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A CLASS OF MULTIVALENT ANALYTIC FUNCTIONS INVOLVING THE DZIOK-SRIVASTAVA OPERATOR

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Abstract. By making use of subordination between analytic functions and the Dziok-Srivastava operator, we introduce a new subclass of multivalent analytic functions. Such results as inclusion relationship, integral presentations and convolution properties for this function class are proved.

1. **Introduction**

Let $\mathcal{A}_p$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \qquad (p \in \mathbb{N} := \{1, 2, 3, \ldots\}), \tag{1.1}$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} := \{z : \ z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Let $f, \ g \in \mathcal{A}_p$, where f is given by (1.1) and g is defined by

$$g(z) = z^p + \sum_{n=1}^{\infty} b_{n+p} z^{n+p}.$$

Then the Hadamard product (or convolution) $f * g$ of the functions f and g is defined by

$$(f * g)(z) := z^p + \sum_{n=1}^{\infty} a_{n+p} b_{n+p} z^{n+p} =: (g * f)(z).$$

For parameters

$$\alpha_j \in \mathbb{C} \ \ (j = 1, \ldots, l) \quad \text{and} \quad \beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^- \ \ (\mathbb{Z}_0^- := \{0, -1, -2, \ldots\}; \ j = 1, \ldots, m),$$

the generalized hypergeometric function

$${}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$$

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is defined by the following infinite series:

$$ {}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_m)_n} \frac{z^n}{n!} $$

$$ (l \leqq m+1;\ l, m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\ z \in \mathbb{U}), $$

where $(\lambda)_n$ is the Pochhammer symbol defined by

$$ (\lambda)_n := \begin{cases} 1, & (n = 0), \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & (n \in \mathbb{N}). \end{cases} $$

Recently, Dziok and Srivastava [1] introduced a linear operator

$$ H_p(\alpha_1, \ldots, \alpha_l; \beta, \ldots, \beta_m) :\ \mathcal{A}_p \longrightarrow \mathcal{A}_p $$

defined by the following Hadamard product:

$$ H_p(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)f(z) := [z^p\ {}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)] * f(z) \tag{1.2} $$

$$ (l \leqq m+1;\ l, m \in \mathbb{N}_0;\ z \in \mathbb{U}). $$

If $f \in \mathcal{A}_p$ is given by (1.1), then we have

$$ H_p(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)f(z) = z^p + \sum_{n=1}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_m)_n}\, a_{n+p} \frac{z^{n+p}}{n!} \qquad (n \in \mathbb{N};\ z \in \mathbb{U}). $$

In order to make the notation simple, we write

$$ H_p^{l,m}(\alpha_j) := H_p(\alpha_1, \ldots, \alpha_j, \ldots, \alpha_l;\ \beta_1, \ldots, \beta_m) $$

$$ (l \leqq m+1;\ l, m \in \mathbb{N}_0;\ j \in \{1, 2, \ldots, l\}). $$

It is easily verified from the definition (1.2) that

$$ z\left(H_p^{l,m}(\alpha_j)f\right)'(z) = \alpha_j H_p^{l,m}(\alpha_j+1)f(z) - (\alpha_j - p)H_p^{l,m}(\alpha_j)f(z) \qquad (f \in \mathcal{A}_p). \tag{1.3} $$

Let $\mathcal{P}$ denote the class of functions of the form:

$$ p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, $$

which are analytic and convex in $\mathbb{U}$ and satisfy the condition:

$$ \Re(p(z)) > 0 \qquad (z \in \mathbb{U}). $$

For two functions f and g, analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$, and write

$$ f(z) \prec g(z) \quad \text{or} \quad f \prec g, $$

if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{U})$$

such that

$$f(z) = g\big(\omega(z)\big) \qquad (z \in \mathbb{U}).$$

Throughout this paper, we assume that

$$p, k \in \mathbb{N}, \quad l, m \in \mathbb{N}_0, \quad \varepsilon_k = \exp\left(\frac{2\pi i}{k}\right),$$

and

$$f_{p,k}^{l,m}(\alpha_j; z) = \frac{1}{k}\sum_{\nu=0}^{k-1} \varepsilon_k^{-\nu p}\left(H_p^{l,m}(\alpha_j)f\right)(\varepsilon_k^{\nu} z) = z^p + \cdots \qquad (f \in \mathcal{A}_p). \tag{1.4}$$

Clearly, for $k = 1$, we have

$$f_{p,1}^{l,m}(\alpha_j; z) = H_p^{l,m}(\alpha_j)f(z).$$

In a recent paper, Patel *et al.* [9] discussed the following subclass of multivalent analytic functions defined by Dziok-Srivastava operator $H_p^{l,m}(\alpha_1)$.

Definition 1.1. (See [9]) A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{S}_p^{l,m}(\alpha_1; \beta; A, B)$ if it satisfies the following subordination condition:

$$\frac{1}{p-\beta}\left(\frac{z\left(H_p^{l,m}(\alpha_1)f\right)'(z)}{H_p^{l,m}(\alpha_1)f(z)} - \beta\right) \prec \frac{1+Az}{1+Bz} \qquad (0 \leqq \beta < p;\ -1 \leqq B < A \leqq 1).$$

In 2007, Polatoğlu *et al.* [8] introduced and investigated the following subclass of the class $\mathcal{A}_p$ of p-valent analytic functions.

Definition 1.2. (See [8]) A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{M}_p(\alpha)$ if it satisfies the following inequality:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) < \alpha \qquad (\alpha > p).$$

Motivated by the function classes $\mathcal{S}_p^{l,m}(\alpha_1; \beta; A, B)$ and $\mathcal{M}_p(\alpha)$, by making use of the operator $H_p^{l,m}(\alpha_j)$ and the above-mentioned principle of subordination between analytic functions, we introduce and investigate the following subclass of the class $\mathcal{A}_p$ of p-valent analytic functions.

Definition 1.3. A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{M}_{p,k}^{l,m}(\alpha_j; \alpha; \phi)$ if it satisfies the following subordination condition:

$$\frac{1}{\alpha-p}\left(\alpha - \frac{z\left(H_p^{l,m}(\alpha_j)f\right)'(z)}{f_{p,k}^{l,m}(\alpha_j; z)}\right) \prec \phi(z) \quad \left(\alpha \geqq 0,\ \alpha \neq p;\ \phi \in \mathcal{P};\ f_{p,k}^{l,m}(\alpha_j; z) \neq 0\right). \tag{1.5}$$

Remark 1.4. It is easy to see that, if we set

$$k = j = 1,\ 0 \leqq \alpha < p \quad \text{and} \quad \phi(z) = \frac{1 + Az}{1 + Bz} \quad (-1 \leqq B < A \leqq 1)$$

in the class $\mathcal{M}_{p,k}^{l,m}(\alpha_j; \alpha; \phi)$, then it reduces to the class $\mathcal{S}_p^{l,m}(\alpha_1; \alpha; A, B)$. Furthermore, if we choose

$$l = 2,\ m = \alpha_1 = \alpha_2 = \beta_1 = 1,\ \alpha > p \quad \text{and} \quad \phi(z) = \frac{1 + z}{1 - z}$$

in the class $\mathcal{M}_{p,k}^{l,m}(\alpha_j; \alpha; \phi)$, then it reduces to the class $\mathcal{M}_p^{(k)}(\alpha)$. We observe that the class $\mathcal{M}_p^{(1)}(\alpha) =: \mathcal{M}_p(\alpha)$ was discussed by Polatoğlu *et al.* [8]. Moreover, the class $\mathcal{M}_1^{(k)}(\alpha)$ was considered recently by Wang *et al.* [12], the class $\mathcal{M}_1^{(1)}(\alpha)$ was studied earlier by Nishiwaki and Owa [4], Owa and Nishiwaki [5], Owa and Srivastava [6], Srivastava and Attiya [10], Uralegaddi and Desai [11].

In this paper, we aim at proving such results as inclusion relationship, integral presentations and convolution properties for the function class $\mathcal{M}_{p,k}^{l,m}(\alpha_j; \alpha; \phi)$.

2. Preliminary results

In order to prove our main results, we need the following lemmas.

Lemma 2.1. (See [2, 3]) *Let $\beta, \gamma \in \mathbb{C}$. Suppose that φ is convex and univalent in $\mathbb{U}$ with*

$$\varphi(0) = 1 \quad \textit{and} \quad \Re(\beta\varphi(z) + \gamma) > 0.$$

If $\mathfrak{p}$ is analytic in $\mathbb{U}$ with $\mathfrak{p}(0) = 1$, then the following subordination:

$$\mathfrak{p}(z) + \frac{z\mathfrak{p}'(z)}{\beta\mathfrak{p}(z) + \gamma} \prec \varphi(z)$$

implies that

$$\mathfrak{p}(z) \prec \varphi(z).$$

Lemma 2.2. (See [7]) *Let $\beta, \gamma \in \mathbb{C}$. Suppose that φ is convex and univalent in $\mathbb{U}$ with*

$$\varphi(0) = 1 \quad \textit{and} \quad \Re(\beta\varphi(z) + \gamma) > 0.$$

Also let

$$\mathfrak{q}(z) \prec \varphi(z).$$

If $\mathfrak{p} \in \mathcal{P}$ and satisfies the following subordination:

$$\mathfrak{p}(z) + \frac{z\mathfrak{p}'(z)}{\beta\mathfrak{q}(z) + \gamma} \prec \varphi(z),$$

then

$$\mathfrak{p}(z) \prec \varphi(z).$$

Lemma 2.3. *Let* $f \in \mathcal{M}_{p,k}^{l,m}(\alpha_j;\alpha;\phi)$. *Then*

$$\frac{1}{\alpha-p}\left(\alpha-\frac{z\left(f_{p,k}^{l,m}(\alpha_j;z)\right)'}{f_{p,k}^{l,m}(\alpha_j;z)}\right)\prec\phi(z). \tag{2.1}$$

Proof. By virtue of (1.4), we replace z by $\varepsilon_k^{\nu}z$ ($\nu=0,1,2,\ldots,k-1$) in $f_{p,k}^{l,m}(\alpha_j;z)$. Then

$$\begin{aligned}
f_{p,k}^{l,m}(\alpha_j;\varepsilon_k^{\nu}z) &= \frac{1}{k}\sum_{n=0}^{k-1}\varepsilon_k^{-np}\left(H_p^{l,m}(\alpha_j)f\right)(\varepsilon_k^{n+\nu}z)\\
&= \varepsilon_k^{\nu p}\cdot\frac{1}{k}\sum_{n=0}^{k-1}\varepsilon_k^{-(n+\nu)p}\left(H_p^{l,m}(\alpha_j)f\right)(\varepsilon_k^{n+\nu}z)\\
&= \varepsilon_k^{\nu p}f_{p,k}^{l,m}(\alpha_j;z).
\end{aligned} \tag{2.2}$$

Differentiating both sides of (1.4) with respect to z, we have

$$\left(f_{p,k}^{l,m}(\alpha_j;z)\right)'=\frac{1}{k}\sum_{\nu=0}^{k-1}\varepsilon_k^{-\nu(p-1)}\left(H_p^{l,m}(\alpha_j)f\right)'(\varepsilon_k^{\nu}z). \tag{2.3}$$

Thus, combining (2.2) and (2.3), we easily find that

$$\begin{aligned}
\frac{1}{\alpha-p}\left(\alpha-\frac{z\left(f_{p,k}^{l,m}(\alpha_j;z)\right)'}{f_{p,k}^{l,m}(\alpha_j;z)}\right) &= \frac{1}{\alpha-p}\left(\alpha-\frac{1}{k}\sum_{\nu=0}^{k-1}\frac{\varepsilon_k^{-\nu(p-1)}z\left(H_p^{l,m}(\alpha_j)f\right)'(\varepsilon_k^{\nu}z)}{f_{p,k}^{l,m}(\alpha_j;z)}\right)\\
&= \frac{1}{\alpha-p}\left(\alpha-\frac{1}{k}\sum_{\nu=0}^{k-1}\frac{\varepsilon_k^{\nu}z\left(H_p^{l,m}(\alpha_j)f\right)'(\varepsilon_k^{\nu}z)}{f_{p,k}^{l,m}(\alpha_j;\varepsilon_k^{\nu}z)}\right).
\end{aligned} \tag{2.4}$$

Moreover, since $f \in \mathcal{M}_{p,k}^{l,m}(\alpha_j;\alpha;\phi)$, it follows that

$$\frac{1}{\alpha-p}\left(\alpha-\frac{\varepsilon_k^{\nu}z\left(H_p^{l,m}(\alpha_j)f\right)'(\varepsilon_k^{\nu}z)}{f_{p,k}^{l,m}(\alpha_j;\varepsilon_k^{\nu}z)}\right)\prec\phi(z) \qquad (\nu\in\{0,1,2,\ldots,k-1\}). \tag{2.5}$$

Finally, by noting that ϕ is convex and univalent in $\mathbb{U}$, from (2.4) and (2.5), we conclude that the assertion (2.1) of Lemma 2.3 holds. □

3. Properties of the function class $\mathcal{M}_{p,k}^{l,m}(\alpha_j;\alpha;\phi)$

We begin by stating the following inclusion relationship for the function class $\mathcal{M}_{p,k}^{l,m}(\alpha_j;\alpha;\phi)$.

Theorem 3.1. *Let $\phi \in \mathcal{P}$ with*

$$\Re(\alpha+\alpha_j-p+(p-\alpha)\phi(z))>0 \qquad (\alpha \geqq 0,\ \alpha \neq p).$$

Then

$$\mathcal{M}_{p,k}^{l,m}(\alpha_j+1;\alpha;\phi) \subset \mathcal{M}_{p,k}^{l,m}(\alpha_j;\alpha;\phi).$$

Proof. Making use of the relationships (1.3) and (1.4), we know that

$$\begin{aligned} z\left(f_{p,k}^{l,m}(\alpha_j;z)\right)' + (\alpha_j-p)f_{p,k}^{l,m}(\alpha_j;z) &= \frac{\alpha_j}{k}\sum_{\nu=0}^{k-1}\varepsilon_k^{-\nu p}\left(H_p^{l,m}(\alpha_j+1)f\right)(\varepsilon_k^{\nu}z) \\ &= \alpha_j f_{p,k}^{l,m}(\alpha_j+1;z). \end{aligned} \tag{3.1}$$

Let $f \in \mathcal{M}_{p,k}^{l,m}(\alpha_j+1;\alpha;\phi)$. Suppose also that

$$h(z) = z\left(\frac{f_{p,k}^{l,m}(\alpha_j;z)}{z^p}\right)^{1/(\alpha-p)} \qquad (\alpha \geqq 0,\ \alpha \neq p). \tag{3.2}$$

Then h is analytic in $\mathbb{U}$. By taking logarithmic differentiation in (3.2), it follows that

$$q(z) = \frac{zh'(z)}{h(z)} = \frac{1}{\alpha-p}\left(\alpha - \frac{z\left(f_{p,k}^{l,m}(\alpha_j;z)\right)'}{f_{p,k}^{l,m}(\alpha_j;z)}\right) \tag{3.3}$$

is analytic in $\mathbb{U}$ with $q(0)=1$. We now find from (3.1) and (3.3) that

$$\alpha+\alpha_j-p+(p-\alpha)q(z) = \alpha_j\frac{f_{p,k}^{l,m}(\alpha_j+1;z)}{f_{p,k}^{l,m}(\alpha_j;z)}. \tag{3.4}$$

Differentiating both sides of (3.4) with respect to z logarithmically and using (3.3), we have

$$q(z) + \frac{zq'(z)}{\alpha+\alpha_j-p+(p-\alpha)q(z)} = \frac{1}{\alpha-p}\left(\alpha - \frac{z\left(f_{p,k}^{l,m}(\alpha_j+1;z)\right)'}{f_{p,k}^{l,m}(\alpha_j+1;z)}\right). \tag{3.5}$$

From (3.5) and Lemma 2.3 (with α_j replaced by α_j+1), we conclude that

$$q(z) + \frac{zq'(z)}{\alpha+\alpha_j-p+(p-\alpha)q(z)} \prec \phi(z). \tag{3.6}$$

Since

$$\Re(\alpha+\alpha_j-p+(p-\alpha)\phi(z))>0,$$

an application of Lemma 2.1 to (3.6) yields

$$q(z) = \frac{1}{\alpha - p}\left(\alpha - \frac{z\left(f_{p,k}^{l,m}(\alpha_j; z)\right)'}{f_{p,k}^{l,m}(\alpha_j; z)}\right) \prec \phi(z). \tag{3.7}$$

We now suppose that

$$q_1(z) = \frac{1}{\alpha - p}\left(\alpha - \frac{z\left(H_p^{l,m}(\alpha_j)f\right)'(z)}{f_{p,k}^{l,m}(\alpha_j; z)}\right). \tag{3.8}$$

Then $q_1(z)$ is analytic in $\mathbb{U}$ with $q_1(0) = 1$. It follows from (1.3) and (3.8) that

$$[(p-\alpha)q_1(z) + \alpha]f_{p,k}^{l,m}(\alpha_j; z) = \alpha_j H_p^{l,m}(\alpha_j + 1)f(z) - (\alpha_j - p)H_p^{l,m}(\alpha_j)f(z). \tag{3.9}$$

Differentiating both sides of (3.9) with respect to z and using (3.8), we have

$$zq_1'(z) + \left(\alpha_j - p + \frac{z\left(f_{p,k}^{l,m}(\alpha_j; z)\right)'}{f_{p,k}^{l,m}(\alpha_j; z)}\right)\left(q_1(z) + \frac{\alpha}{p-\alpha}\right) = \frac{\alpha_j}{p-\alpha}\,\frac{z\left(H_p^{l,m}(\alpha_j+1)f\right)'(z)}{f_{p,k}^{l,m}(\alpha_j; z)}. \tag{3.10}$$

We now easily find from (3.3), (3.4) and (3.10) that

$$q_1(z) + \frac{zq_1'(z)}{\alpha + \alpha_j - p + (p-\alpha)q(z)} = \frac{1}{\alpha - p}\left(\alpha - \frac{z\left(H_p^{l,m}(\alpha_j+1)f\right)'(z)}{f_{p,k}^{l,m}(\alpha_j+1; z)}\right) \prec \phi(z). \tag{3.11}$$

Since

$$q(z) \prec \phi(z)$$

and

$$\Re(\alpha + \alpha_j - p + (p-\alpha)\phi(z)) > 0,$$

it follows from (3.11) and Lemma 2.2 that

$$q_1(z) = \frac{1}{\alpha - p}\left(\alpha - \frac{z\left(H_p^{l,m}(\alpha_j)f\right)'(z)}{f_{p,k}^{l,m}(\alpha_j; z)}\right) \prec \phi(z),$$

that is, that $f \in \mathcal{M}_{p,k}^{l,m}(\alpha_j; \alpha; \phi)$. This implies that

$$\mathcal{M}_{p,k}^{l,m}(\alpha_j + 1; \alpha; \phi) \subset \mathcal{M}_{p,k}^{l,m}(\alpha_j; \alpha; \phi).$$

The proof of Theorem 3.1 is thus completed. □

Next, we derive several integral representations for the function class $\mathcal{M}_{p,k}^{l,m}(\alpha_j; \alpha; \phi)$.

Theorem 3.2. *Let* $f \in \mathcal{M}_{p,k}^{l,m}(\alpha_j;\alpha;\phi)$. *Then*

$$f_{p,k}^{l,m}(\alpha_j;z) = z^p \cdot \exp\left(\frac{(p-\alpha)}{k}\sum_{\nu=0}^{k-1}\int_0^z \frac{\phi\left(\omega(\varepsilon_k^\nu \xi)\right)-1}{\xi}d\xi\right), \tag{3.12}$$

where $f_{p,k}^{l,m}(\alpha_j;z)$ *is defined by* (1.4), ω *is analytic in* $\mathbb{U}$ *with*

$$\omega(0)=0 \quad \textit{and} \quad |\omega(z)|<1 \quad (z\in\mathbb{U}).$$

Proof. Suppose that $f \in \mathcal{M}_{p,k}^{l,m}(\alpha_j;\alpha;\phi)$. We know that the subordination condition (1.5) can be written as follows:

$$\frac{z\left(H_p^{l,m}(\alpha_j)f\right)'(z)}{f_{p,k}^{l,m}(\alpha_j;z)} = (p-\alpha)\phi\left(\omega(z)\right)+\alpha, \tag{3.13}$$

where ω is analytic in $\mathbb{U}$ with

$$\omega(0)=0 \quad \text{and} \quad |\omega(z)|<1 \quad (z\in\mathbb{U}).$$

Replacing z by $\varepsilon_k^\nu z$ $(\nu = 0,1,2,\ldots,k-1)$ in the equation (3.13), we observe that (3.13) also holds, that is,

$$\frac{\varepsilon_k^\nu z\left(H_p^{l,m}(\alpha_j)f\right)'(\varepsilon_k^\nu z)}{f_{p,k}^{l,m}(\alpha_j;\varepsilon_k^\nu z)} = (p-\alpha)\phi\left(\omega(\varepsilon_k^\nu z)\right)+\alpha. \tag{3.14}$$

We note that

$$f_{p,k}^{l,m}(\alpha_j;\varepsilon_k^\nu z) = \varepsilon_k^{\nu p} f_{p,k}^{l,m}(\alpha_j;z).$$

Thus, by letting $\nu = 0,1,2,\ldots,k-1$ in (3.14), successively, and summing the resulting equations, we get

$$\frac{z\left(f_{p,k}^{l,m}(\alpha_j;z)\right)'}{f_{p,k}^{l,m}(\alpha_j;z)} = \frac{(p-\alpha)}{k}\sum_{\nu=0}^{k-1}\phi\left(\omega(\varepsilon_k^\nu z)\right)+\alpha. \tag{3.15}$$

We next find from (3.15) that

$$\frac{\left(f_{p,k}^{l,m}(\alpha_j;z)\right)'}{f_{p,k}^{l,m}(\alpha_j;z)} - \frac{p}{z} = \frac{(p-\alpha)}{k}\sum_{\nu=0}^{k-1}\frac{\phi\left(\omega(\varepsilon_k^\nu z)\right)-1}{z}, \tag{3.16}$$

which, upon integration, yields

$$\log\left(\frac{f_{p,k}^{l,m}(\alpha_j;z)}{z^p}\right) = \frac{(p-\alpha)}{k}\sum_{\nu=0}^{k-1}\int_0^z \frac{\phi\left(\omega(\varepsilon_k^\nu \xi)\right)-1}{\xi}d\xi. \tag{3.17}$$

The assertion (3.12) of Theorem 3.2 can now easily be derived from (3.17). □

Theorem 3.3. *Let* $f \in \mathcal{M}_{p,k}^{l,m}(\alpha_j;\alpha;\phi)$. *Then*

$$H_p^{l,m}(\alpha_j)f(z)=\int_0^z \zeta^{p-1}[(p-\alpha)\phi\left(\omega(\zeta)\right)+\alpha]\cdot\exp\left(\frac{(p-\alpha)}{k}\sum_{\nu=0}^{k-1}\int_0^{\zeta}\frac{\phi\left(\omega(\varepsilon_k^{\nu}\xi)\right)-1}{\xi}d\xi\right)d\zeta, \tag{3.18}$$

where ω *is analytic in* $\mathbb{U}$ *with*

$$\omega(0)=0 \quad \textit{and} \quad |\omega(z)|<1 \quad (z\in\mathbb{U}).$$

Proof. Suppose that $f \in \mathcal{M}_{p,k}^{l,m}(\alpha_j;\alpha;\phi)$. Then, by virtue of (3.12) and (3.13), we get

$$\left(H_p^{l,m}(\alpha_j)f\right)'(z)=\frac{f_{p,k}^{l,m}(\alpha_j;z)}{z}\cdot[(p-\alpha)\phi\left(\omega(z)\right)+\alpha]$$

$$=z^{p-1}[(p-\alpha)\phi\left(\omega(z)\right)+\alpha]\cdot\exp\left(\frac{(p-\alpha)}{k}\sum_{\nu=0}^{k-1}\int_0^{z}\frac{\phi\left(\omega(\varepsilon_k^{\nu}\xi)\right)-1}{\xi}d\xi\right), \tag{3.19}$$

which, upon integration of (3.19), leads us easily to the assertion (3.18) of Theorem 3.3. □

In view of Lemma 2.3 and Theorem 3.1, we get another integral representation for the function class $\mathcal{M}_{p,k}^{l,m}(\alpha_j;\alpha;\phi)$.

Theorem 3.4. *Let* $f \in \mathcal{M}_{p,k}^{l,m}(\alpha_j;\alpha;\phi)$. *Then*

$$H_p^{l,m}(\alpha_j)f(z)=\int_0^z \zeta^{p-1}[(p-\alpha)\phi\big(\omega_2(\zeta)\big)+\alpha]\cdot\exp\left((p-\alpha)\int_0^{\zeta}\frac{\phi\big(\omega_1(\xi)\big)-1}{\xi}d\xi\right)d\zeta, \tag{3.20}$$

where ω_t $(t=1,2)$ *are analytic in* $\mathbb{U}$ *with*

$$\omega_t(0)=0 \quad \textit{and} \quad |\omega_t(z)|<1 \quad (z\in\mathbb{U};\ t=1,2).$$

Proof. Suppose that $f \in \mathcal{M}_{p,k}^{l,m}(\alpha_j;\alpha;\phi)$. We then find from (2.1) that

$$\frac{z\left(f_{p,k}^{l,m}(\alpha_j;z)\right)'}{f_{p,k}^{l,m}(\alpha_j;z)}=(p-\alpha)\phi\left(\omega_1(z)\right)+\alpha, \tag{3.21}$$

where ω_1 is analytic in $\mathbb{U}$ and $\omega_1(0)=0$. Thus, by similarly applying the method of proof of Theorem 3.2, we find that

$$f_{p,k}^{l,m}(\alpha_j;z)=z^p\cdot\exp\left((p-\alpha)\int_0^z\frac{\phi\big(\omega_1(\xi)\big)-1}{\xi}d\xi\right). \tag{3.22}$$

It now follows from (3.13) and (3.22) that

$$\begin{aligned}\left(H_p^{l,m}(\alpha_j)f\right)'(z) &= \frac{f_{p,k}^{l,m}(\alpha_j;z)}{z}\cdot[(p-\alpha)\phi\big(\omega_2(z)\big)+\alpha] \\ &= z^{p-1}[(p-\alpha)\phi\big(\omega_2(z)\big)+\alpha]\cdot\exp\left((p-\alpha)\int_0^z \frac{\phi\big(\omega_1(\xi)\big)-1}{\xi}d\xi\right),\end{aligned} \tag{3.23}$$

where ω_t $(t=1,2)$ are analytic in $\mathbb{U}$ with

$$\omega_t(0)=0 \quad \text{and} \quad |\omega_t(z)|<1 \quad (z\in\mathbb{U};\ t=1,2).$$

Upon integrating both sides of (3.23), we readily arrive at the assertion (3.20) of Theorem 3.4. □

In the following we give some convolution properties for the function class $\mathcal{M}_{p,k}^{l,m}(\alpha_j;\alpha;\phi)$.

Theorem 3.5. *Let $f\in\mathcal{M}_{p,k}^{l,m}(\alpha_j;\alpha;\phi)$. Then*

$$\begin{aligned}f(z) = &\left[\int_0^z \zeta^{p-1}[(p-\alpha)\phi\left(\omega(\zeta)\right)+\alpha]\cdot\exp\left(\frac{(p-\alpha)}{k}\sum_{\nu=0}^{k-1}\int_0^\zeta \frac{\phi\left(\omega(\varepsilon_k^\nu\xi)\right)-1}{\xi}d\xi\right)d\zeta\right] \\ &* \left(\sum_{n=0}^\infty \frac{n!(\beta_1)_n\cdots(\beta_m)_n}{(\alpha_1)_n\cdots(\alpha_j)_n\cdots(\alpha_l)_n}z^{n+p}\right),\end{aligned} \tag{3.24}$$

where ω is analytic in $\mathbb{U}$ with

$$\omega(0)=0 \quad \textit{and} \quad |\omega(z)|<1 \quad (z\in\mathbb{U}).$$

Proof. In view of (1.2) and (3.18), we find that

$$\begin{aligned}&\int_0^z \zeta^{p-1}[(p-\alpha)\phi\left(\omega(\zeta)\right)+\alpha]\cdot\exp\left(\frac{(p-\alpha)}{k}\sum_{\nu=0}^{k-1}\int_0^\zeta \frac{\phi\left(\omega(\varepsilon_k^\nu\xi)\right)-1}{\xi}d\xi\right)d\zeta \\ &\qquad = [z^p\ {}_lF_m(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m)]*f(z).\end{aligned} \tag{3.25}$$

Thus, from (3.25), we easily get the assertion (3.24) of Theorem 3.5. □

Theorem 3.6. *Let $f\in\mathcal{M}_{p,k}^{l,m}(\alpha_j;\alpha;\phi)$. Then*

$$\begin{aligned}f(z) = &\left[\int_0^z \zeta^{p-1}[(p-\alpha)\phi\big(\omega_2(\zeta)\big)+\alpha]\cdot\exp\left((p-\alpha)\int_0^\zeta \frac{\phi\big(\omega_1(\xi)\big)-1}{\xi}d\xi\right)d\zeta\right] \\ &* \left(\sum_{n=0}^\infty \frac{n!(\beta_1)_n\cdots(\beta_m)_n}{(\alpha_1)_n\cdots(\alpha_j)_n\cdots(\alpha_l)_n}z^{n+p}\right),\end{aligned} \tag{3.26}$$

where ω_t $(t = 1, 2)$ *are analytic in* $\mathbb{U}$ *with*

$$\omega_t(0) = 0 \quad \textit{and} \quad |\omega_t(z)| < 1 \quad (z \in \mathbb{U};\ t = 1, 2).$$

Proof. By virtue of (1.2) and (3.20), we know that

$$\int_0^z \zeta^{p-1}[(p-\alpha)\phi(\omega_2(\zeta)) + \alpha] \cdot \exp\left((p-\alpha)\int_0^\zeta \frac{\phi(\omega_1(\xi)) - 1}{\xi} d\xi\right) d\zeta \\ = [z^p\ {}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)] * f(z). \tag{3.27}$$

Thus, from (3.27), we easily arrive at the convolution property (3.26) asserted by Theorem 3.6. □

Theorem 3.7. *Let*

$$f \in \mathcal{A}_p \quad \textit{and} \quad \phi \in \mathcal{P}.$$

Then $f \in \mathcal{M}_{p,k}^{l,m}(\alpha_j; \alpha; \phi)$ *if and only if*

$$\frac{1}{z}\Bigg\{ f * \Bigg[\left(pz^p + \sum_{n=1}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_j)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_m)_n} \frac{n+p}{n!} z^{n+p} \right) \\ -[(p-\alpha)\phi(e^{i\theta}) + \alpha] \left(z^p + \sum_{n=1}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_j)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_m)_n} \frac{1}{n!} z^{n+p} \right) * \left(\frac{1}{k} \sum_{v=0}^{k-1} \frac{z^p}{1 - \varepsilon^v z} \right) \Bigg] \Bigg\} \neq 0 \tag{3.28}$$

$(z \in \mathbb{U};\ \ 0 \leqq \theta < 2\pi)$.

Proof. Suppose that $f \in \mathcal{M}_{p,k}^{l,m}(\alpha_j; \alpha; \phi)$. Since

$$\frac{z\left(H_p^{l,m}(\alpha_j)f\right)'(z)}{f_{p,k}^{l,m}(\alpha_j; z)} \prec (p-\alpha)\phi(z) + \alpha$$

is equivalent to

$$\frac{z\left(H_p^{l,m}(\alpha_j)f\right)'(z)}{f_{p,k}^{l,m}(\alpha_j; z)} \neq (p-\alpha)\phi(e^{i\theta}) + \alpha \qquad (z \in \mathbb{U};\ \ 0 \leqq \theta < 2\pi), \tag{3.29}$$

it is easy to see that the condition (3.29) can be written as follows:

$$\frac{1}{z}\left\{ z\left(H_p^{l,m}(\alpha_j)f\right)'(z) - f_{p,k}^{l,m}(\alpha_j; z)[(p-\alpha)\phi(e^{i\theta}) + \alpha] \right\} \neq 0 \quad (z \in \mathbb{U};\ \ 0 \leqq \theta < 2\pi). \tag{3.30}$$

On the other hand, we know from (1.2) that

$$z\left(H_p^{l,m}(\alpha_j)f\right)'(z) = \left(pz^p + \sum_{n=1}^{\infty} \frac{(\alpha_j)_n \cdots (\alpha_j)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_m)_n} \frac{n+p}{n!} z^{n+p} \right) * f(z). \tag{3.31}$$

Moreover, from the definition of $f_{p,k}^{l,m}(\alpha_j;z)$, we have

$$\begin{aligned} f_{p,k}^{l,m}(\alpha_j;z) &= H_p^{l,m}(\alpha_j)f(z) * \left(\frac{1}{k}\sum_{\upsilon=0}^{k-1}\frac{z^p}{1-\varepsilon^{\upsilon}z}\right) \\ &= \left(z^p + \sum_{n=1}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_j)_n\cdots(\alpha_l)_n}{(\beta_1)_n\cdots(\beta_m)_n}\frac{1}{n!}z^{n+p}\right) * \left(\frac{1}{k}\sum_{\upsilon=0}^{k-1}\frac{z^p}{1-\varepsilon^{\upsilon}z}\right) * f(z). \end{aligned} \tag{3.32}$$

Upon substituting (3.31) and (3.32) into (3.30), we easily deduce the convolution property (3.28) asserted by Theorem 3.7. □

Remark 3.8. By specializing the parameters in Theorems 3.1-3.7, we can get several interesting properties for some special function classes associated with the class $\mathcal{M}_{p,k}^{l,m}(\alpha_j;\alpha;\phi)$. Here, we choose to omit the details involved.

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