STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LV**, Number 2, June 2010

ANALYSIS OF A BILATERAL CONTACT PROBLEM WITH ADHESION AND FRICTION FOR ELASTIC MATERIALS

AREZKI TOUZALINE

Abstract. We consider a mathematical model which describes a contact problem between a deformable body and a foundation. The contact is bilateral and is modelled with Tresca's friction law in which adhesion is taken into account. The evolution of the bonding field is discribed by a first order differential equation and the material's behavior is modelled with a nonlinear elastic constitutive law. We derive a variational formulation of the mechanical problem and prove the existence and uniqueness result of the weak solution. Moreover, we prove that the solution of the contact problem can be obtained as the limit of the solution of a regularized problem as the regularizaton parameter converges to 0. The proof is based on arguments of time-dependent variational inequalities, differential equations and Banach fixed point theorem.

1. Introduction

Contact problems involving deformable bodies are quite frequent in industry as well as in daily life and play an important role in structural and mechanical systems. Because of the importance of this process a considerable effort has been made in its modelling and numerical simulations. A first study of frictional contact problems within the framework of variational inequalities was made in [5]. Recently a new book [18] was appeared such that the aim is to introduce the reader of the theory of variational inequalities with analysis to the study of contact mechanics, and, specifically, with study of antiplane contact problems with linearly elastic and viscoelastic materials. The mathematical, mechanical and numerical state of the art can be found in [15]. The frictional contact problem with normal compliance and adhesion for elastic materials was studied in [11]. In this paper we study a model of an elastic contact problem with Tresca's friction law in which adhesion into contact surfaces was taken into account. We recall that models for dynamic or quasistatic

Received by the editors: 15.01.2009.
2000 *Mathematics Subject Classification.* 47J20, 49J40, 74M10, 74M15.
Key words and phrases. Elastic materials, adhesion, Tresca's friction, fixed point, weak solution.

process of frictionless adhesive contact between a deformable body and a foundation have been studied in [2, 3, 15, 16]. In [1] the unilateral quasistatic contact problem with friction and adhesion was studied and an existence result for a friction coefficient small enough was established. As in [7, 8] we use the bonding field as an additional state variable β, defined on the contact surface of the boundary. The variable is restricted to values $0 \leq \beta \leq 1$, when $\beta = 0$ all the bonds are severed and there are no active bonds; when $\beta = 1$ all the bonds are active; when $0 < \beta < 1$ it measures the fraction of active bonds and partial adhesion takes place. We refer the reader to the extensive bibliography on the subject in [9, 12, 13, 14, 15, 16, 17]. In this work we derive a variational formulation of the mechanical problem for which we prove the existence of a unique weak solution, and obtain a partial regularity result for the solution. Moreover, we study the behavior of the solution of a regularized problem as the regularization parameter converges to 0.
The paper is structured as follows. In Section 2 we present some notations and give the variational formulation. In Section 3 we state and prove our main existence and uniqueness result, Theorem 2.1. Finally, in Section 4, we show that the regularized problem admits a unique solution, Theorem 4.1, and prove a convergence result of this problem, Theorem 4.2.

2. **Problem statement and variational formulation**

Let $\Omega \subset \mathbf{R}^d; (d = 2, 3)$, be the domain occupied by a nonlinear elastic elastic body. Ω is supposed to be open, bounded, with a sufficiently regular boundary Γ. Γ is partitioned into three measurable parts $\Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}_3$ where $\Gamma_1, \Gamma_2, \Gamma_3$ are disjoint open sets and $meas\ \Gamma_1 > 0$. The body is acted upon by a volume force of density φ_1 on Ω and a surface traction of density φ_2 on Γ_2. On Γ_3 the body is in bilateral and adhesive contact with Tresca's friction law with a foundation.
Thus, the classical formulation of the mechanical problem is written as follows.
Problem P_1. Find a displacement field $u : \Omega \times [0, T] \to \mathbf{R}^d$ and a bonding field $\beta : \Gamma_3 \times [0, T] \to [0, 1]$ such that

$$div\sigma + \varphi_1 = 0 \text{ in } \Omega \times (0, T), \tag{2.1}$$

$$\sigma = F\varepsilon(u) \text{ in } \Omega \times (0, T), \tag{2.2}$$

$$u = 0 \quad \text{on } \Gamma_1 \times (0, T), \tag{2.3}$$

$$\sigma\nu = \varphi_2 \quad \text{on } \Gamma_2 \times (0, T), \tag{2.4}$$

$$\left.\begin{array}{c} u_\nu = 0 \\ \\ \left|\sigma_\tau + c_\tau \beta^2 R_\tau (u_\tau)\right| \leq g \\ \\ \left|\sigma_\tau + c_\tau \beta^2 R_\tau (u_\tau)\right| < g \Longrightarrow u_\tau = 0 \\ \\ \left|\sigma_\tau + c_\tau \beta^2 R_\tau (u_\tau)\right| = g \Longrightarrow \\ \\ \exists \lambda \geq 0 \text{ s.t. } u_\tau = -\lambda \left(\sigma_\tau + c_\tau \beta^2 R_\tau (u_\tau)\right) \end{array}\right\} \text{on } \Gamma_3 \times (0,T), \tag{2.5}$$

$$\dot{\beta} = -(c_\tau \beta \left|R_\tau (u_\tau)\right|^2 - \varepsilon_a)_+ \text{ on } \Gamma_3 \times (0,T), \tag{2.6}$$

$$\beta(0) = \beta_0 \text{ on } \Gamma_3. \tag{2.7}$$

We denote by u the displacement field, σ the stress field and $\varepsilon(u)$ the strain tensor. Equation (2.1) represents the equilibrium equation. Equation (2.2) represents the elastic constitutive law of the material in which F is a given nonlinear function. Here and below a dot above a variable represents a time derivative. We recall that in linear elasticity the stress tensor $\sigma = (\sigma_{ij})$ is given by

$$\sigma_{ij} = a_{ijkh} \varepsilon_{kh}(u),$$

where $F = (a_{ijkh})$ is the linear elasticity tensor, for $i, j, k, h = 1, ..., d$; (2.3) and (2.4) are the displacement and traction boundary conditions, respectively, in which ν denotes the unit outward normal vector on Γ and $\sigma\nu$ represents the Cauchy stress vector. Condition (2.5) represents the bilateral contact with Tresca's friction law in which adhesion is taken into account. Here g is a friction bound and the parameters c_τ and ε_a are adhesion coefficients which may depend on $x \in \Gamma_3$. As in [17], R_τ is a truncation operator defined by

$$R_\tau(v) = \begin{cases} v & \text{if } |v| \leq L \\ \\ L\dfrac{v}{|v|} & \text{if } |v| > L \end{cases},$$

where $L > 0$ is a characteristic length of the bonds. Equation (2.6) represents the ordinary differential equation which describes the evolution of the bonding field and it was already used in [17] where $[s]_+ = \max(s, 0)\ \forall s \in \mathbf{R}$. Since $\dot{\beta} \leq 0$ on $\Gamma_3 \times (0, T)$, once debonding occurs, bonding cannot be reestablished. Also we wish to make it clear that from [11] it follows that the model does not allow for complete debonding field in finite time. Finally, (2.7) represents the initial bonding field. We recall that the inner products and the corresponding norms on $\mathbf{R}^d$ and S_d are given by

$$u.v = u_i v_i, \quad |v| = (v.v)^{\frac{1}{2}} \quad \forall u, v \in \mathbf{R}^d,$$
$$\sigma.\tau = \sigma_{ij}\tau_{ij}, \ |\tau| = (\tau.\tau)^{\frac{1}{2}} \quad \forall \sigma, \tau \in S_d,$$

where S_d is the space of second order symmetric tensors on $\mathbf{R}^d$ $(d = 2, 3)$. Here and below, the indices i and j run between 1 and d and the summation convention over repeated indices is adopted. Now, to proceed with the variational formulation, we need the following function spaces:

$$H = \left(L^2(\Omega)\right)^d, \ H_1 = \left(H^1(\Omega)\right)^d, \ Q = \left\{\tau = (\tau_{ij})\,; \tau_{ij} = \tau_{ji} \in L^2(\Omega)\right\},$$

$$Q_1 = \{\sigma \in Q; div\sigma \in H\}.$$

Note that H and Q are real Hilbert spaces endowed with the respective canonical inner products

$$\langle u, v\rangle_H = \int_\Omega u_i v_i dx, \quad \langle \sigma, \tau\rangle_Q = \int_\Omega \sigma_{ij}\tau_{ij} dx.$$

The strain tensor is

$$\varepsilon(u) = (\varepsilon_{ij}(u)) = \frac{1}{2}(u_{i,j} + u_{j,i})\,;$$

$div\sigma = (\sigma_{ij,j})$ is the divergence of σ. For every element $v \in H_1$ we denote by v_ν and v_τ the normal and the tangential components of v on the boundary Γ given by

$$v_\nu = v.\nu, \quad v_\tau = v - v_\nu \nu.$$

Similarly, for a regular function $\sigma \in Q_1$, we define its normal and tangential components by

$$\sigma_\nu = (\sigma\nu).\nu, \quad \sigma_\tau = \sigma\nu - \sigma_\nu \nu$$

and we recall that the following Green's formula holds:

$$\langle \sigma, \varepsilon(v)\rangle_Q + \langle div\sigma, v\rangle_H = \int_\Gamma \sigma\nu.v da \quad \forall v \in H_1,$$

where da is the surface measure element. Let V be the closed subspace of H_1 defined by

$$V = \{v \in H_1 : v = 0 \text{ on } \Gamma_1, \, v_\nu = 0 \text{ on } \Gamma_3\}.$$

Since $meas\ \Gamma_1 > 0$, the following Korn's inequality holds [5],

$$\|\varepsilon(v)\|_Q \geq c_\Omega \|v\|_{H_1} \quad \forall v \in V, \tag{2.8}$$

where the constant $c_\Omega > 0$ depends only on Ω and Γ_1. We equip V with the inner product

$$(u, v)_V = \langle \varepsilon(u), \varepsilon(v)\rangle_Q$$

and $\|.\|_V$ is the associated norm. It follows from Korn's inequality (2.8) that the norms $\|.\|_{H_1}$and $\|.\|_V$ are equivalent on V. Then $(V, \|.\|_V)$ is a real Hilbert space. Moreover by Sobolev's trace theorem, there exists $d_\Omega > 0$ which depends only on the domain Ω, Γ_1 and Γ_3 such that

$$\|v\|_{(L^2(\Gamma_3))^d} \leq d_\Omega \|v\|_V \quad \forall v \in V. \tag{2.9}$$

For $p \in [1, \infty]$, we use the standard norm of $L^p(0, T; V)$. We also use the Sobolev space $W^{1,\infty}(0, T; V)$ equipped with the norm

$$\|v\|_{W^{1,\infty}(0,T;V)} = \|v\|_{L^\infty(0,T;V)} + \|\dot{v}\|_{L^\infty(0,T;V)}.$$

For every real Banach space $(X, \|.\|_X)$ and $T > 0$ we use the notation $C([0,T]; X)$ for the space of continuous functions from $[0,T]$ to X; recall that $C([0,T]; X)$ is a real Banach space with the norm

$$\|x\|_{C([0,T];X)} = \max_{t\in[0,T]} \|x(t)\|_X.$$

We assume that the body forces and surface tractions have the regularity

$$\varphi_1 \in W^{1,\infty}(0, T; H), \qquad \varphi_2 \in W^{1,\infty}\left(0, T; \left(L^2(\Gamma_2)\right)^d\right) \tag{2.10}$$

and we denote by $f(t)$ the element of V defined by

$$(f(t), v)_V = \int_\Omega \varphi_1(t).vdx + \int_{\Gamma_2} \varphi_2(t).vda \quad \forall v \in V, \text{ for } t \in [0,T]. \tag{2.11}$$

Using (2.10) and (2.11) yield

$$f \in W^{1,\infty}(0, T; V).$$

Also we define the functional $j : V \to \mathbf{R}_+$ by

$$j(v) = \int_{\Gamma_3} g|v_\tau| da,$$

where g is assumed to satisfy

$$g \in L^\infty(\Gamma_3), \, g \geq 0 \text{ } a.e. \text{ on } \Gamma_3. \tag{2.12}$$

In the study of Problem P_1 we assume that the elasticity operator F satisfies

$$\left.\begin{array}{l}(a)\ \ F:\Omega\times S_d\rightarrow S_d;\\ \\ (b)\ \text{there exists } M>0 \text{ such that}\\ \qquad |F(x,\varepsilon_1)-F(x,\varepsilon_2)|\leq M|\varepsilon_1-\varepsilon_2|,\\ \qquad \text{for all } \varepsilon_1,\varepsilon_2 \text{ in } S_d,\ a.e.\ x \text{ in } \Omega;\\ \\ (c)\ \text{there exists } m>0 \text{ such that}\\ \qquad (F(x,\varepsilon_1)-F(x,\varepsilon_2)).(\varepsilon_1-\varepsilon_2)\geq m|\varepsilon_1-\varepsilon_2|^2,\\ \qquad \text{for all } \varepsilon_1,\varepsilon_2 \text{ in } S_d,\ a.e.\ x \text{ in } \Omega;\\ \\ (d)\ \text{the mapping } x\rightarrow F(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega,\\ \qquad \text{for any } \varepsilon \text{ in } S_d;\\ \\ (e)\ x\rightarrow F(x,0)\in Q.\end{array}\right\} \tag{2.13}$$

As in [17] we assume that the adhesion coefficients c_τ and ε_a satisfy the conditions

$$c_\tau,\ \varepsilon_a\in L^\infty(\Gamma_3),\ c_\tau,\ \varepsilon_a\geq 0,\ a.e.\text{on } \Gamma_3. \tag{2.14}$$

Next, we define the functional $r:L^2(\Gamma_3)\times V\times V\rightarrow\mathbb{R}$ by

$$r(\beta,u,v)=\int_{\Gamma_3}c_\tau\beta^2R_\tau(u_\tau).v_\tau da.$$

Finally, we assume that the initial data satisfy

$$\beta_0\in L^2(\Gamma_3);\ 0\leq\beta_0\leq 1,\ a.e. \text{ on } \Gamma_3, \tag{2.15}$$

and we need the following set for the bonding field,

$$\mathcal{O}=\left\{\theta:[0,T]\rightarrow L^2(\Gamma_3);\ 0\leq\theta(t)\leq 1\ \forall t\in[0,T],\ a.e. \text{ on } \Gamma_3\right\}.$$

Now by assuming the solution to be sufficiently regular, we obtain by using Green's formula that the problem P_1 has the following variational formulation.

Problem P_2. Find a displacement field $u\in W^{1,\infty}(0,T;\Omega)$ and a bonding field $\beta\in W^{1,\infty}\left(0,T;L^2(\Gamma_3)\right)\cap\mathcal{O}$ such that

$$\begin{aligned}&\langle F\varepsilon(u(t)),\varepsilon(v)-\varepsilon(u(t))\rangle_Q+j(v)-j(u(t))\\ &+r(\beta(t),u(t),v-u(t))\geq(f(t),v-u(t))_V\quad\forall\ v\in V,\ t\in[0,T],\end{aligned} \tag{2.16}$$

$$\dot{\beta}(t)=-(c_\tau\beta(t)|R_\tau(u_\tau(t))|^2-\varepsilon_a)_+\ \ a.e.t\in(0,T), \tag{2.17}$$

$$\beta(0)=\beta_0 \text{ on } \Gamma_3. \tag{2.18}$$

Our main result of this section, which will be established in the next is the following theorem.

Theorem 2.1. *Let* $T > 0$ *and assume* (2.10), (2.12), (2.13), (2.14) *and* (2.15). *Then there exists a unique solution to Problem* P_2.

3. **Existence and uniqueness result**

The proof of Theorem 2.1 will be carried out in several steps. In the first step, let $k > 0$ and consider the space

$$X = \left\{ \beta \in C\left([0,T]\,; L^2\left(\Gamma_3\right)\right); \sup_{t\in[0,T]} \left[\exp\left(-kt\right) \left\|\beta\left(t\right)\right\|_{L^2(\Gamma_3)}\right] < +\infty \right\}.$$

X is a Banach space for the norm

$$\left\|\beta\right\|_X = \sup_{t\in[0,T]} \left[\exp\left(-kt\right) \left\|\beta\left(t\right)\right\|_{L^2(\Gamma_3)}\right],$$

which is equivalent for the standard norm $\|.\|_{C([0,T];L^2(\Gamma_3))}$, and for a given $\beta \in X$ we consider the following variational problem.

Problem $P_{1\beta}$. Find $u_\beta : [0,T] \to V$ such that

$$\begin{aligned} &\left\langle F\varepsilon\left(u_\beta\left(t\right)\right)), \varepsilon\left(v\right) - \varepsilon(u_\beta\left(t\right))\right\rangle_Q + j\left(v\right) - j\left(u_\beta\left(t\right)\right) \\ &+r\left(\beta\left(t\right), u_\beta\left(t\right), v - u_\beta\left(t\right)\right) \geq \left(f\left(t\right), v - u_\beta\left(t\right)\right)_V \quad \forall\, v \in V,\, t \in [0,T]\,. \end{aligned} \tag{3.1}$$

Lemma 3.1. *There exists a unique solution to Problem* $P_{1\beta}$ *and it satisfies* $u_\beta \in C\left([0,T]\,;V\right)$.

Proof. Let $t \in [0,T]$ and let $A_t : V \to V$ be the operator given by

$$\left(A_t v, w\right)_V = \left\langle F\varepsilon\left(v\right), \varepsilon\left(w\right)\right\rangle_Q + r\left(\beta\left(t\right), v, w\right) \quad \forall v, w \in V.$$

Using (2.13) and the properties of R_τ (3.2) (see [15]) such that

$$\begin{aligned} &\left|R_\tau\left(u_\tau\right)\right| \leq L,\, \forall u \in V;\; \left|R_\tau\left(a\right) - R_\tau\left(b\right)\right| \leq \left|a - b\right|,\, \forall a, b \in \mathbf{R}^d, \\ &\left(R_\tau\left(u_\tau\right) - R_\tau\left(v_\tau\right)\right).\left(u_\tau - v_\tau\right) \geq 0,\, a.e. \text{ on } \Gamma_3,\, \forall u, v \in V, \end{aligned} \tag{3.2}$$

it follows that A_t is a strongly monotone and Lipschitz continuous operator. The functional j is a continuous semi-norm on V, then by a classical argument of elliptic variational inequalities [18], we deduce that there exists a unique element $u_\beta\left(t\right) \in V$ which satisfies (3.1). Let now, t_1, $t_2 \in [0,T]$. In inequality (3.1) written for $t = t_1$, take $w = u_\beta(t_2)$ and also in the same inequality written for $t = t_2$, take $w = u_\beta(t_1)$. We find after adding the resulting inequalities that

$$\begin{aligned} &\langle F\varepsilon(u_\beta(t_1))) - F\varepsilon(u_\beta(t_2)), \varepsilon(u_\beta(t_1)) - \varepsilon(u_\beta(t_2))\rangle_Q \leq \\ &r(\beta(t_1), u_\beta(t_1), u_\beta(t_2) - u_\beta(t_1)) + r(\beta(t_2), u_\beta(t_2), u_\beta(t_1) - u_\beta(t_2)). \end{aligned} \tag{3.3}$$

We have

$$\begin{aligned} &r(\beta(t_1), u_\beta(t_1), u_\beta(t_2) - u_\beta(t_1)) + r(\beta(t_2), u_\beta(t_2), u_\beta(t_1) - u_\beta(t_2)) = \\ &\textstyle\int_{\Gamma_3}(c_\tau\beta^2(t_1) - \beta^2(t_2))R_\tau(u_\tau(t_1)).(u_{\beta\tau}(t_2) - u_{\beta\tau}(t_1))\,da \\ &+\textstyle\int_{\Gamma_3} c_\tau\beta^2(t_2)(R_\tau(u_\tau(t_2)) - R_\tau(u_\tau(t_1))).(u_{\beta\tau}(t_1) - u_{\beta\tau}(t_2))\,da. \end{aligned}$$

Using (3.2), (2.13) (c) and, (2.9), it follows that exists a constant $c_1 > 0$ such that

$$\begin{aligned} &\|u_\beta(t_1) - u_\beta(t_2)\|_V \leq \\ &c_1\left(\|f(t_1) - f(t_2)\|_V + \|\beta(t_1) - \beta(t_2)\|_{L^2(\Gamma_3)}\right). \end{aligned} \tag{3.4}$$

As $f \in C([0,T];V)$ and $\beta \in C\left([0,T];L^2(\Gamma_3)\right)$, then (3.4) implies that $u_\beta \in C([0,T];V)$. $\square$

Next, we consider the following problem.

Problem $P_{2\beta}$. Find a bonding field $\beta^* : [0,T] \to L^2(\Gamma_3)$ such that

$$\dot{\beta}^*(t) = -\left(c_\tau\beta^*(t)\left|R_\tau(u_{\beta^*\tau}(t))\right|^2 - \varepsilon_a\right)_+ \quad a.e.\ t \in (0,T), \tag{3.5}$$

$$\beta^*(0) = \beta_0 \text{ on } \Gamma_3. \tag{3.6}$$

We have the following result.

Lemma 3.2. *There exists a unique solution to Problem $P_{2\beta}$ and it satisfies*

$$\beta^* \in W^{1,\infty}\left(0,T;L^2(\Gamma_3)\right) \cap \mathcal{O}.$$

Proof. We consider the mapping $\mathcal{T} : X \to X$ given by

$$\mathcal{T}\beta(t) = \beta_0 - \int_0^t (c_\tau\beta(s)\left|R_\tau(u_{\beta\tau}(s))\right|^2 - \varepsilon_a)_+ ds,$$

where u_β is a solution of Problem $P_{1\beta}$. Using that $|R_\tau(u_{\beta\tau})| \leq L$, it follows that there exists a constant $c_2 > 0$ such that

$$\|\mathcal{T}\beta_1(t) - \mathcal{T}\beta_2(t)\|_{L^2(\Gamma_3)} \leq c_2\textstyle\int_0^t \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma_3)}\,ds.$$

Since

$$\int_0^t \left\|\beta_1(s)-\beta_2(s)\right\|_{L^2(\Gamma_3)} ds = \int_0^t e^{ks}(e^{-ks}\left\|\beta_1(s)-\beta_2(s)\right\|_{L^2(\Gamma_3)})ds$$

$$\leq \left\|\beta_1-\beta_2\right\|_X \frac{e^{kt}}{k},$$

this inequality implies

$$e^{-kt}\left\|\mathcal{T}\beta_1(t)-\mathcal{T}\beta_2(t)\right\|_{L^2(\Gamma_3)} \leq \frac{c_2}{k}\left\|\beta_1-\beta_2\right\|_X \quad \forall t\in[0,T],$$

and then,

$$\left\|\mathcal{T}\beta_1-\mathcal{T}\beta_2\right\|_X \leq \frac{c_2}{k}\left\|\beta_1-\beta_2\right\|_X. \tag{3.7}$$

The inequality (3.7) shows that for $k>c_2$, $\mathcal{T}$ is a contraction. Then we deduce, by the Banach fixed point theorem that $\mathcal{T}$ has a unique fixed point β^* which satisfies (3.5) and (3.6). The regularity $\beta^*\in\mathcal{O}$ is a consequence of (3.6) and (2.15), see [15, 16] for details. □

Now, we provide the existence of the solution of Theorem 2.1. Indeed, let β^* be the fixed point of $\mathcal{T}$ and let u^* be the solution of Problem $P_{1\beta}$ for $\beta=\beta^*$, i-e., $u^*=u_{\beta^*}$. Take $v=u^*(t_2)$ in inequality (3.1) written for $t=t_1$, and also take $v=u^*(t_1)$ in the same inequality written for $t=t_2$ and adding the two inequalities, we obtain using similar arguments to those in the proof of (3.4), that there exists a constant $c_3>0$ such that

$$\left\|u^*(t_1)-u^*(t_2)\right\|_V \leq$$

$$c_3\left(\left\|\beta^*(t_1)-\beta^*(t_2)\right\|_{L^2(\Gamma_3)}+\left\|f(t_1)-f(t_2)\right\|_V\right) \quad \forall t_1,t_2\in[0,T]. \tag{3.8}$$

Now, as $T\beta^*=\beta^*$ we deduce from Lemma 3.2 that $\beta^*\in W^{1,\infty}\left(0,T;L^2(\Gamma_3)\right)$ and moreover as $f\in W^{1,\infty}(0,T;V)$, then (3.8) implies that $u^*\in W^{1,\infty}(0,T;V)$. Thus, we conclude by (3.1), (3.5) and (3.6) that (u^*,β^*) is a solution to Problem P_2. To prove the uniqueness of the solution, suppose that (u,β) is a solution of Problem P_2 which satisfies

$$(u,\beta)\in W^{1,\infty}(0,T;V)\times W^{1,\infty}\left(0,T;L^2(\Gamma_3)\right)\cap\mathcal{O},$$

it follows that $\beta\in\mathcal{O}$. Moreover, we deduce from (3.1) that u is a solution to Problem $P_{1\beta}$, and as by Lemma 3.1, this problem has a unique solution denoted by u_β, we get $u=u_\beta$. Take $u=u_\beta$ in (2.16) and use the initial condition (2.18), we deduce that β is a solution of Problem $P_{2\beta}$. Therefore, we obtain from Lemma 3.2 that $\beta=\beta^*$ and we deduce that (u^*,β^*) is a unique solution to Problem P_2. □

4. **The regularized problem**

In this section we consider the frictional contact problem with adhesion in the case when the contact condition (2.5) is replaced by the contact conditions

$$u_{\delta\nu} = 0, \;\; \sigma_{\delta\tau} = -c_\tau \beta_\delta^2 R_\tau\left(u_{\delta\tau}\right) - g\frac{u_{\delta\tau}}{\sqrt{u_{\delta\tau}^2 + \delta^2}}, \text{ on } \Gamma_3 \times (0,T)\,, \tag{4.1}$$

where $\delta > 0$ is a regularization parameter. The friction law (4.1) describes situation when slip appears for a small shear, this is the case when the contact surfaces are lubricated by a thin layer or non-Newtonian fluid. Thus the regularized problem is defined as follows.

Problem $P_{1\delta}$. Find a displacement field $u_\delta : \Omega \times [0,T] \to \mathbf{R}^d$ and a bonding field $\beta_\delta : \Gamma_3 \times [0,T] \to [0,1]$ such that

$$div\sigma\left(u_\delta\right) + \varphi_1 = 0 \text{ in } \Omega \times (0,T)\,, \tag{4.2}$$

$$\sigma = F\varepsilon\left(u_\delta\right) \text{ in } \Omega \times (0,T)\,, \tag{4.3}$$

$$u_\delta = 0 \quad \text{ on } \Gamma_1 \times (0,T)\,, \tag{4.4}$$

$$\sigma\nu = \varphi_2 \quad \text{ on } \Gamma_2 \times (0,T)\,, \tag{4.5}$$

$$\left.u_{\delta\nu} = 0, \, \sigma_{\delta\tau} = -c_\tau \beta_\delta^2 R_\tau\left(u_{\delta\tau}\right) - g\frac{u_{\delta\tau}}{\sqrt{u_{\delta\tau}^2 + \delta^2}} \right\} \text{on } \Gamma_3 \times (0,T)\,, \tag{4.6}$$

$$\dot\beta_\delta = -(c_\tau \beta_\delta \left|R_\tau\left(u_{\delta\tau}\right)\right|^2 - \varepsilon_a)_+ \text{ on } \Gamma_3 \times (0,T)\,, \tag{4.7}$$

$$\beta_\delta\left(0\right) = \beta_0 \text{ on } \Gamma_3. \tag{4.8}$$

As in [18] let us define the functional $j_\delta : V \to \mathbf{R}$ by

$$j_\delta\left(v\right) = \int_{\Gamma_3} g\left(\sqrt{v_\tau^2 + \delta^2} - \delta\right) da,$$

then the problem (4.2) – (4.8) admits the following variational formulation.

Problem $P_{2\delta}$. Find $\left(u_\delta, \beta_\delta\right) \in W^{1,\infty}(0,T;V) \times W^{1,\infty}\left(0,T;L^2\left(\Gamma_3\right)\right) \cap \mathcal{O}$ such that

$$\begin{aligned} &\left\langle F\varepsilon\left(u_\delta\left(t\right)\right), \varepsilon\left(v\right) - \varepsilon\left(u_\delta\left(t\right)\right)\right\rangle_Q + \; j_\delta\left(v\right) - \; j_\delta\left(u_\delta\left(t\right)\right) \\ &+ r\left(\beta_\delta\left(t\right), u_\delta\left(t\right), v - u_\delta\left(t\right)\right) \geq \left(f\left(t\right), v - u_\delta\left(t\right)\right)_V \quad \forall v \in V,\, t \in [0,T]\,, \end{aligned} \tag{4.9}$$

$$\dot\beta_\delta\left(t\right) = -(c_\tau \beta_\delta\left(t\right) \left|R_\tau\left(u_{\delta\tau}\left(t\right)\right)\right|^2 - \varepsilon_a)_+ \; a.e. \text{ on } \; (0,T)\,, \tag{4.10}$$

$$\beta_\delta\left(0\right) = \beta_0 \text{ on } \Gamma_3. \tag{4.11}$$

Now, our main is to study the behavior of the solution as $\delta \to 0$ and to prove that in the limit we obtain the solution of Problem P_2.

Theorem 4.1. *Assume that* (2.10), (2.12), (2.13), (2.14) *and* (2.15) *hold. Then for each* $\delta > 0$*, there exists a unique solution to Problem* $P_{2\delta}$.

Proof. The proof of Theorem 4.1 is similar to the proof of Theorem 2.1 and it is carried out in several steps. For this reason, we omit the details of the proof. The steps are:
(i) For any $\beta \in X$ we prove that there exists a unique $u_\delta \in C([0,T];V)$ such that

$$\begin{aligned}&\langle F\varepsilon(u_\delta(t)), \varepsilon(v) - \varepsilon(u_\delta(t))\rangle_Q + j_\delta(v) - j_\delta(u_\delta(t)) \\ &+ r(\beta(t), u_\delta(t), v - u_\delta(t)) \geq (f(t), v - u_\delta(t))_V \quad \forall v \in V,\ t \in [0,T].\end{aligned} \tag{4.12}$$

Indeed as the functional j_δ is proper convex and lower semicontinous on V, (see [18]), then using similar arguments to those in the proof of Lemma 3.1, we deduce (i).
(ii) There exists a unique β_δ such that

$$\beta_\delta \in W^{1,\infty}\left(0,T;L^2(\Gamma_3)\right) \cap \mathcal{O}, \tag{4.13}$$

$$\dot{\beta}_\delta(t) = -(c_\tau \beta_\delta(t)\,|R_\tau(u_{\delta\tau}(t))|^2 - \varepsilon_a)_+ \ a.e.\ t \in (0,T), \tag{4.14}$$

$$\beta_\delta(0) = \beta_0. \tag{4.15}$$

The proof of this step is based on Lemma 3.2.
(iii) Let β_δ defined in (ii) and denote again by u_δ the function obtained in step (i) for $\beta = \beta_\delta$. Then, using (4.12) – (4.15) we see that (u_δ, β_δ) is the unique solution to Problem $P_{2\delta}$ and it satisfies

$$(u_\delta, \beta_\delta) \in W^{1,\infty}(0,T;V) \times W^{1,\infty}\left(0,T;L^2(\Gamma_3)\right) \cap \mathcal{O}.$$

We now study the convergence of the solution (u_δ, β_δ) as $\delta \to 0$.
Theorem 4.2. *Assume that* (2.10), (2.12), (2.13), (2.14) *and* (2.15) *hold. Then we have the following convergences:*

$$\lim_{\delta\to 0} \|u_\delta(t) - u(t)\|_V = 0,\ \textit{for all}\ t \in [0,T], \tag{4.16}$$

$$\lim_{\delta\to 0} \|\beta_\delta(t) - \beta(t)\|_{L^2(\Gamma_3)} = 0,\ \textit{for all}\ t \in [0,T]. \tag{4.17}$$

The proof is carried out in several steps. In the first step, we show the following lemma.
Lemma 4.3. *For each $t \in [0,T]$, there exists $\bar{u}(t) \in V$ such that after passing to a subsequence still denoted $(u_\delta(t))$ we have*

$$u_\delta(t) \to \bar{u}(t) \ \textit{ weakly in } V \textit{ as } \delta \to 0. \tag{4.18}$$

Proof. It is well known (see [18]) that the inequality (4.9) is equivalent to the equality

$$\begin{aligned}&\langle F\varepsilon(u_\delta(t)), \varepsilon(v)\rangle_Q + (\nabla j_\delta(u_\delta(t)), v)_{L^2(\Gamma_3)} \\ &+ r(\beta_\delta(t), u_\delta(t), v) = (f(t), v)_V \quad \forall v \in V,\ t \in [0,T],\end{aligned} \tag{4.19}$$

where

$$(\nabla j_\delta (u_\delta (t)), v)_{L^2(\Gamma_3)} = \int_{\Gamma_3} g \frac{u_{\delta\tau}(t) v_\tau}{\sqrt{u_{\delta\tau}(t)^2 + \delta^2}} da.$$

Take $v = u_\delta (t)$ in (4.19), as

$$(\nabla j_\delta (u_\delta (t)), (u_\delta (t)))_{L^2(\Gamma_3)} \geq 0,\ r(\beta_\delta (t), u_\delta (t), u_\delta (t)) \geq 0,$$

then we get from (4.19) that

$$\langle F\varepsilon (u_\delta (t)), \varepsilon (u_\delta (t))\rangle_Q \leq (f(t), u_\delta (t))_V$$

and keeping, in mind (2.13) (c), it follows that there exists a constant $C > 0$ such that

$$\|u_\delta (t)\|_V \leq C \left(\|f(t)\|_V + \|F(0)\|_Q\right).$$

The sequence $(u_\delta (t))$ is bounded in V, then there exists $\bar{u}(t) \in V$ and a subsequence again denoted $(u_\delta (t))$ such that (4.18) holds. Let us now consider the auxiliary problem.

Problem P_a. Find $\beta : [0,T] \to L^2(\Gamma_3)$, such that

$$\dot{\beta}(t) = -\left(c_\tau \beta(t)\right) |R_\tau (\bar{u}_\tau (t))|^2 - \varepsilon_a)_+,\ a.e.\ t \in (0,T),$$

$$\beta(0) = \beta_0.$$

Using similar arguments to those in the proof of Lemma 3.2, we have the following result.

Lemma 4.4. *Problem P_a has a unique solution which satisfies*

$$\beta \in W^{1,\infty}\left(0,T; L^2(\Gamma_3)\right) \cap \mathcal{O}.$$

Next, we have the convergence result.

Lemma 4.5. *Let β be the solution to Problem P_a, then we have*

$$\lim_{\delta \to 0} \|\beta_\delta (t) - \beta(t)\|_{L^2(\Gamma_3)} = 0,\ \textit{for all } t \in [0,T]. \tag{4.20}$$

Proof. Using the properties of the operator R_τ, (see [15]), it follows that there exists a constant $C_1 > 0$ such that

$$\|\beta_\delta (t) - \beta(t)\|_{L^2(\Gamma_3)} \leq C_1 \int_0^t \|u_{\delta\tau}(s) - \bar{u}_\tau (s)\|_{(L^2(\Gamma_3))^d} ds. \tag{4.21}$$

From (4.18) we deduce that $u_{\delta\tau}(t) \to \bar{u}_\tau (t)$ strongly in $\left(L^2(\Gamma_3)\right)^d$, as $\delta \to 0$. On the other hand using (2.9), we have

$$\|u_{\delta\tau}(t) - \bar{u}_\tau (t)\|_{(L^2(\Gamma_3))^d} \leq d_\Omega \|u_\delta (t) - \bar{u}(t)\|_V$$

$$\leq d_\Omega \left(\|f(t)\|_V + \|\bar{u}(t)\|_V\right),$$

which implies that there exists a constant $C_2 > 0$ such that

$$\left\| u_{\delta\tau}\left(t\right) - \bar{u}_{\tau}\left(t\right)\right\|_{(L^2(\Gamma_3))^d} \leq C_2.$$

Then it follows from Lebesgue convergence theorem that

$$\lim_{\delta\to 0}\int_0^t \left\| u_{\delta\tau}\left(s\right) - \bar{u}_{\tau}\left(s\right)\right\|_{(L^2(\Gamma_3))^d} ds = 0.$$

So we deduce from (4.21) that

$$\left\|\beta_\delta\left(t\right) - \beta\left(t\right)\right\|_{L^2(\Gamma_3)} \to 0 \text{ as } \delta \to 0, \text{ for all } t \in [0, T],$$

and so (4.20) is proved.

Now it is necessary to show the following result.

Lemma 4.6. *We have* $\bar{u}\left(t\right) = u\left(t\right)$ *for all* $t \in [0, T]$.

Proof. Let $t \in [0, T]$. We have (see [18]),

$$\begin{aligned} &\left|j_\delta(v) - j\left(v\right)\right| \leq \delta \left\|g\right\|_{L^\infty(\Gamma_3)} meas\ \Gamma_3, \\ &\left|j_\delta(u_\delta\left(t\right)) - j\left(\bar{u}\left(t\right)\right)\right| \leq \delta \left\|g\right\|_{L^\infty(\Gamma_3)} meas\ \Gamma_3 \\ &+ \left\|g\right\|_{L^\infty(\Gamma_3)} \left\| u_{\delta\tau}\left(t\right) - \bar{u}_{\tau}\left(t\right)\right\|_{(L^2(\Gamma_3))^d}. \end{aligned} \tag{4.22}$$

It follows from (4.22), as $\delta \to 0$, that

$$j_\delta\left(v\right) \to j\left(v\right), \text{ for all } v \in V, \tag{4.23}$$

and

$$j_\delta\left(u_\delta\left(t\right)\right) \to j\left(\bar{u}\left(t\right)\right). \tag{4.24}$$

On the other hand we have

$$\begin{aligned} &r\left(\beta_\delta\left(t\right), u_\delta\left(t\right), v - u_\delta\left(t\right)\right) \\ &= r\left(\beta_\delta\left(t\right), u_\delta\left(t\right), v - u_\delta\left(t\right)\right) - r\left(\beta\left(t\right), u_\delta\left(t\right), v - u_\delta\left(t\right)\right) \\ &+ r\left(\beta\left(t\right), u_\delta\left(t\right), v - u_\delta\left(t\right)\right) - r\left(\beta\left(t\right), \bar{u}\left(t\right), v - u_\delta\left(t\right)\right) \\ &+ r\left(\beta\left(t\right), \bar{u}\left(t\right), \bar{u}\left(t\right) - u_\delta\left(t\right)\right) + r\left(\beta\left(t\right), \bar{u}\left(t\right), v - \bar{u}\left(t\right)\right). \end{aligned} \tag{4.25}$$

Using that $0 \leq \beta(t) \leq 1$, $0 \leq \beta_\delta(t) \leq 1$, for all $t \in [0,T]$, and the properties of the operator R_τ, we have as $\delta \to 0$,

$$\begin{aligned}
&r(\beta_\delta(t), u_\delta(t), v - u_\delta(t)) - r(\beta(t), u_\delta(t), v - u_\delta(t)) \to 0,\\
&r(\beta(t), u_\delta(t), v - u_\delta(t)) - r(\beta(t), \bar{u}(t), v - u_\delta(t)) \to 0, \qquad (4.26)\\
&r(\beta(t), \bar{u}(t), \bar{u}(t) - u_\delta(t)) \to 0.
\end{aligned}$$

So, we deduce from (4.25) and (4.26) that

$$r(\beta_\delta(t), u_\delta(t), v - u_\delta(t)) \to r(\beta(t), \bar{u}(t), v - \bar{u}(t)), \text{ as } \delta \to 0. \qquad (4.27)$$

Therefore, using (4.23), (4.24), (4.27), and passing to the limit in (4.9) as $\delta \to 0$, we obtain

$$\begin{aligned}
&\langle F\varepsilon(\bar{u}(t)), \varepsilon(v) - \varepsilon(\bar{u}(t))\rangle_Q + j(v) - j(\bar{u}(t))\\
&+r(\beta(t), \bar{u}(t), v - \bar{u}(t)) \geq (f(t), v - \bar{u}(t))_V \quad \forall v \in V.
\end{aligned} \qquad (4.28)$$

Take now $v = u(t)$ in (4.28) and $v = \bar{u}(t)$ in (2.16) and add them up, we obtain using (2.13)(c) that

$$m\|\bar{u}(t) - u(t)\|_V^2 \leq r(\beta(t), \bar{u}(t), u(t) - \bar{u}(t)) + r(\beta(t), u(t), \bar{u}(t) - u(t)). \quad (4.29)$$

So as

$$r(\beta(t), \bar{u}(t), u(t) - \bar{u}(t)) + r(\beta(t), u(t), \bar{u}(t) - u(t)) \leq 0,$$

it follows from (4.29) that

$$\bar{u}(t) = u(t). \qquad (4.30)$$

We have now all the ingredients to prove Theorem 4.2. Indeed, from (4.20) and (4.30), we deduce immediatly (4.17). To prove (4.16), take $v = u(t)$ in (4.28), and using (2.13)(c), it follows

$$\begin{aligned}
&m\|u_\delta(t) - u(t)\|_V^2 \leq\\
&j_\delta(u(t)) - j_\delta(u_\delta(t)) + r(\beta_\delta(t), u_\delta(t), u(t) - u_\delta(t)) \qquad (4.31)\\
&+\langle F\varepsilon(u(t)), \varepsilon(u(t) - u_\delta(t))\rangle_Q + (f(t), u_\delta(t) - u(t))_V.
\end{aligned}$$

Passing to the limit as $\delta \to 0$ in the previous inequality and using the convergences

$$j_\delta(u(t)) - j_\delta(u_\delta(t)) \to 0,$$

$$r(\beta_\delta(t), u_\delta(t), u(t) - u_\delta(t)) \to 0,$$

$$\langle F\varepsilon(u(t)), \varepsilon(u(t) - u_\delta(t))\rangle_Q + (f(t), u_\delta(t) - u(t))_V \to 0,$$

we see immediately that (4.16) follows from (4.31).

References

[1] L. Cangémi, *Frottement et adhérence: modèle, traitement numérique et application à l'interface fibre/matrice,* Ph. D. Thesis, Univ. Méditerranée, Aix Marseille I, 1997.

[2] O. Chau, J. R. Fernandez, M. Shillor, M. Sofonea, *Variational and numerical analysis of a quasistatic viscoelastic contact problem with adhesion*, Journal of Computational and Applied Mathematics, **159**(2003), 431-465.

[3] O. Chau, M. Shillor, M. Sofonea, *Dynamic frictionless contact with adhesion*, J. Appl. Math. Phys. (ZAMP), **55**(2004), 32-47.

[4] M. Cocou, R. Rocca, *Existence results for unilateral quasistatic contact problems with friction and adhesion*, Math. Model. Num. Anal., **34**(2000), 981-1001.

[5] G. Duvaut, J.-L. Lions, *Les inéquations en mécanique et en physique*, Dunod, Paris, 1972.

[6] J. R. Fernandez, M. Shillor, M. Sofonea, *Analysis and numerical simulations of a dynamic contact problem with adhesion*, Math. Comput. Modelling, **37**(2003), 1317-1333.

[7] M. Frémond, *Adhérence des solides*, J. Mécanique Théorique et Appliquée, **6**(1987), 383-407.

[8] M. Frémond, *Equilibre des structures qui adhèrent à leur support*, C. R. Acad. Sci. Paris, 295, série II, (1982), 913-916.

[9] Frémond, *Non Smooth Thermomechanics*, Springer, Berlin 2002.

[10] S. A. Nassar, T. Andrews, S. Kruk, M. Shillor, *Modelling and simulations of a bonded rod*, Math. Comput. Modelling, **42**(2005), 553-572.

[11] Z. Lerguet, M. Sofonea, S. Drabla, *Analysis of a frictional contact problem with adhesion*, Acta Mathematica Universitatis Comenianae, **77**(2008), 181-198.

[12] M. Raous, L. Cangémi and M. Cocu, *A consistent model coupling adhesion, friction, and unilateral contact*, Comput. Meth. Appl. Mech. Engng., **177**(1999), 383-399.

[13] J. Rojek, J. Telega, *Contact problems with friction, adhesion and wear in orthopeadic biomechanics. I: General developements*, J. Theor. Appl. Mech., **39**(2001), 655-677.

[14] M. Shillor, M. Sofonea, J. J. Telega, *Models and Variational Analysis of Quasistatic Contact*, Lecture Notes Physics, **655**(2004), Springer, Berlin.

[15] M. Sofonea, W. Han, M. Shillor, *Analysis and Approximations of Contact Problems with Adhesion or Damage*, Pure and Applied Mathematics 276, Chapman & Hall / CRC Press, Boca Raton, Florida, 2006.

[16] M. Sofonea, T. V. Hoarau-Mantel, *Elastic frictionless contact problems with adhesion*, Adv. Math. Sci. Appl., **15**(2005), no. 1, 49-68.

[17] M. Sofonea, R. Arhab, R. Tarraf, *Analysis of electroelastic frictionless contact problems with adhesion*, Journal of Applied Mathematics, (2006), ID 64217, 1-25.

[18] M. Sofonea, A. Matei, *Variational inequalities with applications*, Advances in Mathematics and Mechanics, **18**(2009), Springer, New York.

Laboratoire de Systèmes Dynamiques
Faculté de Mathématiques, USTHB
BP 32 EL ALIA
Bab-Ezzouar, 16111, Algérie
E-mail address: `ttouzaline@yahoo.fr`