STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LV**, Number 2, June 2010

AN INVERSION OF ONE CLASS OF INTEGRAL OPERATOR BY L. A. SAKHNOVICH'S OPERATOR IDENTITY METHOD

RAED HATAMLEH, AHMAD QAZZA, AND MOHAMMAD AL-HAWARI

Abstract. An inversion problem of integral operator in the form

$$Sf = \frac{d^3}{dx^3}\int_0^{\omega} S(x,t)f(t)dt$$

under the condition that the kernel $S(x,t)$ satisfies the equation

$$(\partial_x^3 + \partial_t^3)S(x,t) = 0$$

is investigated. It was proved that the operator $A_0S - SA_0^*$ is finite if $A_0 = J^3$, where $Jf = i\int_0^x f(t)dt$. Presentation for the inverse operator $T = S^{-1}$ is obtained and it's structure is studied.

1. Introduction

An inversion of some classes of the integral operators S is based on use of operator identities in the form $A_0S - SA_0^*$ or $S - T_0ST_0^*$. The main idea of the operator identity method lies in the fact, that, if the operator $B = A_0S - SA_0^*$ is a projector on a finite-dimensional subspace, then the inversion of the integral operator reduced to the inversion on a finite number of specific functions, the number of function is equal to the dimension of the finite-dimensional subspace, mentioned above. Thus, in general case the inversion of the integral operator is reduced to the selection of the operator A_0 and is determined by the finite number of partial solutions of corresponding integral equation.

The concept, first, was realized by V. A. Ambartzumyan. However, as the operator A_0, for the integral equation with kernel, depending on the difference, he used the operator of differentiation that leads to some difficulties in verification of

Received by the editors: 07.01.2009.
2000 *Mathematics Subject Classification.* 47G10.
Key words and phrases. Linear bounded operator, operator identity, inverse operator, operator representation.

the operator identity method. For the kernel, depending on the difference, L. A. Sakhnovich [3] proposed to use

$$A_0 = i \int_0^x f(t)dt,$$

the integral operator acting in $L^2[0, \omega]$ space.
The significant point here is the fact that the integral equation kernel, depending on the difference, satisfies the equation

$$\frac{\partial}{\partial x} S(x,t) = -\frac{\partial}{\partial t} S(x,t),$$

that allows to use the operator identity method effectively and to find the structure of the inverse integral operator.
Later Sakhnovich's idea was generalized in different directions [4]-[2].

The problem, concerning the inversion of the integral operator in the form

$$Sf = \frac{d^3}{dx^3} \int_0^\omega S(x,t)f(t)dt,$$

is investigated in this article under the condition that the kernel $S(x,t)$ satisfies the equation

$$\frac{\partial^3}{\partial x^3} S(x,t) + \frac{\partial^3}{\partial t^3} S(x,t) = 0.$$

It was proved that if the operator A_0 is in the form

$$A_0 f = -i \int_0^x \int_0^y \int_0^z f(t)dtdzdy = -\frac{i}{2} \int_0^x (x-t)^2 f(t)dt,$$

then the operator

$$A_0 S - S A_0^*,$$

is finite-dimensional.

The representation of the inverse operator is obtained and it is structure is investigated.

2. **The operator identity**

The general idea of this method can be summarized as follows. Consider an operator kernel S such that, $S(x,t) \in L^2([0,\omega] \times [0,\omega])$ and satisfying the equation

$(D_x \pm D_t)S(x,t) = 0$, where D_x is differential or integro-differential operator. Then, if

$$Sf = D_x \int_0^\omega S(x,t)f(t)dt$$

and the corresponding form of the operator A is chosen (often the operator D_x^{-1} may be used as operator A) so that $A_0S - SA_0$ is finite-dimensional. Then the evaluation of the inverse operator S^{-1} is reduced to the inversion of the operator S on the finite numbers of functions.

Currently, we suppose that,

$$\begin{aligned} f(x) &\in L^2[0,\omega], \\ S(x,t) &\in L^2([0,\omega]\times[0,\omega]), \end{aligned}$$

and that

$$g(x) = \int_0^\omega f(t)S(x,t)\ dt,$$

is absolutely continuous on the segment $[0,\omega]$.

Let

$$Jf = i\int_0^x f(t)dt,$$

then

$$J^*f = -i\int_x^\omega f(t)\ dt, \qquad J^2f = \int_0^x (t-x)\ f(t)\ dt,$$

$$J^{*^2}f = \int_x^\omega (x-t)\ f(t)\ dt, \qquad J^3f = -\tfrac{i}{2}\int_0^x (x-t)^2\ f(t)\ dt, \text{ and}$$

$$J^{*^3}f = \tfrac{i}{2}\int_x^\omega (x-t)^2\ f(t)\ dt.$$

Lemma 2.1. (On representation of a linear bounded operator in $L^2[0,\omega]$) *Any bounded operator $S \in [L^2[0,\omega]\times L^2[0,\omega]]$ is representable in the form*

$$Sf = \frac{d^3}{dx^3}\int_0^\omega S(x,t)f(t)dt,$$

where $S(x,t) \in L^2[0,\omega]$ at any fixed x.

Proof. Consider the function

$$\ell_x(t) = \begin{cases} \frac{(x-t)^2}{2}, & \text{when } t \le x \\ 0, & \text{when } t > x. \end{cases}$$

Then for the scalar product we have

$$\langle Sf, \ell_x \rangle = \int_0^x (Sf) dt = \langle f, S^* \ell_x \rangle .$$

Let us denote $S(x,t)$ by $S^*\ell_x$ at any fixed x.
Then

$$\langle f, S^* \ell_x \rangle = \int_0^\omega S(x,t) f(t) dt.$$

On the other hand, denoting $g(x)$ by Sf, we get

$$\langle Sf, \ell_x \rangle = \langle g, \ell_x \rangle = \frac{1}{2} \int_0^x (x-t)^2 g(t) dt.$$

So that,

$$\frac{1}{2} \int_0^x (x-t)^2 g(t) dt = \int_0^\omega S(x,t) g(t) dt.$$

And differentiating by x three times we get the representation

$$g(x) = Sf = \frac{d^3}{dx^3} \int_0^\omega S(x,t) f(t) dt.$$ □

Let $Df = \frac{d}{dx} f(x)$, $A_0 f = J^3 f$. Consider the operator

$$Sf = \frac{d^3}{dx^3} \int_0^\omega S(x,t) f(t) dt. \tag{2.1}$$

Then the next theorem holds.

Theorem 2.2. *For a bounded operator of the form (2.1) with the kernel $S(x,t)$, satisfying the equation*

$$\left(D_x^3 + D_t^3\right) S(x,t) = 0, \tag{2.2}$$

there holds an equality (operator identity)

$$\begin{aligned} \left(A_0 S - S A_0^*\right) f \quad &= \quad i \int_0^\omega f(t) (\frac{x^2}{2} N''(t) - \frac{t^2}{2} M''(t) + x N'(t) \\ &\qquad -t M'(x) + N(t) - M(x)) dt, \end{aligned} \tag{2.3}$$

where,

$$\begin{aligned} M(x) &= S(x,0), & N(t) &= S(0,t),\\ M'_x(x) &= S'_t(x,0), & N'(t) &= S'_t(0,t),\\ M''_x(x) &= S''_t(x,0), & N''(t) &= S''_t(0,t). \end{aligned}$$

Proof. Integrating by parts and using the equation for the kernel we obtain

$$\begin{aligned} A_0Sf &= i\int_0^x \left(xt - \frac{x^2}{2} - \frac{t^2}{2}\right)\frac{d^3}{dt^3}\int_0^\omega S(t,y)f(y)dydt\\ &= i\int_0^\omega f(t)\left(\frac{x^2}{2}S''_{xx}(0,t) + xS'_x(0,t) + S(0,t) - S(x,t)\right)dt. \end{aligned}$$

Similarly,

$$\begin{aligned} SA_0^*f &= i\frac{d^3}{dx^3}\int_0^\omega\left(\int_t^\omega\left(\frac{y^2}{2} + \frac{t^2}{2} - t_y\right)f(y)dy\right)S(x,t)dt\\ &= i\int_0^\omega f(t)\left(\frac{t^2}{2}S''_{tt}(x,0) + tS'_t(x,0) - S(x,t)\right)dt. \end{aligned}$$

And subtracting the equalities obtained above and using (2.3) we get the assertion of the Theorem 2.2. □

From the above Theorem it follows that the operator $A_0S - SA_0^*$ maps $L^2[0,\omega]$ onto six-dimensional space, stretched on the functions

$$1, x, \frac{x^2}{2}, M(x), M'(x), M''(x).$$

Really,

$$\begin{aligned} (AS - SA_0^*)f &= i\Big\{\left(f,\overline{N''}\right)\frac{x^2}{2} - \left(f,\frac{t^2}{2}\right)M'' + \left(f,\overline{N'}\right)x\\ &\qquad -(f,t)M'(x) + \left(f,\overline{N}\right)1 - (f,1)\,M(x)\Big\}. \end{aligned}$$

Corollary 2.3. *If there exists a bounded operator T, which is the inverse to the operator S, then the following equality holds*

$$(TA_0 - A_0^*T)\,f = i\int_0^\omega f(t)\sum_{i=1}^{6}\overline{M_i(t)}N_i(t)dt, \tag{2.4}$$

where,

$$\begin{array}{cc} S^*M_1(t)=\overline{N''}(x), & SN_1(t)=\frac{x^2}{2} \\ S^*M_2(t)=-\frac{x^2}{2}, & SN_2(t)=M''(x) \\ S^*M_3(t)=\overline{N'}(x), & SN_3(t)=x \\ S^*M_4(t)=-x, & SN_4(t)=M'(x) \\ S^*M_5(t)=\overline{N}(x), & SN_5(t)=1 \\ S^*M_6(t)=-1, & SN_6(t)=M(x). \end{array} \tag{2.5}$$

Proof.

$$(TA_0-A_0^*T)\,f=T\,(A_0S-SA_0^*)\,Tf$$

$$=iT\int\limits_0^\omega Tf\Big[\frac{x^2}{2}N''(t)-\frac{t^2}{2}M''(x)+xN'(t)-tM'(x)+N(t)-M(x)\Big]dt$$

$$=iT\Big\{\left(Tf,\overline{N''}\right)\frac{x^2}{2}-(Tf,\frac{t^2}{2})M''+(Tf,\overline{N'})x-(Tf,t)M'(x)$$
$$+(Tf,\overline{N})-(Tf,1)M(x)\Big\}$$

$$=iT\Big\{\left(f,T^*\overline{N''}\right)\frac{x^2}{2}-(f,T^*\frac{x^2}{2})M''(x)+(f,T^*\overline{N'})x$$
$$-(f,T^*x)M'(x)+(f,T^*\overline{N})-(f,T^*1)M(x)\Big\}$$

$$=iT\int\limits_0^\omega f(t)\Big[\overline{T^*\overline{N''}}\frac{x^2}{2}-\overline{T^*\frac{x^2}{2}}M''(x)+\overline{T^*\overline{N'}}x-\overline{T^*x}M'(x)$$
$$+\overline{T^*\overline{N}}-\overline{T^*1}M(x)\Big]dt$$

$$=i\int\limits_0^\omega f(t)\Big[T\frac{x^2}{2}\overline{T^*\overline{N''}}-TM''(x)\overline{T^*\frac{x^2}{2}}+Tx\overline{T^*\overline{N'}}$$
$$-TM'(x)\overline{T^*x}+T1\overline{T^*\overline{N}}-TM(x)\overline{T^*1}\Big]dt$$

$$=i\int\limits_0^\omega f(t)\sum_{i=1}^{6}\overline{M_i(t)}\;N_i(t)\;dt. \qquad \square$$

3. Representation for the inverse operator

Let $N_k(x)$, $M_k(x)$ $(k=\overline{1,6})$ be functions in $L^2[0,\omega]$.
Let us introduce the function

$$Q(x,t)=\sum_{i=1}^{6}\overline{M_i(t)}N_i(x), \tag{3.1}$$

then

$$Qf=\int_0^\omega f(t)Q(x,t)dt.$$

Theorem 3.1. *If a bounded operator T, acting in $L^2[0,\omega]$, satisfies the operator equation $TA_0-A_0^*T=iQ$, then*

$$Tf=\frac{d^3}{dx^3}\int_0^\omega f(t)\frac{\partial^3}{\partial t^3}\ \Phi(x,t)\ dt, \tag{3.2}$$

holds, where $\frac{\partial^3}{\partial t^3}\Phi(x,t)$ is the solution of the equation

$$\frac{\partial^3 F(x,t)}{\partial x^3}-\frac{\partial^3 F(x,t)}{\partial t^3}=\frac{\partial^6 q(x,t)}{\partial t^3\partial x^3}.$$

Proof. The operator T may be represented in the form

$$Tf=\frac{d^3}{dx^3}\int_0^\omega f(t)F(x,t)dt.$$

The operator equation $TA_0-A_0^*T=iQ$ means, that

$$\begin{aligned} & i\int_0^\omega\int_0^t\int_0^y\int_0^z f(s)F(x,t)dsdzdydt+i\int_x^\omega\int_y^\omega\int_x^\omega\int_0^\omega f(t)F(x,t)dtdsdzdy \\ = {} & i\int_0^\omega f(t)q(x,t)dt. \end{aligned}$$

Consequently,

$$\frac{\partial^3 F(x,t)}{\partial x^3}-\frac{\partial^3 F(x,t)}{\partial t^3}=\frac{\partial^6 q(x,t)}{\partial t^3\partial x^3}.$$

Then the solution is

$$F(x,t)=H(t,x,q(x,t))=\frac{\partial^3}{\partial t^3}\Phi(x,t). \qquad \square$$

4. Relation between $N_k(x)$ and $M_k(x)$

Let us define the involution operator Uf by

$$Uf = \overline{f(\omega - x)}.$$

Lemma 4.1. *$USU = S^*$.*

Proof. Let

$$\begin{aligned} g(x) &\in C^3(0,\omega), \\ g(0) &= g(\omega) = 0, \\ g'(0) &= g'(\omega) = 0, \\ g''(0) &= g''(\omega) = 0, \end{aligned}$$

since

$$(Sf,g) = \int_0^\omega \frac{d^3}{dt^3} \int_0^\omega f(y)S(t,y)dy\overline{g(t)}dt = \begin{bmatrix} \overline{g(t)} = U, & \int_0^\omega f(y)S'''_{ttt}(t,y)dy = V'_t \\ \overline{g'(t)} = U', & \int_0^\omega f(y)S''_{tt}(t,y)dy = V \end{bmatrix}$$

$$= -\int_0^\omega \int_0^\omega f(y)S''_{tt}(t,y)dy\overline{g'(t)}dt = \begin{bmatrix} \overline{g'(t)} = U, & \int_0^\omega f(y)S''_{tt}(t,y)dy = V'_t \\ \overline{g''(t)} = U', & \int_0^\omega f(y)S'_t(t,y)dy = V \end{bmatrix}$$

$$= \int_0^\omega \int_0^\omega f(y)S'_t(t,y)dy\overline{g''(t)}dt = \begin{bmatrix} \overline{g''(t)} = U, & \int_0^\omega f(y)S'_t(t,y)dy = V'_t \\ \overline{g'''(t)} = U', & \int_0^\omega f(y)S(t,y)dy = V \end{bmatrix}$$

$$= -\int_0^\omega \int_0^\omega f(y)S(t,y)dy\overline{g'''(t)}dt = -\int_0^\omega \int_0^\omega f(y)S(t,y)\overline{g'''(t)}dtdy$$

$$= -\int_0^\omega f(y) \int_0^\omega S(t,y)\overline{g'''(t)}dtdy,$$

it follows that

$$\begin{aligned} S^*g &= -\int_0^\omega \overline{S(t,x)}g'''(t)dt = \begin{bmatrix} \overline{S(t,x)} = U, & g'''(t) = V' \\ \overline{S'_t(t,x)} = U'_t, & g''(t) = V \end{bmatrix} \\ &= \int_0^\omega \overline{S'_t(t,x)}g''(t)dt = \begin{bmatrix} \overline{S'_t(t,x)} = U, & g''(t) = V' \\ \overline{S''_{tt}(t,x)} = U'_t, & g'(t) = V \end{bmatrix} \\ &= -\int_0^\omega \overline{S''_{tt}(t,x)}g'(t)dt = \begin{bmatrix} \overline{S''_{tt}(t,x)} = U, & g'(t) = V' \\ \overline{S'''_{ttt}(t,x)} = U'_t, & g(t) = V \end{bmatrix} \\ &= \int_0^\omega \overline{S'''_{ttt}(t,x)}g(t)dt = -\frac{d^3}{dx^3}\int_0^\omega g(t)\overline{S(t,x)}dt. \end{aligned}$$

Then it is easy to see, that

$$USUg = -\frac{d^3}{dx^3}\int_0^\omega g(t)\overline{S(t,x)}dt. \qquad \square$$

In what follows, for simplicity, we restrict our study to those solution of equation for the kernel $S(x,t)$ which depends only on the difference $x-t$. More general case, require cumbersome computations while the reasoning is the same as for the case when the kernel depends only on the difference.

Theorem 4.2. *Suppose that there exists such N_i $(i=\overline{1,6})$ from $L^2[0,\omega]$ such that*

$$\begin{aligned} SN_1(t) &= \tfrac{x^2}{2}, \\ SN_2(t) &= M''(x), \\ SN_3(t) &= x, \\ SN_4(t) &= M'(x), \\ SN_5(t) &= 1, \\ SN_6(t) &= M(x), \end{aligned}$$

holds, then

$$\begin{aligned} S^*M_1(t) &= \overline{N''(t)}, \\ S^*M_2(t) &= -\tfrac{x^2}{2}, \\ S^*M_3(t) &= \overline{N'(t)}, \\ S^*M_4(t) &= -x, \\ S^*M_5(t) &= \overline{N(x)}, \\ S^*M_6(t) &= -1, \end{aligned}$$

are valid, where

$$\begin{aligned} M(x) &= S(x), & N(t) &= S(-t),\\ M'(x) &= S'_t(x), & N'(t) &= S'_x(-t),\\ M''(x) &= S''_{tt}(x), & N''(t) &= S''_{xx}(-t), \end{aligned}$$

and

$$\begin{aligned} M_1(x) = &\ \overline{N_2(\omega-x)}-1,\\ M_2(x) = &\ \overline{N_1(\omega-x)}+\omega\overline{N_3(\omega-x)}+\tfrac{\omega^2}{2}\overline{N_6(\omega-x)},\\ M_3(x) = &\ \omega\overline{N_2(\omega-x)}+\overline{N_4(\omega-x)}+x,\\ M_4(x) = &\ \omega\overline{N_5(\omega-x)}-\overline{N_3(\omega-x)},\\ M_5(x) = &\ \overline{N_6(\omega-x)}-\tfrac{(\omega-x)^2}{2}+(\tfrac{\omega^2}{2}+\omega)(\omega\overline{N_2(\omega-x)}+\overline{N_4(\omega-x)}+x),\\ M_6(x) = &\ -\overline{N_5(\omega-x)}. \end{aligned}$$

Proof. By direct integration by parts we verify, that

$$\begin{aligned} S1 &= \frac{d^3}{dx^3}\int\limits_0^\omega S(x-t)dt = -\omega S''_{tt}(x-\omega)+S'_t(x-\omega)-S'_t(x)\\ &= -\omega U\overline{N''(x)}-U\overline{N'(x)}-M'(x). \end{aligned}$$

Similarly,

$$\begin{aligned} S\frac{t^2}{2} &= \frac{d^3}{dx^3}\int\limits_0^\omega \frac{t^2}{2}S(x-t)dt = -\int\limits_0^\omega \frac{t^2}{2}S'''_{ttt}(x-t)dt\\ &= -\frac{\omega^2}{2}U\overline{N''(x)}-\omega U\overline{N'(x)}-U\overline{N(x)}+M(x). \end{aligned}$$

That is

$$\begin{aligned} S1 &= SN_2-U\overline{N''(x)},\\ St &= -SN_4-\omega U\overline{N''(x)}-U\overline{N'(x)},\\ S\frac{t^2}{2} &= -\left(\frac{\omega^2}{2}+\omega\right)U\overline{N'(x)}-U\overline{N(x)}+SN_6. \end{aligned}$$

Consequently,

$$\begin{aligned} U\overline{N''(x)} &= S[N_2-1],\\ U\overline{N'(x)} &= -S[t+N_4]-\omega\left(U\overline{N(x)}\right)'' = S[\omega-\omega N_2-N_4-t],\\ U\overline{N(x)} &= S[N_6-\frac{t^2}{2}]-\left(\frac{\omega^2}{2}+\omega\right)\left(U\overline{N(x)}\right)'\\ &= S\left[N_6-\frac{t^2}{2}-\left(\frac{\omega^2}{2}+\omega\right)(\omega-\omega N_2-N_4-t)\right]. \end{aligned}$$

Then,

1)

$$\begin{aligned}
M_1(x) &= \overline{N_2(\omega - x)} - 1,\\
M_1(x) &= U\left[N_2(x) - 1\right],\\
US^*M_1 &= US^*U\left[N_2 - 1\right],\\
US^*M_1 &= S\left[N_2 - 1\right],\\
US^*M_1 &= U\overline{N''(x)},\\
S^*M_1 &= \overline{N''(x)}.
\end{aligned}$$

2)

$$\begin{aligned}
M_2(x) &= \left[\overline{N_1(\omega - x)} + \omega\overline{N_3(\omega - x)} + \tfrac{\omega^2}{2}\overline{N_5(\omega - x)}\right]\\
M_2(x) &= U\left[N_1(x) + \omega N_3(x) + \tfrac{\omega^2}{2}N_5(x)\right]\\
US^*M_2 &= US^*U\left[N_1 + \omega N_3 + \tfrac{\omega^2}{2}N_5\right]\\
US^*M_2 &= S\left[N_1 + \omega N_3 + \tfrac{\omega^2}{2}N_5\right]\\
US^*M_2 &= \tfrac{-x^2+2x\omega-\omega^2}{2}\\
US^*M_2 &= -\tfrac{(\omega-x)^2}{2}\\
S^*M_2 &= -\tfrac{x^2}{2}.
\end{aligned}$$

3)

$$\begin{aligned}
M_3(x) &= \omega\overline{N_2(\omega - x)} + \overline{N_4(\omega - x)} + x\\
M_3(x) &= U\left[\omega - \omega N_2(x) - N_4(x) - x\right]\\
US^*M_3 &= US^*U\left[\omega - \omega N_2 - N_4 - t\right]\\
US^*M_3 &= S\left[\omega - \omega N_2 - N_4 - t\right]\\
US^*M_3 &= U\overline{N'(x)}\\
S^*M_1 &= \overline{N'(x)}.
\end{aligned}$$

4)

$$\begin{aligned}
M_4(x) &= \omega\overline{N_5(\omega - x)} - \overline{N_3(\omega - x)}\\
M_4(x) &= U\left[\omega N_5(x) - N_3(x)\right]\\
US^*M_4 &= US^*U\left[\omega N_5 - N_3\right]\\
US^*M_4 &= S\left[\omega N_5 - N_3\right]\\
US^*M_4 &= \omega - x\\
S^*M_4 &= x.
\end{aligned}$$

5)

$$\begin{aligned}
M_5(x) &= \overline{N_6(\omega,t)} - \tfrac{(\omega-x)^2}{2} + \left(\tfrac{\omega^2}{2}+\omega\right)\left(\omega\overline{N_2(\omega-x)} + \overline{N_4(\omega-x)} + x\right)\\
M_5(x) &= U\left[N_6(x) - \tfrac{x^2}{2} - \left(\tfrac{\omega^2}{2}+\omega\right)(\omega-\omega N_2 - N_4 - t)\right]\\
US^*M_5 &= US^*U\left[N_6 - \tfrac{t^2}{2} - \left(\tfrac{\omega^2}{2}+\omega\right)(\omega-\omega N_2 - N_4 - t)\right]\\
US^*M_5 &= S\left[N_6 - \tfrac{t^2}{2} - \left(\tfrac{\omega^2}{2}+\omega\right)(\omega-\omega N_2 - N_4 - t)\right]\\
US^*M_5 &= U\overline{N(x)}\\
S^*M_5 &= \overline{N(x)}.
\end{aligned}$$

6)

$$\begin{aligned}
M_6(x) &= -\overline{N_5(\omega-x)}\\
M_6(x) &= -UN_5(x)\\
US^*M_6 &= -US^*UN_5\\
US^*M_6 &= -SN_5\\
S^*M_6 &= -1.
\end{aligned}$$

□

If the operator S is invertible then from formula (3.1) it follows that

$$\begin{aligned}
Q(x,t) &= \sum_{i=1}^{6}\overline{M_i(t)}N_i(t)\\
&= [N_2(\omega-t)-1]\,N_1(x) + [N_1(\omega-t)+\omega N_3(\omega-t)\\
&\quad + \frac{\omega^2}{2}N_6(\omega-t)]N_2(x) + [\omega N_2(\omega-t)+N_4(\omega-t)+t]\,N_3(x)\\
&\quad + [\omega N_5(\omega-t) - N_3(\omega-t)]\,N_4(x) + [N_6(\omega-t) - \frac{(\omega-t)^2}{2}\\
&\quad + \left(\frac{\omega^2}{2}+\omega\right)(\omega N_2(\omega-t)+N_4(\omega-t)+t)]N_5(x)\\
&\quad - [N_5(\omega-t)]\,N_6(x).
\end{aligned}$$

Using $Q(x,t)$ one may construct the operator T.
Thus to construct operator $T = S^{-1}$ it is sufficiently to know it's action upon

$$1, x, \frac{x^2}{2}, M(x), M'(x), M''(x).$$

Thus, a method, proposed by L. A. Sakhnovich, and it's generalizations are analogs of construction of the general solution for the linear differential equation by it's particular solutions.However, in the theory of differential equations there exist general methods for solutions representation by partial solutions for any linear differential equation with variable coefficients of any finite order, while it was not possible to extend Sakhnovich's method for linear integral equations with any arbitrary kernel, i.e, it was not possible to prove that the operator $A_0S - SA_0^*$ is finite dimensional, where $Sf = D_x \int\limits_0^{\omega} S(x,t)f(t)dt$, and $A_0 = (D_x)^{-1}$, such that $f(x) \in L^2[0,\omega]$, D_x is a linear integro-differential operator, and the kernel $S(x,t)$ satisfies the equation

$$(D_x + D_t)\, S(x,t) = 0.$$

As it is obvious from the results obtained, Sakhnovich's method can be extended to include a case where D_x is a general linear differential operator of the order 3 as in the form

$$D_x = \sum_{k=0}^{3} a_k \frac{d^k}{dx^k}.$$

Sakhnovich's method may be also applied when $D_x = \frac{d^4}{dx^4}$.

References

[1] I. I. Kalmushevski, *On the solution of some integral equations with kernels depending on difference and sum of the arguments*, (Russian), Differential Equations, **16**(1980), no. 5, 941-943.

[2] M. G. Kreine, *On a New method of Linear Differential Equations of the First and the Second Type*, (Russian), Dokl. Acad. Nauk, **100**(1955), 413-416.

[3] L. A. Sakhnovich, *Equations with Difference Kernel on a Finite Interval*, Russian Math Surveys, **35**(1980), no. 4, 81-152.

[4] L. A. Sakhnovich, *Factorization Problems and Operator Kernel Identities*, Russian Math. Surveys, **41**(1986), no. 1, 1-64.

[5] L. A. Sakhnovich, *Integral Equations with Difference Kernel on Finite Intervals*, Birkhäuser Verlag, Basel-Boston-Berlin, 175, 1996.

[6] L. A. Sakhnovich, *Spectral Theory of Canonical Differential Systems. Method of Operator Identities,* Birkhäuser Verlag, Basel-Boston-Berlin, 202, 1999.

[7] V. A. Zolatarev, *La Factorization des Fonctions*, Preprint, Universite Pierré et Marie Curie, 1983, 1-35.

Dept. of Mathematics, Jadara University Irbid-Jordan
E-mail address: raedhat@yahoo.com

Dept. of Mathematics, Jadara University Irbid-Jordan
E-mail address: dr_qazza@yahoo.co.uk

Dept. of Mathematics, Jadara University Irbid-Jordan
E-mail address: analysi2003@yahoo.com