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A COLLOCATION METHOD USING CUBIC B-SPLINES FUNCTIONS FOR SOLVING SECOND ORDER LINEAR VALUE PROBLEMS WITH CONDITIONS INSIDE THE INTERVAL $[0,1]$

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Abstract. Consider the problem:

$$\begin{aligned} &y''(x) - Q(x)y(x) = R(x), \qquad x \in [0,1] \\ &y(a) = \alpha \\ &y(b) = \beta, \qquad a, b \in (0,1). \end{aligned}$$

where $Q(x), R(x) \in C[0,1]; y \in C^2[0,1]$. The aim of this paper is to present an approximate solution of this problem based on cubic B-splines. The approximate solution uses a mesh based on Legendre points.A numerical solution is also given.

1. **Introduction**

Consider the problem(PVP):

$$\begin{aligned} y''(x) - Q(x)y(x) = R(x), \qquad & x \in [0,1] \\ y(a) = \alpha & \\ y(b) = \beta, \qquad & a, b \in (0,1). \end{aligned} \tag{1.1}$$

where $Q(x), R(x) \in C[0,1]; y \in C^2[0,1], a, b, \alpha, \beta \in \mathbb{R}$.This is not a two point boundary value problem (BVP), since $a, b \in (0,1)$.

If the solution of the two-point boundary value problem (BVP):

$$\begin{aligned} y''(x) - Q(x)y(x) = r(x), \qquad & x \in [a,b] \\ y(a) = \alpha & \\ y(b) = \beta, & \end{aligned} \tag{1.2}$$

exists and it is unique, then the requirement $y \in C^2[0,1]$ assures the existence and the uniqueness of (1.1).

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I have two initial value problems on $[0,a]$ and $[b,1]$, respectively, and the existence and the uniqueness for (1.2) assure existence and uniqueness of these problems. It is possible to solve this problem by dividing it into the three above-mentioned problems and to solve each of these problem separately, but I am interested to a unitary approach that solve it as a whole.

Remark 1.1. • If $a = 0$ and $b = 1$ the problem (PVP) becomes a classical (BVP).

• If $a = 0$ or $b = 1$ the problem (PVP) may be decomposed into an (BVP) and one initial value problem(IVP).

Historical Note

In 1966, two researchers from *Tiberiu Popoviciu Institute of Romanian Academy Cluj Napoca*, D. Rîpianu and O. Aramă published a paper on polylocal problem (see [10]).

2. **Preliminaries**

Consider a partition of $[0,1]$ like:

$$\pi : 0 = x_0 < x_1 < \cdots < x_N < x_{N+1} = 1, \tag{2.1}$$

and the step sizes:

$$H_i := x_{i+1} - x_i, \qquad i = 0, \ldots, N. \tag{2.2}$$

In each subinterval $[x_i, x_{i+1}]$ we construct the collocation points as follows

$$\xi_{ij} := x_i + H_i\rho_j;\ i = 0,1,...,N,\ j = 0,1,2,...,k, \tag{2.3}$$

where

$$0 \leq \rho_0 < \rho_1 < \rho_2 < ... < \rho_k \leq 1 \tag{2.4}$$

are the roots of k-th Legendre polynomial on each subintervals:$[x_i, x_{i+1}], i = 0,1,...,N$ with the stepsize given by (2.2) (see [1] for more details). I insert the points a, b so I obtained $N(k+1)+2$ points. One renumbers the collocation points such that the first is $\xi_0 := x_0 + H_0\rho_0 = 0$, and the last is $\xi_{n+2} := x_N + H_N\rho_k = 1$, where $n = N(K+1)$. Therefore the partition of $[0,1]$ becomes:

$$\Delta := 0 \leq \xi_0 < \xi_1 < ... < \xi_{n+2} = 1$$

We augment the above partition Δ to form:

$$\overline{\Delta} : \xi_{-2} < \xi_{-1} < \xi_0 = 0 < \xi_1 < ... < \xi_{n+2} = 1 < \xi_{n+3} < \xi_{n+4} \tag{2.5}$$

where: $\xi_l := a$; $\xi_{l+p} := b$; $0 < l < n+1$; $1 < l+p < n+2$, $\xi_{-1} - \xi_{-2} = \xi_0 - \xi_{-1} = \xi_1 - \xi_0$, $\xi_{n+4} - \xi_{n+3} = \xi_{n+3} - \xi_{n+2} = \xi_{n+2} - \xi_{n+1}$.

Remark 2.1. If $a = \xi_i$ or $b = \xi_{i+p}$, $1 \leq i \leq n-2$, $1 < p < n+1-i$ we increment k.

Notation 2.2.

$$Q_i := Q(\xi_i)\ ;\ h_i := \xi_{i+1} - \xi_i;\ H := \max_{0\leq\ i\leq\ n+1}(\xi_{i+1} - \xi_i);\ h := \min_{0\leq\ i\leq n+1}(\xi_{i+1} - \xi_i).$$

Definition 2.3. Given the meshpoint (2.5) I define the vector space:

$S\left(\overline{\Delta}\right) = \{p(x) \in C^2[0,1] : p(x)$ is a cubic polynomial of each subinterval $[\xi_{i-2}\ ,\xi_{i+2}],\ 0 \leq i \leq n+2\}$.

$\dim S\left(\overline{\Delta}\right) = n+2$ (numbers of subintervals, see [12, pp. 73])

Definition 2.4. For $x \in \mathbb{R}\ ; 0 \leq i \leq n$, the cubic B-splines with the five knots: ξ_{i-2}, ξ_{i-1}, ξ, ξ, ξ are given by:

$$B_{i,3}(x) = \frac{x - \xi_{\ i-2}}{h_{i-2} + h_{i-1} + h_i} B_{i,2}(x) + \frac{\xi_{\ i+2} - x}{h_{i+1} + h_i + h_{i-1}} B_{i+1,2}(x) \tag{2.6}$$

where

$$B_{i,0} = \begin{cases} 1 \text{ if } \xi_{i-2} \leq x < \xi_{i-1} \\ 0 \text{ otherwise} \end{cases}$$

$$B_{i,2}(x) = \begin{cases} \frac{(x-\xi_{i-2})^2}{h_{i-2}\ (h_{i-2}+h_{i-1})}, \text{ if } \xi_{i-2} \leq\ x\ \leq \xi_{i-1} \\ \frac{(x-\xi_{i-2})(\xi_i-x)}{h_{i-1}(h_{i-1}+h_{i-2})} + \frac{(\xi_{i\ +1}-x)(x-\xi_{i-1})}{h_{i-1}(h_{i-1}+h_i)}, \text{if } \xi_{i-1} \leq x \leq \xi_i \\ \frac{(\xi_{i+1}-x)^2}{(h_{i-1}+h_i)h_i}, \text{ if } \xi_i \leq x \leq \xi_{i+1} \\ 0 \qquad , \qquad \text{otherwise}\ . \end{cases}$$

We need a bases from $S(\overline{\Delta})$ having $(n+2)$ cubic B-splines.Our choice is based on some special properties of cubic B-splines (see [11, pp.19-21] for details):

- The set
$$\{B_i\}\ i = 0, ..., n+1 \tag{2.7}$$
form a basis for $S(\overline{\Delta})$.
- $$\{B_i\} \text{ is positive on } (\xi_{i-2}, \xi_{i+2}) \text{ and zero elsewhere.} \tag{2.8}$$
- $\{B_i\}$ has local support (ξ_{i-2}, ξ_{i+2}) so computations using B-splines lead to linear system of equations with banded matrices.
- $$\sum_{i\ =\ 0}^{n+1} B_{i,3}(x) = 1 \text{ for every } x \in [0,1] \tag{2.9}$$

I recall some results from matrix theory ([7, pp. 359-361], [8, pp. 50-55]):

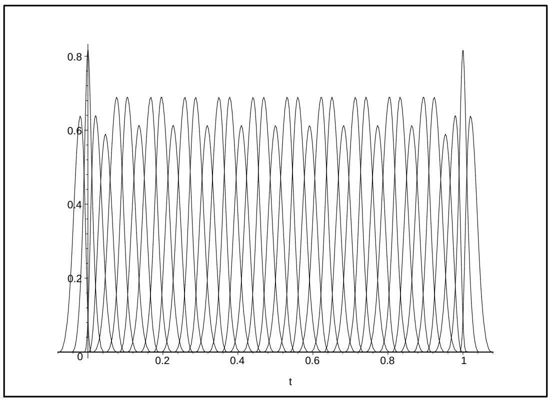

FIGURE 1. **B-spline bases**

Definition 2.5. A matrix $A = [a_{i\ j}], i = 1, 2, ..., m, j = 1, 2, ..., n$ is called *reducible* if there is a permutation that puts it into the form

$$\widetilde{A} = \begin{pmatrix} B & 0 \\ C & D \end{pmatrix},$$

where B and D are square matrices. Otherwise A is called *irreducible.*

Definition 2.6. A matrix $A = [a_{i\ j}], i = 1, 2, ..., m, j = 1, 2, ..., n$ is called *monotone* if $Az \geq 0$ implies $z \geq 0$.

Theorem 2.7. *A square tridiagonal matrix* $A = [a_{ij}]$ $i, j = 1, 2, ..., n$ *is* irreducible *iff:*

$$a_{i,i-1} \neq 0 \ (i = 2, 3, ..., n) \text{ and } a_{i,i+1} \neq 0 \ (i = 1, 2, ..., n-1)$$

and is reducible *iff:*

$$a_{i,i-1} = 0 \text{ or } a_{i,i+1} = 0 \text{ for some } i = 2, 3, ..., n$$

Theorem 2.8. *A monotone matrix is nonsingular.*

3. Main Results

3.1. **Consistency of the method.** I wish to find a approximate solution of the problem (1.1) in the following form:

$$u_{\overline{\Delta}}(x) = \sum_{i=0}^{n+1} c_i B_{i,3}(x). \tag{3.1}$$

where $B_{i,3}(x)$ is a cubic B-splines with knots $\{\xi_{i+k}\}_{k=-2}^{2}$.

Remark 3.1. My approximation method is inspired from ([3], chap. 2,5)

I impose the conditions:

(c1) The approximate solution (3.1) verifies the differential equation (1.1) at ξ_j, $j = 1, ..., n+2$, $j \neq l$, $j \neq l+p$.

(c2) The solution verifies $u_{\overline{\Delta}}(\xi_l) = \alpha, u_{\overline{\Delta}}(\xi_{l+p}) = \beta$ (we recall that $a = \xi_l, b = \xi_{l+p}$).

Conditions **(c1)** and **(c2)** yield to a linear system:

$$A \cdot c = \gamma \tag{3.2}$$

with $(n+2)$ equations and $(n+2)$ unknowns c_i, $i = 0, ..., n+1$.The system matrix A is tridiagonal with 3 nonzero elements on each row.

We denote by:

$$f_i(x) := B"_{i,3}(x) - Q(x)B_{i,3}(x), \; i = 0, 1, ..., n+1;$$

then

$$A = \begin{bmatrix} f_i(\xi_j); i \in \{0, 1, 2..., n+1\}, \quad j \in \{1, 2, ..., n+2\} \backslash \{l, l+p\} \\ B_{i,3}(\xi_l); i = l-1, l, l+1 \\ B_{i,3}(\xi_{l+p}); i = l+p-1, l+p, l+p+1 \end{bmatrix}$$

The right hand side of (3.2) is:

$$\gamma = [R(\xi_1), ..., R(\xi_{l-1}), \alpha, R(\xi_{l+1}), ..., R(\xi_{l+p-1}), \beta, R(\xi_{l+p+1}), ..., R(\xi_{n+2})]$$

Lemma 3.2. (see [11, p. 23]) *For each $l > 0$, and $x \in [0,1]$, we have $B_{i,l}(x) \in C^1[0,1]$ and*

$$B'_{i,l}(x) = l \left[\frac{B_{i,l-1}(x)}{\xi_{i+l-2} - \xi_{i-2}} - \frac{B_{i+1,l-1}(x)}{\xi_{i+l-1} - \xi_{i-1}} \right]. \tag{3.3}$$

First I prove the next lemmas:

Lemma 3.3. *For each $x \in [0,1]$, $B_{i,3}(x) \in C^2[0,1]$ and*

$$B^{"}_{i,3}(x) = 3! \left[\frac{B_{i,1}(x)}{(h_i + h_{i-1} + h_{i-2})(h_{i-1} + h_{i-2})} - \right. \tag{3.4a}$$

$$- \frac{B_{i+1,1}(x)(h_{i-2} + 2h_{i-1} + 2h_i + h_{i+1})}{(h_i + h_{i-1})(h_i + h_{i-1} + h_{i-2})(h_{i+1} + h_i + h_{i-1})} + \tag{3.4b}$$

$$\left. + \frac{B_{i+2,1}(x)}{(h_{i+1} + h_i + h_{i-1})(h_i + h_{i+1})} \right], \tag{3.4c}$$

where

$$B_{i,1}(x) = \begin{cases} \frac{x - \xi_{i-2}}{h_{i-2}}, \text{ if } \xi_{i-2} \le x < \xi_{i-1} \\ \frac{\xi_i - x}{h_{i-1}}, \text{ if } \xi_{i-1} \le x < \xi_i \\ 0, \quad \text{otherwise.} \end{cases}$$

Proof. For $l = 3$ we obtain from (3.3)

$$B'_{i,3}(x) = 3\left[\frac{B_{i,2}(x)}{h_i + h_{i-1} + h_{i-2}} - \frac{B_{i+1,2}(x)}{(h_{i+1} + h_i + h_{i-1}}\right].$$

Then

$$B^{"}_{i,3}(x) = 3\left[\frac{B'_{i,2}(x)}{h_i + h_{i-1} + h_{i-2}} - \frac{B'_{i+1,2}(x)}{h_{i+1} + h_i + h_{i-1}}\right]. \tag{3.5}$$

Using again (3.3) for $l = 2$, it results:

$$B'_{i,2}(x) = 2\left[\frac{B_{i,1}(x)}{h_{i-1} + h_{i-2}} - \frac{B_{i+1,1}(x)}{h_i + h_{i-1}}\right], \tag{3.6}$$

$$B'_{i+1,2}(x) = 2\left[\frac{B_{i+1,1}(x)}{h_i + h_{i-1}} - \frac{B_{i+2,1}(x)}{h_{i+1} + h_i}\right]. \tag{3.7}$$

By substituting (3.6) and (3.7) into (3.5), I obtain (3.4a), □

Lemma 3.4. *For every* $i = 0, 1, ..., n + 1$, *it holds*

$$\frac{h^2}{3H^2} < B_{i,3}(\xi_i) < \frac{H^2}{3h^2} \tag{3.8}$$

$$-\frac{2}{h^2} < B''_{i,3}(\xi_i) < -\frac{2}{H^2} \tag{3.9}$$

Proof. By substituting ξ_i into (2.6) I obtain:

$$B_{i,3}(\xi_i) = \frac{1}{(h_{i-1} + h_i)}\left[\frac{h_i(h_{i-1} + h_{i-2})}{(h_i + h_{i-1} + h_{i-2})} + \frac{h_{i-1}(h_{i+1} + h_i)}{(h_i + h_{i-1} + h_{i+1})}\right]$$

But since

$$h \le h_i \le H, \text{ for every } i = 0, 1, ..., n \tag{3.10}$$

we obtain (3.8). Also substituting ξ_i into (3.4a) we have:

$$B''_{i,3}(\xi_i) = -\frac{1}{(h_{i-1} + h_i)}\left[\frac{1}{(h_i + h_{i-1} + h_{i-2})} + \frac{1}{(h_i + h_{i-1} + h_{i+1})}\right]$$

Using again (3.10), it results (3.9). □

Lemma 3.5. *If* $Q(x) < -1$ *for all* $x \in [0, 1]$, *then the elements of the matrix* A *are strictly positive.*

Proof. From (2.8)

$$B_{i,3}(\xi_l) > 0; i = l - 1, l, l + 1$$

$$B_{i,3}(\xi_{l+p}) > 0; i = l + p - 1, l + p, l + p + 1.$$

Using (3.4a)

$$B''_{i,3}(\xi_{i-1}) = \frac{3!}{(h_i + h_{i-1} + h_{i-2})(h_{i-1} + h_{i-2})} > 0,$$
$$B''_{i,3}(\xi_{i+1}) = \frac{3!}{(h_{i+1} + h_i + h_{i-1})(h_i + h_{i+1})} > 0$$

and:

$$Q(x) < 0, B_{i,3}(\xi_{i-1}) > 0, B_{i,3}(\xi_{i+1}) > 0 \text{ then } f_i(\xi_{i-1}) > 0, f_i(\xi_{i+1}) > 0.$$

Also since

$$f_i(\xi_i) = B''_{i,3}(\xi_i) - Q_i \cdot B_{i,3}(\xi_i)$$

it follows:

$$\text{If } Q_i < \frac{B''_{i,3}(\xi_i)}{B_{i,3}(\xi_i)} < -\frac{2}{H^2}\frac{3H^2}{h^2} < -\frac{1}{h^2} < -1; \text{then for all } i = 0, 1, 2, ..., n : f_i(\xi_i) > 0$$

□

Lemma 3.6. *If* $A = [a_{i,j}]$ *is a square tridiagonal matrix with all elements strict positive then* A *is* monotone.

Proof. By hypothesis $a_{i,i-1} > 0; a_{i,i} > 0; a_{i,i+1} > 0$ then, cf. Theorem 2.7, the matrix A is irreducible, and moreover

$$a_{i,i-1} + a_{i,i} + a_{i,i+1} > 0 \tag{3.11}$$

*Reductio ad absurdum.*I assume that there exists a vector z with a negative component $z_q < 0$ but such $Az \geq 0$. This assumption is equivalent to assuming that A is not monotone.I shall show that this contradicts the assumption that A is irreducible. Denote by $W := \{1, 2, ..., n\}$ and e the vector whose components are all 1. Then from (3.11) we have

$$A \cdot e > 0, A \cdot e \neq 0. \tag{3.12}$$

Since the sum of two nonnegative vectors is nonnegative, it follows that for $0 \leq \lambda \leq 1$

$$\lambda Az + (1 - \lambda)Ae = A[\lambda z + (1 - \lambda)e] > 0 \tag{3.13}$$

Consider the vector $w_\lambda = \lambda z + (1 - \lambda)e$ as a function of λ.For $\lambda = 0$ all components w_λ are positive, namely +1. For $\lambda = 1$ there is a least one negative component, namely $z_q, q \in W$. The components of w_λ are continuous functions of λ. Since $0 \leq \lambda \leq 1$, at least one component of w_λ must pass thought the value 0. Let δ the smallest value of λ such that w_λ has a zero component $(0 < \delta < 1)$. Now let S be a set of indices of zero components of w_λ and let $T = W - S$. (By construction,

$S \neq \Phi, T \neq \Phi$). For if all components of w_λ were zero, then the vectors z and e would be proportional:

$$e = -\frac{\delta}{1-\delta} z, \tag{3.14}$$

and from $Az \geq 0$ it would followed that:

$$Ae = -\frac{\delta}{1-\delta} Az \leq 0$$

contradicting (3.12). By (3.13), $Aw_\delta \geq 0$, so in particular, if $i \in S$:

$$(Aw_\delta)_i = \sum_{j \in T} a_{i\ j} w_{\delta\ j} \geq 0 \tag{3.15}$$

by construction $w_{\delta\ j} > 0$, if $j \in T$. In view of $a_{i,j} > 0$ if $j \in T$, (3.15) is thus possible if $a_{i,i-1} = a_{i,i} = a_{i,i+1} = 0$.Then A is reducible, contradicting our assumption, that implies A is monotone. □

Theorem 3.7. *If $Q(x) < -1$ the system(3.2) has a unique solution.*

Proof. Using above lemmas the system matrix A is monotone. By Theorem 2.8 A is nonsingular and moreover $\det A \neq 0$. □

To solve the system (3.2), I use *Crout Reduction for Tridiagonal Linear Systems Algorithm* (see [5, pp. 336-340]). This algorithm requires only $(5n-4)$ multiplications/divisions and $(3n-3)$ addition/subtractions, and consequently it has considerable computational advantages over the methods that do not consider the tridiagonality of the matrix, especially for large values of n.

3.2. **Error analysis.** I recall ([2, pp. 58-62]):

Theorem 3.8. *If the exact solution of (PVP) $y(x) \in C^2[0,1]$, then there exists a B-spline $B(x) \in S(\overline{\Delta})$ determined locally as follows*

$$\max_{\xi_{i-2} \leq x \leq \xi_{i+2}} |y(x) - B_i(x)| := \|y - B_i\|_{[\xi_{i-2},\xi_{i+2}]} \leq K \cdot H_1^2 \cdot \left\| y^{(2)} \right\|_{[\xi_{i-2},\xi_{i+2}]}, \tag{3.16}$$

where $H_1 := \max\{h_{i-2}, h_{i-1}, h_i, h_{i+1}\}$ and K is a real constant independent of $\overline{\Delta}$ and $y(x)$.

Since the points of $\overline{\Delta}$, except $\xi_l = a$ and $\xi_{l+p} = b$ are the roots of the kth Legendre polynomial, the orthogonality relation

$$\int_0^1 \rho(t) \prod_{j=1}^{k} (t - \rho_j) dt = 0$$

holds for all polynomials $\rho(t)$ of degree $q (2 \leq q \leq k)$, and then the superconvergence occurs at the meshpoints:

$$\left| y^{(j)}(\xi_i) - u^{(j)}_{\overline{\Delta}}(\xi_i) \right| = \mathbb{O}(H^{k+q}); 0 \leq i \leq n+2, 0 \leq j \leq 1 \tag{3.17}$$

(see [1], [4]). I use as collocation points the Gaussian points taking $q = k$.Then the *superconvergence* of my method at the meshpoints $\xi_i, i \in \{0, 1, 2, ..., n+2\} \setminus \{l, l+p\}$ is assured.

$$\left| y^{(j)}(\xi_i) - u^{(j)}_{\bar{\Delta}}(\xi_i) \right| = \mathbb{O}(H^{2k}); 0 \le i \le n+2, 0 \le j \le 1$$

Since $Q(x) \in C^1[0, 1]$, then there exists $N = \max_{0 \le x \le 1} |Q(x)|$ such that

$$\left| y''(\xi_i) - u''_{\bar{\Delta}}(\xi_i) \right| \le N \left| y(\xi_i) - u_{\bar{\Delta}}(\xi_i) \right| = N \cdot \mathbb{O}(H^{2k}).$$

In $\xi_l = a, \xi_{l+p} = b$ cf(3.16)

$$|y(\xi_l) - B_i(\xi_l)|_{[\xi_{l-2},\xi_{l+2}]} \le K_1 \cdot H^2 \cdot \left\| y^{(2)} \right\|_{[\xi_{l-2},\xi_{l+2}]}$$

$$|y(\xi_{l+p}) - B_i(\xi_{l+p})|_{[\xi_{l+p-2},\xi_{l+p+2}]} \le K_1 \cdot H^2 \cdot \left\| y^{(2)} \right\|_{[\xi_{l+p-2},\xi_{l+p+2}]}$$

where K_1, K_2 are constants, independent of $\overline{\Delta}$ and $y(x)$. It follows that my method is *superconvergent* of order $\mathbb{O}(H^2)$.

3.3. **Numerical examples.** I shall give one example. For this example, I plot the approximate solution, error in semilogarithmic scale and I generate the execution profile with the pair `profile` − `showprofile`, see ([6]).

I want to approximate the oscillating solution of the following problem:

$$Z''(t) - 50 \cdot Z(t) = \sin(t); 0 \le t \le 1 \tag{3.18}$$

with conditions:

$$Z\left(\frac{1}{6}\right) = \frac{1}{49} \frac{-\sin(\frac{5\sqrt{2}}{6}) \sin 1 + \sin \frac{1}{6} \sin(5\sqrt{2})}{\sin(5\sqrt{2})} \tag{3.19}$$

$$Z\left(\frac{3}{4}\right) = \frac{1}{49} \frac{-\sin(\frac{15\sqrt{2}}{4}) \sin 1 + \sin \frac{3}{4} \sin(5\sqrt{2})}{\sin(5\sqrt{2})}$$

The exact solution provided by `dsolve` is:

$$Z(t) = \frac{1}{49} \frac{-\sin(5\sqrt{2}t) \sin 1 + \sin t \sin(5\sqrt{2})}{\sin(5\sqrt{2})}$$

Since

$$\int_0^1 |Q(x)|\, dx > 4,$$

due to disconjugate criteria given by *Lyapunov* (1893), the problem (3.18) has an oscillatory solution. I used Maple 8 to solve the problem exactly and to approximate the solution, for $n = 10$ and $k = 3$. I obtained a very good approximation, but I must increase the number of decimals with Maple command:

> `Digits := 18;`

If I use a method based on orthogonal polynomials, for example first kind Chebyshev polynomials, I observe that the B-spline method is faster and requires less memory. The reason is that for the B-spline method the matrix of the system that provides the coefficients is a band matrix with at most 3 nonzero elements per line, while for Chebyshev method the matrix is dense. This example with oscillating solution supports this conclusion (see for more details [9]).

Here are the profiles for the procedures `genspline` and `genceb` in the case of oscillating solution to problem (3.18):

function	depth	calls	time	time	bytes	bytes
`genspline`	1	1	7.691	100.0	156424156	100.00
`genceb`	1	1	17115	100.0	156424156	100.00

The the graphs of approximate solution and the error in semilogarithmic scale are given in Figure 2 and Figure 3, respectively.

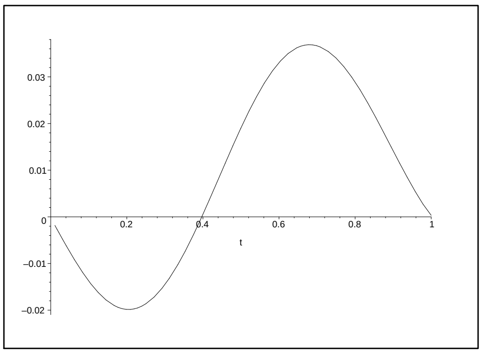

FIGURE 2. **Approximate solution** $n = 10$, $k = 3$

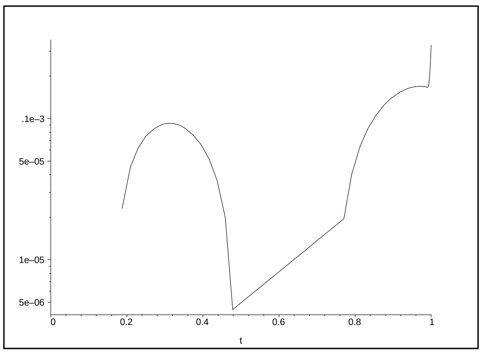

FIGURE 3. **Error plot,** $n = 10$, $k = 3$

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