_0^z\frac{IR^m_{\lambda,l}f\left(t\right)-t}{t^2}dt.$$

Then (2.1) becomes

$$h(z) \prec p(z) + \frac{\lambda(l+1)}{\lambda(l-m+2)-(l+1)} zp'(z), \quad z \in U.$$

By using Lemma 1.9 for $\gamma = 1 - \frac{m-1}{l+1} - \frac{1}{\lambda}$, we have

$$q(z) \prec p(z), \quad z \in U, \quad \text{i.e.} \quad q(z) \prec \left(IR^m_{\lambda,l} f(z)\right)', \quad z \in U,$$

where

$$q(z) = \frac{\lambda(l-m+2)-(l+1)}{\lambda(l+1) nz^{\frac{\lambda(l-m+2)-(l+1)}{\lambda(l+1)n}}} \int_0^z h(t)\, t^{\frac{\lambda(l-m-nl-n+2)-(l+1)}{\lambda(l+1)n}} dt.$$

The function q is convex and it is the best subordinant. □

Corollary 2.5. [3] *Let h be a convex function, $h(0) = 1$. Let $\lambda \geq 0$, $m, n \in \mathbb{N}$, $f \in \mathcal{A}_n$ and suppose that*

$$\frac{m+1}{(m\lambda+1)z} DR^{m+1}_\lambda f(z) - \frac{m(1-\lambda)}{(m\lambda+1)z} DR^m_\lambda f(z)$$

is univalent and $(DR^m_\lambda f(z))' \in \mathcal{H}[1,n] \cap Q$. If

$$h(z) \prec \frac{m+1}{(m\lambda+1)z} DR^{m+1}_\lambda f(z) - \frac{m(1-\lambda)}{(m\lambda+1)z} DR^m_\lambda f(z), \quad z \in U, \tag{2.2}$$

then

$$q(z) \prec (DR^m_\lambda f(z))', \quad z \in U,$$

where

$$q(z) = \frac{m\lambda+1}{n\lambda z^{\frac{m\lambda+1}{n\lambda}}} \int_0^z h(t)\, t^{\frac{(m-n)\lambda+1}{n\lambda}} dt.$$

The function q is convex and it is the best subordinant.

Corollary 2.6. [2] *Let h be a convex function, $h(0) = 1$. Let $m, n \in \mathbb{N}$, $f \in \mathcal{A}_n$ and suppose that*

$$\frac{1}{z} SR^{m+1} f(z) + \frac{m}{m+1} z \left(SR^m f(z)\right)''$$

is univalent and $(SR^m f(z))' \in \mathcal{H}[1,n] \cap Q$. If

$$h(z) \prec \frac{1}{z} SR^{m+1} f(z) + \frac{m}{m+1} z \left(SR^m f(z)\right)'', \quad z \in U, \tag{2.3}$$

then

$$q(z) \prec (SR^m f(z))', \quad z \in U,$$

where

$$q(z)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt.$$

The function q is convex and it is the best subordinant.

Theorem 2.7. *Let q be convex in U and let h be defined by*

$$h\left(z\right)=q\left(z\right)+\frac{\lambda\left(l+1\right)}{\lambda\left(l-m+2\right)-\left(l+1\right)}zq'\left(z\right),$$

$m,n,\lambda,l\in\mathbb{N}$. *If $f\in\mathcal{A}_n$, suppose that*

$$\frac{l+1}{\left[\lambda\left(l-m+2\right)-\left(l+1\right)\right]z}\left[\left(m+1\right)IR_{\lambda,l}^{m+1}f\left(z\right)-\left(m-2\right)IR_{\lambda,l}^{m}f\left(z\right)\right]$$

$$+\left(1-\frac{l+1}{\lambda\left(l-m+2\right)-\left(l+1\right)}\right)-\frac{2\left(l+1\right)\left(m-1\right)-2\lambda m}{\lambda\left(l-m+2\right)-\left(l+1\right)}\int_0^z\frac{IR_{\lambda,l}^{m}f\left(t\right)-t}{t^2}dt$$

is univalent, $\left(IR_{\lambda,l}^{m}f\left(z\right)\right)'\in\mathcal{H}\left[1,n\right]\cap Q$ and satisfies the differential superordination

$$h(z)=q\left(z\right)+\frac{\lambda\left(l+1\right)}{\lambda\left(l-m+2\right)-\left(l+1\right)}zq'\left(z\right)\prec\frac{l+1}{\left[\lambda\left(l-m+2\right)-\left(l+1\right)\right]z}\cdot \qquad (2.4)$$

$$\left[\left(m+1\right)IR_{\lambda,l}^{m+1}f\left(z\right)-\left(m-2\right)IR_{\lambda,l}^{m}f\left(z\right)\right]+\left(1-\frac{l+1}{\lambda\left(l-m+2\right)-\left(l+1\right)}\right)$$

$$-\frac{2\left(l+1\right)\left(m-1\right)-2\lambda m}{\lambda\left(l-m+2\right)-\left(l+1\right)}\int_0^z\frac{IR_{\lambda,l}^{m}f\left(t\right)-t}{t^2}dt,\qquad z\in U,$$

then

$$q(z)\prec\left(IR_{\lambda,l}^{m}f\left(z\right)\right)',\qquad z\in U,$$

where

$$q(z)=\frac{\lambda\left(l-m+2\right)-\left(l+1\right)}{\lambda\left(l+1\right)nz^{\frac{\lambda\left(l-m+2\right)-\left(l+1\right)}{\lambda\left(l+1\right)n}}}\int_0^z h\left(t\right)t^{\frac{\lambda\left(l-m-nl-n+2\right)-\left(l+1\right)}{\lambda\left(l+1\right)n}}dt.$$

The function q is the best subordinant.

Proof. Let

$$p\left(z\right)=\left(IR_{\lambda,l}^{m}f\left(z\right)\right)'=1+\sum_{j=n+1}^{\infty}\left(\frac{1+\lambda\left(j-1\right)+l}{l+1}\right)^m C_{m+j-1}^{m}ja_j^2z^{j-1}.$$

Differentiating, we obtain

$$p\left(z\right)+zp'\left(z\right)=\frac{1}{z}\left(\frac{m+1}{\lambda}IR_{\lambda,l}^{m+1}f\left(z\right)-\frac{m-2}{\lambda}IR_{\lambda,l}^{m}f\left(z\right)\right)$$

$$+\frac{\lambda\left(m-1\right)-\left(l+1\right)}{\lambda\left(l+1\right)}\left(IR_{\lambda,l}^{m}f\left(z\right)\right)'+\left(1-\frac{m-1}{l+1}-\frac{2}{\lambda}\right)$$

$$-\frac{2(l+1)(m-1)-2\lambda m}{\lambda(l+1)}\int_0^z \frac{IR_{\lambda,l}^m f(t)-t}{t^2}dt$$

and

$$p(z)+\frac{\lambda(l+1)}{\lambda(l-m+2)-(l+1)}zp'(z)=$$

$$\frac{l+1}{[\lambda(l-m+2)-(l+1)]z}\left[(m+1)IR_{\lambda,l}^{m+1}f(z)-(m-2)IR_{\lambda,l}^m f(z)\right]$$

$$+\left(1-\frac{l+1}{\lambda(l-m+2)-(l+1)}\right)$$

$$-\frac{2(l+1)(m-1)-2\lambda m}{\lambda(l-m+2)-(l+1)}\int_0^z \frac{IR_{\lambda,l}^m f(t)-t}{t^2}dt,\ z\in U$$

and (2.4) becomes

$$q(z)+\frac{\lambda(l+1)}{\lambda(l-m+2)-(l+1)}zq'(z)\prec p(z)+\frac{\lambda(l+1)}{\lambda(l-m+2)-(l+1)}zp'(z),\quad z\in U.$$

Using Lemma 1.10 for $\gamma=1-\frac{m-1}{l+1}-\frac{1}{\lambda}$, we have $q(z)\prec p(z)$, $z\in U$, i.e.

$$q(z)=\frac{\lambda(l-m+2)-(l+1)}{\lambda(l+1)nz^{\frac{\lambda(l-m+2)-(l+1)}{\lambda(l+1)n}}}\int_0^z h(t)\,t^{\frac{\lambda(l-m-nl-n+2)-(l+1)}{\lambda(l+1)n}}dt$$

$$\prec\left(IR_{\lambda,l}^m f(z)\right)',\ z\in U,$$

and q is the best subordinant. □

Corollary 2.8. [3] *Let q be convex in U and let h be defined by*

$$h(z)=q(z)+\frac{\lambda}{m\lambda+1}zq'(z),$$

$\lambda\geq 0$, $m,n\in\mathbb{N}$. *If $f\in\mathcal{A}_n$, suppose that*

$$\frac{m+1}{(m\lambda+1)z}DR_\lambda^{m+1}f(z)-\frac{m(1-\lambda)}{(m\lambda+1)z}DR_\lambda^m f(z)$$

is univalent and $(DR_\lambda^m f(z))'\in\mathcal{H}[1,n]\cap Q$ and satisfies the differential superordination

$$h(z)=q(z)+\frac{\lambda}{m\lambda+1}zq'(z)\prec\frac{m+1}{(m\lambda+1)z}DR_\lambda^{m+1}f(z)-\frac{m(1-\lambda)}{(m\lambda+1)z}DR_\lambda^m f(z),\tag{2.5}$$

$z\in U$, then

$$q(z)\prec(DR_\lambda^m f(z))',\qquad z\in U,$$

where

$$q(z)=\frac{m\lambda+1}{n\lambda z^{\frac{m\lambda+1}{n\lambda}}}\int_0^z h(t)\,t^{\frac{(m-n)\lambda+1}{n\lambda}}dt.$$

The function q is the best subordinant.

Corollary 2.9. [2] *Let q be convex in U and let h be defined by $h(z)=q(z)+zq'(z)$. If $m,n\in\mathbb{N}$, $f\in\mathcal{A}_n$, suppose that $\frac{1}{z}SR^{m+1}f(z)+\frac{m}{m+1}z\left(SR^m f(z)\right)''$ is univalent, $\left(SR^m f(z)\right)'\in\mathcal{H}[1,n]\cap Q$ and satisfies the differential superordination*

$$h(z)=q(z)+zq'(z)\prec\frac{1}{z}SR^{m+1}f(z)+\frac{m}{m+1}z\left(SR^m f(z)\right)'',\qquad z\in U,\tag{2.6}$$

then

$$q(z)\prec\left(SR^m f(z)\right)',\qquad z\in U,$$

where

$$q(z)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt.$$

The function q is the best subordinant.

Theorem 2.10. *Let h be a convex function, $h(0)=1$. Let $m,n,\lambda,l\in\mathbb{N}$, $f\in\mathcal{A}_n$ and suppose that $\left(IR^m_{\lambda,l}f(z)\right)'$ is univalent and $\frac{IR^m_{\lambda,l}f(z)}{z}\in\mathcal{H}[1,n]\cap Q$. If*

$$h(z)\prec\left(IR^m_{\lambda,l}f(z)\right)',\qquad z\in U,\tag{2.7}$$

then

$$q(z)\prec\frac{IR^m_{\lambda,l}f(z)}{z},\qquad z\in U,$$

where

$$q(z)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt.$$

The function q is convex and it is the best subordinant.

Proof. Consider

$$p(z)=\frac{IR^m_{\lambda,l}f(z)}{z}=\frac{z+\sum_{j=n+1}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m C^m_{m+j-1}a_j^2z^j}{z}$$
$$=1+\sum_{j=n+1}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m C^m_{m+j-1}a_j^2z^{j-1}.$$

Evidently $p\in\mathcal{H}[1,n]$.

We have $p(z)+zp'(z)=\left(IR^m_{\lambda,l}f(z)\right)'$, $z\in U$.

Then (2.7) becomes

$$h(z) \prec p(z) + zp'(z), \quad z \in U.$$

By using Lemma 1.9 for $\gamma = 1$, we have

$$q(z) \prec p(z), \quad z \in U, \quad \text{i.e.} \quad q(z) \prec \frac{IR^m_{\lambda,l} f(z)}{z}, \quad z \in U,$$

where $q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt$. The function q is convex and it is the best subordinant. □

Corollary 2.11. [3] *Let h be a convex function, $h(0) = 1$. Let $\lambda \geq 0$, $m, n \in \mathbb{N}$, $f \in \mathcal{A}_n$ and suppose that $(DR^m_\lambda f(z))'$ is univalent and $\frac{DR^m_\lambda f(z)}{z} \in \mathcal{H}[1, n] \cap Q$. If*

$$h(z) \prec (DR^m_\lambda f(z))', \quad z \in U, \tag{2.8}$$

then

$$q(z) \prec \frac{DR^m_\lambda f(z)}{z}, \quad z \in U,$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt.$$

The function q is convex and it is the best subordinant.

Corollary 2.12. [2] *Let h be a convex function, $h(0) = 1$. Let $m, n \in \mathbb{N}$, $f \in \mathcal{A}_n$ and suppose that $(SR^m f(z))'$ is univalent and $\frac{SR^m f(z)}{z} \in \mathcal{H}[1, n] \cap Q$. If*

$$h(z) \prec (SR^m f(z))', \quad z \in U, \tag{2.9}$$

then

$$q(z) \prec \frac{SR^m f(z)}{z}, \quad z \in U,$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt.$$

The function q is convex and it is the best subordinant.

Theorem 2.13. *Let q be convex in U and let h be defined by $h(z) = q(z) + zq'(z)$. If $m, n, \lambda, l \in \mathbb{N}$, $f \in \mathcal{A}_n$, suppose that $\left(IR^m_{\lambda,l} f(z)\right)'$ is univalent, $\frac{IR^m_{\lambda,l} f(z)}{z} \in \mathcal{H}[1,n] \cap Q$ and satisfies the differential superordination*

$$h(z) = q(z) + zq'(z) \prec \left(IR^m_{\lambda,l} f(z)\right)', \qquad z \in U, \tag{2.10}$$

then

$$q(z) \prec \frac{IR^m_{\lambda,l} f(z)}{z}, \qquad z \in U,$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt.$$

The function q is the best subordinant.

Proof. Let

$$p(z) = \frac{IR^m_{\lambda,l} f(z)}{z} = \frac{z + \sum_{j=n+1}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m C^m_{m+j-1} a_j^2 z^j}{z}$$
$$= 1 + \sum_{j=n+1}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m C^m_{m+j-1} a_j^2 z^{j-1}.$$

Evidently $p \in \mathcal{H}[1,n]$.

Differentiating, we obtain $p(z) + zp'(z) = \left(IR^m_{\lambda,l} f(z)\right)'$, $z \in U$ and (2.10) becomes

$$q(z) + zq'(z) \prec p(z) + zp'(z), \quad z \in U.$$

Using Lemma 1.10 for $\gamma = 1$, we have

$$q(z) \prec p(z), z \in U, \quad \text{i.e.} \quad q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt \prec \frac{IR^m_{\lambda,l} f(z)}{z}, z \in U,$$

and q is the best subordinant. □

Corollary 2.14. [3] *Let q be convex in U and let h be defined by $h(z) = q(z) + zq'(z)$. If $\lambda \geq 0$, $m, n \in \mathbb{N}$, $f \in \mathcal{A}_n$, suppose that $(DR^m_\lambda f(z))'$ is univalent,*

$$\frac{DR^m_\lambda f(z)}{z} \in \mathcal{H}[1,n] \cap Q$$

and satisfies the differential superordination

$$h(z) = q(z) + zq'(z) \prec (DR^m_\lambda f(z))', \qquad z \in U, \tag{2.11}$$

then

$$q(z) \prec \frac{DR_{\lambda}^{m} f(z)}{z}, \qquad z \in U,$$

where $q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt$. *The function q is the best subordinant.*

Corollary 2.15. [2] *Let q be convex in U and let h be defined by* $h(z) = q(z) + zq'(z)$. *If* $m, n \in \mathbb{N}$, $f \in \mathcal{A}_n$, *suppose that* $(SR^m f(z))'$ *is univalent,* $\frac{SR^m f(z)}{z} \in \mathcal{H}[1,n] \cap Q$ *and satisfies the differential superordination*

$$h(z) = q(z) + zq'(z) \prec (SR^m f(z))', \qquad z \in U, \tag{2.12}$$

then

$$q(z) \prec \frac{SR^m f(z)}{z}, \qquad z \in U,$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt.$$

The function q is the best subordinant.

Theorem 2.16. *Let*

$$h(z) = \frac{1 + (2\beta - 1)z}{1 + z}$$

be a convex function in U, where $0 \le \beta < 1$. *Let* $m, n, \lambda, l \in \mathbb{N}$, $f \in \mathcal{A}_n$ *and suppose that* $\left(IR_{\lambda,l}^{m} f(z)\right)'$ *is univalent and*

$$\frac{IR_{\lambda,l}^{m} f(z)}{z} \in \mathcal{H}[1,n] \cap Q.$$

If

$$h(z) \prec \left(IR_{\lambda,l}^{m} f(z)\right)', \qquad z \in U, \tag{2.13}$$

then

$$q(z) \prec \frac{IR_{\lambda,l}^{m} f(z)}{z}, \qquad z \in U,$$

where q is given by

$$q(z) = 2\beta - 1 + \frac{2(1-\beta)}{nz^{\frac{1}{n}}} \int_0^z \frac{t^{\frac{1}{n}-1}}{1+t} dt, \ z \in U.$$

The function q is convex and it is the best subordinant.

Proof. Following the same steps as in the proof of Theorem 2.10 and considering

$$p(z)=\frac{IR_{\lambda,l}^{m}f\left(z\right)}{z},$$

the differential superordination (2.13) becomes

$$h(z)=\frac{1+(2\beta-1)z}{1+z}\prec p(z)+zp'(z),\quad z\in U.$$

By using Lemma 1.9 for $\gamma=1$, we have $q(z)\prec p(z)$, i.e.,

$$q(z)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h\left(t\right)t^{\frac{1}{n}-1}dt=\frac{1}{nz^{\frac{1}{n}}}\int_0^z t^{\frac{1}{n}-1}\frac{1+(2\beta-1)\,t}{1+t}dt$$

$$=2\beta-1+\frac{2(1-\beta)}{nz^{\frac{1}{n}}}\int_0^z\frac{t^{\frac{1}{n}-1}}{1+t}dt\prec\frac{IR_{\lambda,l}^{m}f\left(z\right)}{z},\quad z\in U.$$

The function q is convex and it is the best subordinant. □

Theorem 2.17. *Let h be a convex function, $h(0)=1$. Let $m,n,\lambda,l\in\mathbb{N}$, $f\in\mathcal{A}_n$ and suppose that $\left(\frac{zIR_{\lambda,l}^{m+1}f(z)}{IR_{\lambda,l}^{m}f(z)}\right)'$ is univalent and $\frac{IR_{\lambda,l}^{m+1}f(z)}{IR_{\lambda,l}^{m}f(z)}\in\mathcal{H}\left[1,n\right]\cap Q$. If*

$$h(z)\prec\left(\frac{zIR_{\lambda,l}^{m+1}f\left(z\right)}{IR_{\lambda,l}^{m}f\left(z\right)}\right)',\qquad z\in U,\tag{2.14}$$

then

$$q(z)\prec\frac{IR_{\lambda,l}^{m+1}f\left(z\right)}{IR_{\lambda,l}^{m}f\left(z\right)},\qquad z\in U,$$

where

$$q(z)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h\left(t\right)t^{\frac{1}{n}-1}dt.$$

The function q is convex and it is the best subordinant.

Proof. Consider

$$p\left(z\right)=\frac{IR_{\lambda,l}^{m+1}f\left(z\right)}{IR_{\lambda,l}^{m}f\left(z\right)}=\frac{z+\sum_{j=n+1}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m+1}C_{m+j}^{m+1}a_j^2z^j}{z+\sum_{j=n+1}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m}C_{m+j-1}^{m}a_j^2z^j}$$

$$=\frac{1+\sum_{j=n+1}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m+1}C_{m+j}^{m+1}a_j^2z^{j-1}}{1+\sum_{j=n+1}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m}C_{m+j-1}^{m}a_j^2z^{j-1}}.$$

Evidently $p\in\mathcal{H}[1,n]$.

We have

$$p'(z) = \frac{\left(IR_{\lambda,l}^{m+1} f(z)\right)'}{IR_{\lambda,l}^{m} f(z)} - p(z) \cdot \frac{\left(IR_{\lambda,l}^{m} f(z)\right)'}{IR_{\lambda,l}^{m} f(z)}.$$

Then

$$p(z) + zp'(z) = \left(\frac{zIR_{\lambda,l}^{m+1} f(z)}{IR_{\lambda,l}^{m} f(z)}\right)'.$$

Then (2.14) becomes

$$h(z) \prec p(z) + zp'(z), \quad z \in U.$$

By using Lemma 1.9 for $\gamma = 1$, we have

$$q(z) \prec p(z), \quad z \in U, \quad \text{i.e.} \quad q(z) \prec \frac{IR_{\lambda,l}^{m+1} f(z)}{IR_{\lambda,l}^{m} f(z)}, \quad z \in U,$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t)\, t^{\frac{1}{n}-1} dt.$$

The function q is convex and it is the best subordinant. □

Corollary 2.18. [3] *Let h be a convex function, $h(0) = 1$. Let $\lambda \geq 0$, $m, n \in \mathbb{N}$, $f \in \mathcal{A}_n$ and suppose that $\left(\frac{zDR_{\lambda}^{m+1} f(z)}{DR_{\lambda}^{m} f(z)}\right)'$ is univalent and $\frac{DR_{\lambda}^{m+1} f(z)}{DR_{\lambda}^{m} f(z)} \in \mathcal{H}[1, n] \cap Q$. If*

$$h(z) \prec \left(\frac{zDR_{\lambda}^{m+1} f(z)}{DR_{\lambda}^{m} f(z)}\right)', \quad z \in U, \tag{2.15}$$

then

$$q(z) \prec \frac{DR_{\lambda}^{m+1} f(z)}{DR_{\lambda}^{m} f(z)}, \quad z \in U,$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t)\, t^{\frac{1}{n}-1} dt.$$

The function q is convex and it is the best subordinant.

Corollary 2.19. [2] *Let h be a convex function, $h(0) = 1$. Let $m, n \in \mathbb{N}$, $f \in \mathcal{A}_n$ and suppose that $\left(\frac{zSR^{m+1} f(z)}{SR^{m} f(z)}\right)'$ is univalent and $\frac{SR^{m+1} f(z)}{SR^{m} f(z)} \in \mathcal{H}[1, n] \cap Q$. If*

$$h(z) \prec \left(\frac{zSR^{m+1} f(z)}{SR^{m} f(z)}\right)', \quad z \in U, \tag{2.16}$$

then

$$q(z) \prec \frac{SR^{m+1}f(z)}{SR^{m}f(z)}, \qquad z \in U,$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t)\, t^{\frac{1}{n}-1} dt.$$

The function q is convex and it is the best subordinant.

Theorem 2.20. *Let q be convex in U and let h be defined by* $h(z) = q(z) + zq'(z)$. *If* $m, n, \lambda, l \in \mathbb{N}$, $f \in \mathcal{A}_n$, *suppose that* $\left(\frac{zIR^{m+1}_{\lambda,l}f(z)}{IR^{m}_{\lambda,l}f(z)}\right)'$ *is univalent,*

$$\frac{IR^{m+1}_{\lambda,l}f(z)}{IR^{m}_{\lambda,l}f(z)} \in \mathcal{H}[1,n] \cap Q$$

and satisfies the differential superordination

$$h(z) = q(z) + zq'(z) \prec \left(\frac{zIR^{m+1}_{\lambda,l}f(z)}{IR^{m}_{\lambda,l}f(z)}\right)', \qquad z \in U, \tag{2.17}$$

then

$$q(z) \prec \frac{IR^{m+1}_{\lambda,l}f(z)}{IR^{m}_{\lambda,l}f(z)}, \qquad z \in U,$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t)\, t^{\frac{1}{n}-1} dt.$$

The function q is the best subordinant.

Proof. Let

$$p(z) = \frac{IR^{m+1}_{\lambda,l}f(z)}{IR^{m}_{\lambda,l}f(z)} = \frac{z + \sum_{j=n+1}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m+1} C^{m+1}_{m+j} a_j^2 z^j}{z + \sum_{j=n+1}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m} C^{m}_{m+j-1} a_j^2 z^j}$$

$$= \frac{1 + \sum_{j=n+1}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m+1} C^{m+1}_{m+j} a_j^2 z^{j-1}}{1 + \sum_{j=n+1}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m} C^{m}_{m+j-1} a_j^2 z^{j-1}}.$$

Evidently $p \in \mathcal{H}[1,n]$.

Differentiating, we obtain

$$p(z) + zp'(z) = \left(\frac{zIR^{m+1}_{\lambda,l}f(z)}{IR^{m}_{\lambda,l}f(z)}\right)', \ z \in U$$

and (2.17) becomes

$$q(z)+zq'(z)\prec p(z)+zp'\left(z\right),\quad z\in U.$$

Using Lemma 1.10 for $\gamma=1$, we have

$$q(z)\prec p(z),\quad z\in U,\quad \text{i.e.}\quad q(z)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h\left(t\right)t^{\frac{1}{n}-1}dt\prec\frac{IR^{m+1}_{\lambda,l}f\left(z\right)}{IR^{m}_{\lambda,l}f\left(z\right)},\ \ z\in U,$$

and q is the best subordinant. □

Corollary 2.21. [3] *Let q be convex in U and let h be defined by $h\left(z\right)=q\left(z\right)+zq'\left(z\right)$. If $\lambda\geq 0$, $m,n\in\mathbb{N}$, $f\in\mathcal{A}_n$, suppose that $\left(\frac{zDR^{m+1}_{\lambda}f(z)}{DR^{m}_{\lambda}f(z)}\right)'$ is univalent,*

$$\frac{DR^{m+1}_{\lambda}f\left(z\right)}{DR^{m}_{\lambda}f\left(z\right)}\in\mathcal{H}\left[1,n\right]\cap Q$$

and satisfies the differential superordination

$$h(z)=q\left(z\right)+zq'\left(z\right)\prec\left(\frac{zDR^{m+1}_{\lambda}f\left(z\right)}{DR^{m}_{\lambda}f\left(z\right)}\right)',\qquad z\in U,\tag{2.18}$$

then

$$q(z)\prec\frac{DR^{m+1}_{\lambda}f\left(z\right)}{DR^{m}_{\lambda}f\left(z\right)},\qquad z\in U,$$

where

$$q(z)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h\left(t\right)t^{\frac{1}{n}-1}dt.$$

The function q is the best subordinant.

Corollary 2.22. [2] *Let q be convex in U and let h be defined by $h\left(z\right)=q\left(z\right)+zq'\left(z\right)$. If $m,n\in\mathbb{N}$, $f\in\mathcal{A}_n$, suppose that $\left(\frac{zSR^{m+1}f(z)}{SR^{m}f(z)}\right)'$ is univalent, $\frac{SR^{m+1}f(z)}{SR^{m}f(z)}\in\mathcal{H}\left[1,n\right]\cap Q$ and satisfies the differential superordination*

$$h(z)=q\left(z\right)+zq'\left(z\right)\prec\left(\frac{zSR^{m+1}f\left(z\right)}{SR^{m}f\left(z\right)}\right)',\qquad z\in U,\tag{2.19}$$

then

$$q(z)\prec\frac{SR^{m+1}f\left(z\right)}{SR^{m}f\left(z\right)},\qquad z\in U,$$

where

$$q(z)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h\left(t\right)t^{\frac{1}{n}-1}dt.$$

The function q is the best subordinant.

Theorem 2.23. *Let*

$$h(z) = \frac{1 + (2\beta - 1)z}{1 + z}$$

be a convex function in U, where $0 \le \beta < 1$. Let $m, n, \lambda, l \in \mathbb{N}$, $f \in \mathcal{A}_n$ and suppose that $\left(\frac{zIR_{\lambda,l}^{m+1} f(z)}{IR_{\lambda,l}^{m} f(z)}\right)'$ is univalent, $\frac{IR_{\lambda,l}^{m+1} f(z)}{IR_{\lambda,l}^{m} f(z)} \in \mathcal{H}[1, n] \cap Q$. If

$$h(z) \prec \left(\frac{zIR_{\lambda,l}^{m+1} f(z)}{IR_{\lambda,l}^{m} f(z)}\right)', \qquad z \in U, \tag{2.20}$$

then

$$q(z) \prec \frac{IR_{\lambda,l}^{m+1} f(z)}{IR_{\lambda,l}^{m} f(z)}, \qquad z \in U,$$

where q is given by

$$q(z) = 2\beta - 1 + \frac{2(1-\beta)}{nz^{\frac{1}{n}}} \int_0^z \frac{t^{\frac{1}{n}-1}}{1+t} dt, \ z \in U.$$

The function q is convex and it is the best subordinant.

Proof. Following the same steps as in the proof of Theorem 2.17 and considering

$$p(z) = \frac{IR_{\lambda,l}^{m+1} f(z)}{IR_{\lambda,l}^{m} f(z)},$$

the differential superordination (2.20) becomes

$$h(z) = \frac{1 + (2\beta - 1)z}{1 + z} \prec p(z) + zp'(z), \quad z \in U.$$

By using Lemma 1.9 for $\gamma = 1$, we have $q(z) \prec p(z)$, i.e.,

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t)\, t^{\frac{1}{n}-1} dt = \frac{1}{nz^{\frac{1}{n}}} \int_0^z t^{\frac{1}{n}-1} \frac{1 + (2\beta - 1)t}{1+t} dt$$

$$= 2\beta - 1 + \frac{2(1-\beta)}{nz^{\frac{1}{n}}} \int_0^z \frac{t^{\frac{1}{n}-1}}{1+t} dt \prec \frac{IR_{\lambda,l}^{m+1} f(z)}{IR_{\lambda,l}^{m} f(z)}, \quad z \in U.$$

The function q is convex and it is the best subordinant. □

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Department of Mathematics and Computer Science
University of Oradea
Str. Universităţii nr. 1, 410087 Oradea, Romania
E-mail address: dalb@uoradea.ro