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ALEXANDER TRANSFORM OF CLOSE-TO-CONVEX FUNCTIONS

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Abstract. In this paper a result concerning the starlikeness of the image of the Alexander Operator is deduced. The technique of differential subordinations is used.

1. Introduction

Let $U = \{z \in \mathbb{C} : |z| < 1$ be the open unit disc of the complex plane.

We denote by $\mathcal{A}$ the class of analytic functions defined on the unit disc U and having the form $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$.

The subclass of $\mathcal{A}$ consisting of functions for which the domain $f(U)$ is starlike with respect to 0, is called the class of starlike functions, and is denoted by S^*. An analytic description of S^* is

$$S^* = \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{zf'(z)}{f(z)} > 0,\ (\forall)\ z \in U \right\}.$$

Let $\alpha \in [0, 1)$. The class of starlike functions of order α denoted by $S^*(\alpha)$, is defined by the equality:

$$S^*(\alpha) = \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha,\ (\forall)\ z \in U \right\}.$$

Another subclass of $\mathcal{A}$ which we deal with, is defined by

$$C = \left\{ f \in \mathcal{A} \mid (\exists)\ g \in S^* : \operatorname{Re} \frac{zf'(z)}{g(z)} > 0,\ z \in U \right\}.$$

This is the class of close-to-convex functions.

We mention that C, S^* and $S^*(\alpha)$ contain univalent functions.

The Operator of Alexander is defined by

$$F(z) = A(f)(z) = \int_0^z \frac{f(t)}{t} dt. \tag{1.1}$$

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In [3] it has been proved that $A(C) \not\subset S^*$.

This result put the problem to determine suitable conditions which ensure that subclasses of C are mapped by the Alexander operator to S^*.

In [2] (pg. 310-311), the authors proved the following theorem concerning this question:

Theorem 1.1. *Let A be the operator of Alexander defined by (1.1) and let $g \in \mathcal{A}$ satisfy*

$$\operatorname{Re} \frac{zg'(z)}{g(z)} \geq \left| \operatorname{Im} \frac{z(zg'(z))'}{g(z)} \right|, \ z \in U. \tag{1.2}$$

If $f \in \mathcal{A}$ satisfies

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > 0, \ z \in U,$$

then $F = A(f) \in S^*$.

We will prove another result regarding this problem. We will need the following definitions and lemmas in our work.

2. **Preliminaries**

The class $\mathcal{P}$ is defined by the equality:

$$\mathcal{P} = \{f | f \text{ analytic in U}, \ f(0) = 1, \text{ and } \operatorname{Re} f(z) > 0, \ z \in \text{U}\}.$$

Lemma 2.1. [1]*(The Herglotz formula) For every $f \in \mathcal{P}$ there exists a measure μ on the interval $[0, 2\pi]$ so that $\mu([0, 2\pi]) = 1$ (a probability measure) and*

$$f(z) = \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} d\mu(t),$$

or in developed form

$$f(z) = 1 + 2 \sum_{n=1}^{\infty} \int_0^{2\pi} z^n e^{-in} d\mu(t).$$

The converse of the theorem is also valid.

Lemma 2.2. [2] *p.26 Let $p(z) = a + \sum\limits_{k=n}^{\infty} a_k z^k$, $p(z) \not\equiv a$ and $n \geq 1$. If $z_0 \in U$ and*

$$\operatorname{Re} p(z_0) = \min\{\operatorname{Re} p(z) : |z| \leq |z_0|\},$$

then

(i) $z_0 p'(z_0) \leq -\dfrac{n}{2} \dfrac{|p(z_0) - a|^2}{\operatorname{Re}(a - p(z_0))}$

and

(ii) $\operatorname{Re}[z_0^2 p''(z_0)] + z_0 p'(z_0) \leq 0.$

Lemma 2.3. *If $f, g \in \mathcal{A}$ and*

$$\mathrm{Re}\Big[\frac{1}{g'(z)}\int_0^1\int_0^1 g'(uvz)\frac{1+uvze^{-it}}{1-uvze^{-it}}dudv\Big] \geq 0,\ z \in U,\ t \in \mathbb{R}, \tag{2.1}$$

then the inequality $\mathrm{Re}\frac{f'(z)}{g'(z)} > 0,\ z \in U$ *implies that*

$$\mathrm{Re}\frac{F(z)}{zg'(z)} > 0,\ z \in U, \tag{2.2}$$

where F is defined by (1.1).

Proof. The developments

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

hold for $z \in U$.

The conditions of the lemma imply $\frac{f'}{g'} \in \mathcal{P}$ and from the Herglotz formula it follows that:

$$\frac{f'(z)}{g'(z)} = 1 + 2\int_0^{2\pi}\Big(\sum_{n=1}^{\infty} z^n e^{-in}\Big)d\mu(t),\ z \in U$$

for a suitable probability measure μ.

Denoting $c_n = 2\int_0^{2\pi} e^{-in} d\mu(t)$, we get:

$$f'(z) = g'(z)(1 + \sum_{n=1}^{\infty} c_n z^n)$$

$$= (1 + \sum_{n=2}^{\infty} nb_n z^{n-1})(1 + \sum_{n=1}^{\infty} c_n z^n) = 1 + \sum_{n=1}^{\infty} d_n z^n, \tag{2.3}$$

$$f(z) = z + \sum_{n=1}^{\infty} \frac{d_n}{n+1} z^{n+1}$$

and

$$\frac{F(z)}{z} = 1 + \sum_{n=1}^{\infty} \frac{d_n}{(n+1)^2} z^n.$$

Thus we have

$$\frac{F(z)}{zg'(z)} = \frac{1}{g'(z)}\int_0^1\int_0^1 \big(1 + \sum_{n=1}^{\infty} d_n u^n v^n z^n\big)dudv,$$

and according to (2.3), this is equivalent to

$$\frac{F(z)}{zg'(z)} = \frac{1}{g'(z)}\int_0^{2\pi}\int_0^1\int_0^1 g'(uvz)\frac{1+uvze^{-it}}{1-uvze^{-it}}dudvd\mu(t),$$

and the proof is finished. □

Lemma 2.4. *The following inequality holds:*

$$1+r^4u^2\rho^2-r^2\rho^2\sqrt{(1+u^2)^2+\frac{4u^2\rho^2(1-u^2)^2\sin^2\alpha}{(1-u^2\rho^2)^2(1+u^2)^2}}\cos(2\theta+\gamma)\geq$$
$$1+r^4u^2-r^2\sqrt{1+6u^2+u^4},\quad \rho,u\in[0,1];\theta,\alpha\in\mathbb{R}.$$

Proof. It is easily seen that:

$$1+r^4u^2\rho^2-r^2\rho^2\sqrt{(1+u^2)^2+\frac{4u^2\rho^2(1-u^2)^2\sin^2\alpha}{(1-u^2\rho^2)^2(1+u^2)^2}}\cos(2\theta+\gamma)\geq$$
$$1+r^4u^2\rho^2-r^2\rho^2\sqrt{(1+u^2)^2+\frac{4u^2\rho^2(1-u^2)^2}{(1-u^2\rho^2)^2(1+u^2)^2}}\tag{2.4}$$

Since

$$1\geq\rho^2$$

and

$$-r^4u^2+r^2\sqrt{1+6u^2+u^4}\geq -r^4u^2+r^2\sqrt{(1+u^2)^2+\frac{4u^2\rho^2(1-u^2)^2}{(1-u^2\rho^2)^2(1+u^2)^2}}\geq 0$$
$$r,u,\rho\in[0,1]$$

it follows that

$$-r^4u^2+r^2\sqrt{1+6u^2+u^4}\geq -r^4u^2\rho^2+r^2\rho^2\sqrt{(1+u^2)^2+\frac{4u^2\rho^2(1-u^2)^2}{(1-u^2\rho^2)^2(1+u^2)^2}}$$
$$r,u,\rho\in[0,1].$$

Thus

$$1+r^4u^2\rho^2-r^2\rho^2\sqrt{(1+u^2)^2+\frac{4u^2\rho^2(1-u^2)^2}{(1-u^2\rho^2)^2(1+u^2)^2}}\geq$$
$$1+r^4u^2-r^2\sqrt{1+6u^2+u^4}\quad r,u,\rho\in[0,1].\tag{2.5}$$

The desiderated inequality follows by (2.4) and (2.5). □

3. Main result

Theorem 3.1. *Let $g\in\mathcal{A}$ be a function having the property:*

$$Re\frac{g'(uz)}{g'(z)}\frac{1+uw}{1-uw}>0,\ \textit{for all}\ \ u\in(0,1)\ \textit{and}\ z,w\in U,|z|=|w|.\tag{3.1}$$

Provided that $f \in \mathcal{A}$, and the function h defined by $h(z) = zg'(z)$ satisfies the inequality

$$Re\frac{zf'(z)}{h(z)} > 0 \quad z \in U, \tag{3.2}$$

then $F = A(f) \in S^*$.

Proof. We differentiate twice the equality $F(z) = \int_0^z \frac{f(t)}{t}$ and we get: $zF''(z) + F'(z) = f'(z)$. If we set $p(z) = \frac{zF'(z)}{F(z)}$, then this equality can be rewritten as follows:

$$\frac{F(z)}{zg'(z)}(zp'(z) + p^2(z)) = \frac{zf'(z)}{h(z)}.$$

The conditions of the theorem imply:

$$\text{Re}\big[\frac{F(z)}{zg'(z)}(zp'(z) + p^2(z))\big] > 0, \text{ for all } z \in U. \tag{3.3}$$

If the inequality Re$p(z) > 0$ does not hold for all $z \in U$, then according to Lemma 2 (in case of $a = 1$) there is a point $z_0 \in U$ and there are two real numbers $x, y \in \mathbb{R}$ having the property:

$$p(z_0) = ix$$

$$z_0 p'(z_0) = y \leq -\frac{x^2+1}{2}.$$

Thus it follows that:

$$\text{Re}\big[\frac{F(z_0)}{z_0 g'(z_0)}(z_0 p'(z_0) + p^2(z_0))\big] = \text{Re}\frac{F(z_0)}{z_0 g'(z_0)}(y - x^2). \tag{3.4}$$

Since $\text{Re}\frac{f'(z)}{g'(z)} = \text{Re}\frac{zf'(z)}{h(z)} > 0, z \in U$, Lemma 3 and condition (3.1) lead to the inequality $\text{Re}\frac{F(z)}{zg'(z)} > 0,\ z \in U$. This inequality and (3.4) imply

$$\text{Re}\frac{z_0 f'(z_0)}{h(z_0)} = \text{Re}\big[\frac{F(z_0)}{z_0 g'(z_0)}(z_0 p'(z_0) + p^2(z_0))\big] \leq 0$$

which contradicts (3.3). The contradiction shows that Re$p(z) > 0$ for all $z \in U$, and this is equivalent to $F \in S^*$. □

Corollary 3.2. *If $Re\frac{f'(z)}{e^z} > 0$ for all $z \in U$, then $A(f) \in S^*$.*

Proof. We apply Theorem 2 to prove this assertion. In case of $g(z) = e^z - 1$, $z = re^{i\theta}$ and $w = re^{i\alpha}$, $r \in (0,1)$ the following equality holds:

$$\text{Re}\frac{g'(uz)}{g'(z)}\frac{1+uw}{1-uw} = \frac{e^{r(u-1)\cos\theta}(1-u^2r^2)}{1+u^2r^2-2ur\cos\alpha}\big\{\cos[r(1-u)\sin\theta] + \frac{2ur\sin\alpha}{1-u^2r^2}\sin[r(1-u)\sin\theta]\big\} \tag{3.5}$$

There is a real number $v \in (-\frac{\pi}{2}, \frac{\pi}{2})$ having the property $\tan v = \frac{2ur\sin\alpha}{1-u^2r^2}$. Therefore the equality (3.5) can be rewritten in the following way:

$$\mathrm{Re}\frac{g'(uz)}{g'(z)}\frac{1+uw}{1-uw} = \frac{e^{r(u-1)\cos\theta}(1-u^2r^2)}{(1+u^2r^2-2ur\cos\alpha)\cos v}\cos[r(1-u)\sin\theta - v].$$

This means that in order to prove condition (3.1) of Theorem 2, we have to prove the inequality: $\cos[r(1-u)\sin\theta - v] > 0,\ r, u \in (0,1),\ \alpha, \theta \in \mathbb{R}$.

Since $|r(1-u)\sin\theta - v| = |r(1-u)\sin\theta - \arctan\frac{2ur\sin\alpha}{1-u^2r^2}| \le r(1-u) + \arctan\frac{2ur}{1-u^2r^2} < 1-u+\arctan\frac{2u}{1-u^2}$, and $\varphi'(u) = \frac{1-u^2}{1+u^2} > 0$ where $\varphi : (0,1) \to \mathbb{R}$, $\varphi(u) = 1-u+\arctan\frac{2u}{1-u^2}$, the inequality follows $|r(1-u)\sin\theta - v| < \lim_{u\to 1}\varphi(u) = \frac{\pi}{2}$.

Thus condition (3.1) also holds, and applying Theorem 2 the proof is done. □

Remark 3.3. In case of $g(z) = e^z - 1$, it is easily seen that $g \in \mathcal{A}$ and $h(z) = zg'(z) = ze^z$ and $\mathrm{Re}\big(\frac{zh'(z)}{h(z)}\big) = \mathrm{Re}(1+z) > 0,\ z \in U$, consequently $h \in S^*$ holds. Thus the differential inequality $Re\frac{zf'(z)}{h(z)} = Re\frac{f'(z)}{e^z} > 0,\ z \in U$, defines a subclass of C and this subclass is mapped by the Operator of Alexander in S^*.

Corollary 3.4. *If* $0 < r \le \big(3 - 8^{\frac{1}{2}}\big)^{\frac{1}{4}} = 0,643...$ *and*

$$Re(1-r^2z^2)f'(z) > 0, \quad z \in U, \tag{3.6}$$

then $A(f) \in S^*$.

Proof. We apply again Theorem 2 to prove this assertion. Let $g : U \to \mathbb{C}$ be the mapping defined by the equality: $g(z) = \frac{1}{2r}\log\frac{1+rz}{1-rz},\ r \in (0,1]$, and $h(z) = zg'(z) = \frac{z}{1-r^2z^2}$. We have to prove condition (3.1) in case of $z = \rho e^{i\theta}$ and $w = \rho e^{i\alpha}$. The following equalities hold:

$$\mathrm{Re}\frac{g'(uz)}{g'(z)}\frac{1+uw}{1-uw} = \mathrm{Re}\frac{1-r^2\rho^2e^{2i\theta}}{1-r^2u^2\rho^2e^{2i\theta}}\frac{1+u\rho e^{i\alpha}}{1-u\rho e^{i\alpha}} =$$

$$\frac{(1-u^2\rho^2)[1+r^4u^2\rho^2-r^2\rho^2(1+u^2)\cos 2\theta + 2\frac{1-u^2}{1-u^2\rho^2}ur^2\rho^3\sin 2\theta\sin\alpha]}{|1-r^2u^2e^{2i\theta}|^2|1-ue^{i\alpha}|^2}. \tag{3.7}$$

According to (3.7) condition (3.1) holds if and only if:

$$1+r^4u^2\rho^2-r^2\rho^2(1+u^2)\cos 2\theta + 2\frac{1-u^2}{1-u^2\rho^2}ur^2\rho^3\sin 2\theta\sin\alpha \ge 0,$$
$$\rho, u \in [0,1]; \theta, \alpha \in \mathbb{R},$$

and this is equivalent to

$$1+r^4u^2\rho^2-r^2\rho^2(1+u^2)\big[\cos 2\theta - 2\frac{1-u^2}{(1-u^2\rho^2)(1+u^2)}u\rho\sin 2\theta\sin\alpha\big] \ge 0,$$
$$\rho, u \in [0,1]; \theta, \alpha \in \mathbb{R}.$$

Using the notation $\tan\gamma = \frac{2u\rho(1-u^2)\sin\alpha}{(1-u^2\rho^2)(1+u^2)},\ \gamma \in (-\frac{\pi}{2}, \frac{\pi}{2})$ it can be rewritten as follows:

$$1 + r^4u^2\rho^2 - r^2\rho^2\sqrt{(1+u^2)^2 + \frac{4u^2\rho^2(1-u^2)^2\sin^2\alpha}{(1-u^2\rho^2)^2(1+u^2)^2}}\cos(2\theta+\gamma) \geq 0,$$
$$u, \rho \in [0,1]; \theta, \alpha \in \mathbb{R}. \qquad (3.8)$$

According to Lemma 4 we have:

$$1 + r^4u^2\rho^2 - r^2\rho^2\sqrt{(1+u^2)^2 + \frac{4u^2\rho^2(1-u^2)^2\sin^2\alpha}{(1-u^2\rho^2)^2(1+u^2)^2}}\cos(2\theta+\gamma) \geq$$
$$1 + r^4u^2 - r^2\sqrt{1+6u^2+u^4}, \rho, u \in [0,1]; \theta, \alpha \in \mathbb{R}.$$

Inequality (3.8) holds provided that:

$$1 + r^4u^2 - r^2\sqrt{1+6u^2+u^4} \geq 0,\ u \in [0,1].$$

The last inequality is equivalent to

$$1 - r^4 - 4r^4u^2 - r^4(1-r^4)u^4 \geq 0,\ u \in [0,1],$$

which holds for all $u \in [0,1]$ if and only if:

$$1 - 6r^4 + r^8 \geq 0, \quad r \in (0,1]$$

and this leads to $0 < r \leq \left(3 - 8^{\frac{1}{2}}\right)^{\frac{1}{4}}$. □

Remark 3.5. 1. Since $g, h \in \mathcal{A}$ and

$$\mathrm{Re}\frac{zh'(z)}{h(z)} = \mathrm{Re}\frac{1+r^2z^2}{1-r^2z^2} > 0, z \in U,\ \ r \in [0,1],$$

follows that $h \in S^*$. Thus condition (3.6) defines a subclass of C.

2. It remains an interesting open question to determine the biggest $r \in [0,1]$ for which the class of analytic functions defined by the conditions

$$f \in \mathcal{A},\ \mathrm{Re}(1 - r^2z^2)f'(z) > 0,\ z \in U$$

is mapped in S^*, by the Alexander Operator.

3. Since Corollary 1 and Corollary 2 can not be proved using Theorem 1, we may assert that Theorem 2 is independent from Theorem 1, in spite of the fact, that the ideas of their proofs are analogous.

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