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THE RATE OF APPROXIMATION OF FUNCTIONS IN AN INFINITE INTERVAL BY POSITIVE LINEAR OPERATORS

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Abstract. We obtain an estimation, in the uniform norm, of the rate of the approximation by positive linear operators of functions defined on the positive half line that have a finite limit at the infinity.

1. **Introduction**

Let us denote by $C^*[0,\infty)$, the Banach space of all real-valued continuous functions on $[0,\infty)$ with the property that $\lim_{x\to\infty} f(x)$ exists and is finite, endowed with the uniform norm. In [2], it is proved the following theorem:

Theorem 1.1. *If the sequence $A_n : C^*[0,\infty) \to C^*[0,\infty)$ of positive linear operators satisfies the conditions*

$$\lim_{n\to\infty} A_n(e^{-kt}, x) = e^{-kx}, \quad k = 0, 1, 2,$$

uniformly in $[0,\infty)$, then

$$\lim_{n\to\infty} A_n f(x) = f(x),$$

uniformly in $[0,\infty)$, for every $f \in C^*[0,\infty)$.

In [1], it is proved the above theorem in a more general setting. In the same book, the authors give the results for the particular operators of Szász-Mirakjan, of Baskakov and of Bernstein-Chlodovsky.

In the following, we obtain an estimation of the rate of convergence of operators satisfying the conditions from the above theorem, first, in the general form and then, for the particular cases presented above. For this estimation, we use the following modulus of continuity:

$$\omega^*(f,\delta) = \sup_{\substack{x,t\geq 0 \\ |e^{-x}-e^{-t}|\leq\delta}} |f(x) - f(t)|,$$

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defined for every $\delta \geq 0$ and every function $f \in C^*[0,\infty)$. This modulus can be expressed in terms of the usual modulus of continuity, by the relation:

$$\omega^*(f,\delta) = \omega(f^*,\delta),$$

where f^* is the continuous function defined on $[0,1]$ by

$$f^*(x) = \begin{cases} f(-\ln x), & x \in (0,1] \\ \lim_{t\to\infty} f(t), & x = 0. \end{cases}$$

Remark 1.2. Because $|e^{-t} - e^{-x}| \leq |t-x|$, for every $t, x \geq 0$, we have for $\delta \geq 0$

$$\omega(f,\delta) \leq \omega^*(f,\delta),$$

and because $|e^{-t} - e^{-x}| = e^{-\theta}|t-x| \geq e^{-M}|t-x|$, for every $t, x \in [0,M]$, we have

$$\omega^*(f,\delta) \leq \omega(f, e^M\delta) \leq (1+e^M)\cdot\omega(f,\delta).$$

2. Main result

Theorem 2.1. *If $A_n : C^*[0,\infty) \to C^*[0,\infty)$ is a sequence of positive linear operators with*

$$\begin{aligned} \|A_n 1 - 1\|_\infty &= a_n, \\ \|A_n(e^{-t},x) - e^{-x}\|_\infty &= b_n, \\ \left\|A_n(e^{-2t},x) - e^{-2x}\right\|_\infty &= c_n, \end{aligned}$$

where a_n, b_n and c_n tend to zero as n goes to the infinity, then

$$||A_n f - f||_\infty \leq \|f\|_\infty a_n + (2+a_n)\cdot\omega^*\left(f, \sqrt{a_n + 2b_n + c_n}\right),$$

for every function $f \in C^*[0,\infty)$.

Proof. Using the property of the usual modulus of continuity

$$|F(u) - F(v)| \leq \left(1 + \frac{(u-v)^2}{\delta^2}\right)\omega(F,\delta),$$

for the function $F = f^*$ and for $u = e^{-t}$ and $v = e^{-x}$ and using the relation $f^*(e^{-t}) = f(t)$, we obtain

$$|f(t) - f(x)| \leq \left(1 + \frac{(e^{-t}-e^{-x})^2}{\delta^2}\right)\omega^*(f,\delta).$$

Because

$$A_n((e^{-t}-e^{-x})^2,x) = [A_n(e^{-2t},x) - e^{-2x}] - 2e^{-x}[A_n(e^{-t},x) - e^{-x}] + e^{-2x}[A_n(1,x) - 1]$$

we obtain

$$\begin{aligned} A_n(|f(t)-f(x)|,x) &\leq \left(A_n(1,x)+\frac{A_n((e^{-t}-e^{-x})^2,x)}{\delta^2}\right)\omega^*(f,\delta) \\ &\leq \left(1+a_n+\frac{a_n+2b_n+c_n}{\delta^2}\right)\omega^*(f,\delta). \end{aligned}$$

Choosing $\delta=\sqrt{a_n+2b_n+c_n}$ and using the inequality

$$|A_nf(x)-f(x)|\leq |f(x)|\cdot|A_n(1,x)-1|+A_n(|f(t)-f(x)|,x),$$

we obtain, in the uniform norm, the estimation stated in the theorem. □

Remark 2.2. Because all positive linear operators L can be modified to preserve constant functions, $\tilde{L}f=\frac{1}{L1}Lf$, we can take $a_n=0$ in the theorem above and obtain:

$$||A_nf-f||_\infty\leq 2\cdot\omega^*(f,\sqrt{2b_n+c_n}).$$

Remark 2.3. If we restrict ourselves on a compact interval $[0,M]$ and if we use the Remark 1.2, we obtain an estimation using the usual modulus of continuity:

$$||A_nf-f||_\infty\leq C\cdot\omega\left(f,\sqrt{2b_n+c_n}\right).$$

We have used the Korovkin subset $\left\{1,e^{-x},e^{-2x}\right\}$ for $C^*[0,\infty)$, but as suggested in the article [3], we can use any other Korovkin subset for this space, such as for example $\left\{1,\frac{x}{1+x},\frac{x^2}{(1+x)^2}\right\}$. In this case we can introduce

$$\omega^\#(f,\delta)=\sup_{\substack{x,t\geq 0\\ \left|\frac{x}{1+x}-\frac{t}{1+t}\right|\leq\delta}}|f(x)-f(t)|,$$

defined for every $\delta\geq 0$ and every function $f\in C^*[0,\infty)$. This modulus can be expressed in terms of the usual modulus of continuity, by the relation:

$$\omega^\#(f,\delta)=\omega(f^\#,\delta),$$

where $f^\#$ is the continuous function defined on $[0,1]$ by

$$f^\#(x)=\begin{cases} f\left(\frac{x}{1-x}\right), & x\in[0,1) \\ \lim_{t\to\infty}f(t), & x=1. \end{cases}$$

Because of $\left|\frac{x}{1+x}-\frac{t}{1+t}\right|\leq|x-t|$, where $x,t\geq 0$, we have

$$\omega(f,\delta)\leq\omega^\#(f,\delta),$$

and because $\left|\frac{x}{1+x}-\frac{t}{1+t}\right|\geq\frac{|x-t|}{(1+M)^2}$, for $x,t\in[0,M]$, we obtain

$$\omega^\#(f,\delta)\leq\omega(f,(1+M)^2\delta)\leq(1+M)^2\cdot\omega(f,\delta),$$

where $M>0$, is an integer. We have the following

Theorem 2.4. *If $A_n : C[0,\infty) \to C[0,\infty)$ is a sequence of positive linear operators which preserves linear functions and*

$$\sup_{x\geq 0} \frac{|A_n(t^2,x)-x^2|}{(1+x)^2} = d_n,$$

is a sequence which tends to zero as n goes to the infinity, then

$$||A_n f - f||_\infty \leq 2\cdot \omega^{\#}\left(f, \sqrt{d_n}\right),$$

for every function $f \in C^*[0,\infty)$.

Proof. Using the property of the usual modulus of continuity

$$|F(u)-F(v)| \leq \left(1+\frac{(u-v)^2}{\delta^2}\right)\omega(F,\delta),$$

for the function $F = f^{\#}$ and for $u = t/(1+t)$ and $v = x/(1+x)$ and using the relation $f^{\#}(t/(1+t)) = f(t)$, we obtain

$$|f(t)-f(x)| \leq \left[1+\frac{1}{\delta^2}\left(\frac{t}{1+t}-\frac{x}{1+x}\right)^2\right]\omega^{\#}(f,\delta) \leq \left(1+\frac{(t-x)^2}{\delta^2(1+x)^2}\right)\omega^{\#}(f,\delta).$$

Because

$$A_n(t-x)^2,x) = A_n(t^2,x)-x^2$$

we obtain

$$|A_n f(x)-f(x)| \leq A_n(|f(t)-f(x)|,x) \leq \left(1+\frac{d_n}{\delta^2}\right)\omega^{\#}(f,\delta).$$

Choosing $\delta = \sqrt{d_n}$ we obtain, in the uniform norm, the estimation stated in the theorem. □

3. **Applications**

In order to obtain particular results, we use the following

Lemma 3.1. *For every $x > 0$ we have*

$$e^{-x\alpha_n} - e^{-x} < \frac{x_n}{2e}, \quad \textit{for every } n \geq 1,$$

where $\alpha_n = \frac{1-e^{-x_n}}{x_n}$ and $x_n > 0$, for every $n \geq 1$.

Proof. First, let us notice that

$$\max_{x>0} xe^{-cx} = \frac{1}{ec}, \text{ for every } c > 0. \tag{3.1}$$

Indeed, the point $t = 1/c$ is a maximum point for $f(t) = te^{-ct}$, $t > 0$.

Secondly, let us notice that $0 < a_n < 1$, for every $n \geq 1$. This is true, because of the inequality $1 - e^{-x} < x$, for $x \neq 0$.

Next, using the inequalities between geometric, logarithmic and arithmetic means

$$\sqrt{uv} < \frac{u-v}{\ln u - \ln v} < \frac{u+v}{2}, \quad \text{for } 0 < v < u,$$

for the values $u = e^{-x\alpha_n} > v = e^{-x} > 0$, we obtain

$$e^{-x\alpha_n} - e^{-x} < \frac{e^{-x\alpha_n} + e^{-x}}{2} \cdot x(1-\alpha_n) = \frac{1-\alpha_n}{2}\left(xe^{-x\alpha_n} + xe^{-x}\right).$$

Using (3.1), we obtain

$$e^{-x\alpha_n} - e^{-x} \leq \frac{1-\alpha_n}{2}\left(\frac{1}{e\alpha_n} + \frac{1}{e}\right) = \frac{1-\alpha_n^2}{2e\alpha_n}.$$

It remain to prove that $\frac{1-\alpha_n^2}{\alpha_n} < x_n$, which is a particular case of

$$\frac{1-\left(\frac{1-e^{-x}}{x}\right)^2}{\frac{1-e^{-x}}{x}} < x, \quad \text{for } x > 0.$$

This is equivalent with $x^2e^{-x} + 2e^{-x} - 1 - e^{-2x} < 0$, for $x > 0$, which is true by an elementary calculus argument. □

Corollary 3.2. *For the Szász-Mirakjan operators* $M_n : C^*[0,\infty) \to C^*[0,\infty)$ *defined by*

$$M_nf(x) = e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right),$$

we have for $f \in C^*[0,\infty)$, *the estimations*

$$||M_nf - f||_\infty \leq 2\cdot\omega^*\left(f, \frac{1}{\sqrt{n}}\right), \quad n \geq 1,$$

and

$$||M_nf - f||_\infty \leq 2\cdot\omega^{\#}\left(f, \frac{1}{2\sqrt{n}}\right), \quad n \geq 1.$$

Proof. We have $M_n(1,x) = 1$, so $a_n = 0$. We, also, have

$$M_n(e^{-\lambda t}, x) = e^{-\lambda x\frac{1-e^{-\lambda/n}}{\lambda/n}},$$

which gives, by Lemma 3.1

$$|M_n(e^{-\lambda t}, x) - e^{-\lambda x}| \leq \frac{\lambda}{2en}.$$

It follows that

$$b_n \leq \frac{1}{2en} \text{ and } c_n \leq \frac{1}{en}, \text{ for } n \geq 1,$$

and because

$$a_n + 2b_n + c_n \leq \frac{2}{2en} + \frac{1}{en} \leq \frac{1}{n}, \text{ for } n \geq 1,$$

we obtain the estimation stated in the theorem.

Because $M_n(t,x)=x$ and $M_n(t^2,x)=x^2+\frac{x}{n}$, we obtain

$$d_n=\sup_{x\geq 0}\frac{|M_n(t^2,x)-x^2|}{(1+x)^2}=\sup_{x\geq 0}\frac{x}{n(1+x)^2}=\frac{1}{4n}.$$

□

Corollary 3.3. *For the Baskakov operators $V_n: C^*[0,\infty)\to C^*[0,\infty)$ defined by*

$$V_nf(x)=\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}f\left(\frac{k}{n}\right),$$

we have for $f\in C^*[0,\infty)$, the estimations

$$||V_nf-f||_\infty\leq 2\cdot\omega^*\left(f,\frac{5}{2\sqrt{n}}\right),\quad n\geq 2,$$

and

$$||V_nf-f||_\infty\leq 2\cdot\omega^\#\left(f,\frac{1}{\sqrt{n}}\right),\quad n\geq 1.$$

Proof. From the identity $V_n(1,x)=1$, we deduce $a_n=0$. Computing

$$V_n(e^{-\lambda t},x)=\sum_{k=0}^{\infty}\binom{-n}{k}(-xe^{-\lambda/n})^k(1+x)^{-n-k}=\left(-xe^{-\lambda/n}+1+x\right)^{-n},$$

we obtain

$$\begin{aligned}|V_n(e^{-\lambda t},x)-e^{-\lambda x}| &= |[1+x(1-e^{-\lambda/n})]^{-n}-e^{-\lambda x}|\\ &= e^{-\lambda x}\left|e^{-n\ln\left(1+x(1-e^{-\lambda/n})\right)+\lambda x}-1\right|\\ &\leq \left[-n\ln\left(1+x(1-e^{-\lambda/n})\right)+\lambda x\right]\cdot e^{-n\ln\left(1+x(1-e^{-\lambda/n})\right)},\end{aligned}$$

where, we have used the inequality $e^t-1\leq te^t$ for

$$t=-n\ln\left(1+x(1-e^{-\lambda/n})\right)+\lambda x\geq -nx(1-e^{-\lambda/n})+\lambda x\geq -nx\cdot\frac{\lambda}{n}+\lambda x=0.$$

Because $\ln(1+t)\geq t/(1+t)$, for every $t\geq 0$, we obtain

$$\begin{aligned}|V_n(e^{-\lambda t},x)-e^{-\lambda x}| &\leq \frac{-nx(1-e^{-\lambda/n})+\lambda x+\lambda x^2(1-e^{-\lambda/n})}{\left(1+x(1-e^{-\lambda/n})\right)^{n+1}}\\ &\leq \frac{-nx(1-e^{-\lambda/n})+\lambda x+\lambda x^2(1-e^{-\lambda/n})}{1+(n+1)x(1-e^{-\lambda/n})+\frac{n(n+1)}{2}x^2(1-e^{-\lambda/n})^2}.\end{aligned}$$

Because $1-e^{-\lambda/n}\geq \lambda/n-\lambda^2/(2n^2)$, we get from the above inequality

$$\sup_{x\geq 0}|V_n(e^{-\lambda t},x)-e^{-\lambda x}|\leq\frac{2\lambda}{n(n+1)(1-e^{-\lambda/n})}.$$

Using the same inequality, we obtain

$$b_n = \sup_{x\geq 0} |V_n(e^{-t}, x) - e^{-x}| \leq \frac{2}{n(n+1)\left(\frac{1}{n} - \frac{1}{2n^2}\right)} \leq \frac{2}{n}, \quad \text{for } n \geq 1$$

and using $1 - e^{-2/n} \geq 2/n - 2/n^2 + 4/(3n^3) - 2/(3n^4)$, we have

$$c_n = \sup_{x\geq 0} |V_n(e^{-2t}, x) - e^{-2x}| \leq \frac{4}{n(n+1)\left(\frac{2}{n} - \frac{2}{n^2} + \frac{4}{3n^3} - \frac{2}{3n^4}\right)} = \frac{h(n)}{n},$$

where $h(t) = 6t^4/((t+1)(3t^3 - 3t^2 + 2t - 1))$. Because

$$h'(t) = \frac{6t^3}{(t+1)^2(3t^3 - 3t^2 + 2t - 1)}(-2t^2 + 3t - 4) < 0, \quad t \geq 1,$$

we obtain $h(n) \leq h(2) = 32/15$, for $n \geq 2$. Finally, we obtain

$$\sqrt{a_n + 2b_n + c_n} \leq \frac{1}{\sqrt{n}}\sqrt{4 + \frac{32}{15}} \leq \frac{5}{2\sqrt{n}}.$$

Because $V_n(t, x) = x$ and $V_n(t^2, x) = x^2 + x(1 + x)/n$, we obtain

$$d_n = \sup_{x\geq 0} \frac{|V_n(t^2, x) - x^2|}{(1 + x)^2} = \sup_{x\geq 0} \frac{x}{n(1 + x)} = \frac{1}{n}.$$

□

Corollary 3.4. *For the Bernstein-Chlodovsky operators* $C_n : C^*[0, \infty) \to C^*[0, \infty)$ *defined by*

$$C_n f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\beta_n\right)\binom{n}{k}\left(\frac{x}{\beta_n}\right)^k\left(1 - \frac{x}{\beta_n}\right)^{n-k},$$

for $0 \leq x \leq \beta_n$ *and* $C_n f(x) = f(x)$, *for* $x > \beta_n$, *where* β_n *is a sequence of positive numbers such that*

$$\lim_{n\to\infty} \beta_n = \infty \text{ and } \lim_{n\to\infty} \frac{\beta_n}{n} = 0,$$

we have for $f \in C^*[0, \infty)$, *the estimations*

$$||C_n f - f||_\infty \leq 2 \cdot \omega^*\left(f, \sqrt{\frac{\beta_n}{n}}\right), \quad n \geq 1,$$

and

$$||C_n f - f||_\infty \leq 2 \cdot \omega^\#\left(f, \sqrt{\frac{\beta_n}{4n}}\right), \quad n \geq 1.$$

Proof. From the identity $C_n(1, x) = 1$, we deduce $a_n = 0$. Computing

$$C_n(e^{-\lambda t}, x) = \sum_{k=0}^{n}\binom{n}{k}\left(\frac{x}{\beta_n}e^{-\lambda\beta_n/n}\right)^k\left(1 - \frac{x}{\beta_n}\right)^{n-k} = \left(e^{-\lambda\beta_n/n}\frac{x}{\beta_n} + 1 - \frac{x}{\beta_n}\right)^n,$$

we obtain

$$\begin{aligned}|C_n(e^{-\lambda t},x)-e^{-\lambda x}| &= \left|\left(1-\lambda x\frac{1-e^{-\lambda\beta_n/n}}{\lambda\beta_n}\right)^n - e^{-\lambda x}\right| \\ &= \left|e^{n\ln\left(1-\frac{x}{\beta_n}(1-e^{-\lambda\beta_n/n})\right)} - e^{-\lambda x}\right| \\ &\le e^{-\lambda x\frac{1-e^{-\lambda\beta_n/n}}{\lambda\beta_n/n}} - e^{-\lambda x},\end{aligned}$$

because $\ln(1-t)\le -t$, for every $t\in(0,1)$. Using Lemma 3.1, we obtain

$$|C_n(e^{-\lambda t},x)-e^{-\lambda x}| \le \frac{\lambda\beta_n}{2en}.$$

This gives the estimations

$$b_n \le \frac{\beta_n}{2en} \text{ and } c_n \le \frac{\beta_n}{en}, \text{ so } a_n+2b_n+c_n \le \frac{\beta_n}{n}.$$

Because $C_n(t,x)=x$ and $C_n(t^2,x)=x^2+\frac{x(\beta_n-x)}{n}$, we obtain

$$d_n = \sup_{x\ge 0}\frac{|C_n(t^2,x)-x^2|}{(1+x)^2} = \sup_{x\in[0,\beta_n]}\frac{x(\beta_n-x)}{n(1+x)^2} = \frac{\beta_n^2}{4n(1+\beta_n)} \le \frac{\beta_n}{4n}.$$

□

Corollary 3.5. *For the Bleimann-Butzer-Hahn operators* $L_n\colon C^*[0,\infty)\to C^*[0,\infty)$ *defined by*

$$L_n(f,x)=\sum_{k=0}^n \binom{n}{k} x^k(1+x)^{-n} f\left(\frac{k}{n-k+1}\right)$$

we have

$$\|L_nf-f\|_\infty \le 2\cdot\omega^{\#}\left(f,\frac{2}{\sqrt{n+1}}\right),\quad n\ge 1,\quad f\in C^*[0,\infty).$$

Proof. For the proof, we use the argument from Theorem 2.1 for the test functions $x^k/(x+1)^k$ instead of e^{-kx} and the modulus $\omega^{\#}(f,\delta)$ instead of $\omega^*(f,\delta)$.

Because $L_n(1,x)=1$ we have $a_n=\|L_n1-1\|_\infty=0$. From the equalities (see [5])

$$\begin{aligned}L_n\left(\frac{t}{1+t},x\right) &= \frac{nx}{(1+n)(1+x)} \\ L_n\left(\left(\frac{t}{1+t}\right)^2,x\right) &= \frac{n^2x^2}{(1+n)^2(1+x)^2}+\frac{nx}{(1+n)^2(1+x)^2}\end{aligned}$$

we obtain

$$b_n=\sup_{x\ge0}\left|L_n\left(\frac{t}{1+t},x\right)-\frac{x}{1+x}\right| = \frac{1}{n+1}$$

and

$$c_n = \sup_{x\geq 0}\left|L_n\left(\left(\frac{t}{1+t}\right)^2, x\right) - \left(\frac{x}{1+x}\right)^2\right| = \sup_{x\geq 0}\frac{\left|nx - x^2(2n+1)\right|}{(1+n)^2(1+x)^2}.$$

After some computations $c_n = \frac{2n+1}{(n+1)^2}$, which gives

$$a_n + 2b_n + c_n \leq \frac{4}{n+1},$$

and so the corollary is proved. □

Remark 3.6. In the papers [5] and [4], it is defined the space H_w: for a function w of the type of modulus of continuity, having the properties:

(i) w is non-negative increasing function on $[0,\infty)$,

(ii) $\lim_{\delta\to 0} w(\delta) = 0$,

the space H_w consists of all real-valued functions f defined on the semiaxis $[0,\infty)$, satisfying the following condition:

$$|f(x) - f(y)| \leq w\left(\left|\frac{x}{1+x} - \frac{y}{1+y}\right|\right), \quad \text{for all } x, y \geq 0.$$

It is proved that $H_w \subset C[0,\infty) \cap B[0,\infty)$ and $\|L_n f - f\|_\infty \to 0$, for $f \in H_w$. But, let us notice that $H_w \subset C^*[0,\infty)$. Indeed, considering $\varphi(x) = x/(1-x)$, $x \in [0,1)$, the inverse of the function $t \mapsto t/(1+t)$ and considering $f \in H_w$, we have

$$\left|f\left(\frac{u}{1-u}\right) - f\left(\frac{v}{1-v}\right)\right| \leq w(|u-v|), \quad \text{for all } u, v \in [0,1).$$

Using the property (ii) of w, we deduce that $f \circ \varphi$ is uniformly continuous on $[0,1)$. From this, it follows that $f\circ\varphi$ has finite limit at $x = 1$, which proves that f has finite limit at infinity.

So, the result obtained in Corollary 3.5 for the space $C^*[0,\infty)$ is more general than the results obtained in the papers mentioned above.

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