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THE ORDER OF CONVEXITY OF TWO INTEGRAL OPERATORS

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Abstract. In this paper, we obtain the order of convexity of the integral operators $\int_0^z \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{\frac{1}{\beta_i}} dt$ and $\int_0^z \left(te^{f(t)}\right)^{\gamma} dt$, where f_i and f satisfy the condition $\left|f'(z)\left(\frac{z}{f(z)}\right)^{\mu} - 1\right| < 1-\alpha$.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form :

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$. A function $f(z)$ belonging to $\mathcal{S}$ is said to be starlike of order α if it satisfies

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathcal{U}) \tag{1.2}$$

for some $\alpha (0 \le \alpha < 1)$. We denote by $\mathcal{S}^*(\alpha)$ the subclass of $\mathcal{A}$ consisting of functions which are starlike of order α in $\mathcal{U}$. Also, a function $f(z)$ belonging to $\mathcal{S}$ is said to be convex of order α if it satisfies

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (z \in \mathcal{U}) \tag{1.3}$$

for some $\alpha (0 \le \alpha < 1)$. We denote by $\mathcal{K}(\alpha)$ the subclass of $\mathcal{A}$ consisting of functions which are convex of order α in $\mathcal{U}$. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}(\alpha)$ iff

$$\mathrm{Re}\left(f'(z)\right) > \alpha, \qquad (z \in \mathcal{U}). \tag{1.4}$$

It is well known that $\mathcal{K}(\alpha) \subset \mathcal{S}^*(\alpha) \subset \mathcal{S}$.

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Very recently, Frasin and Jahangiri [4] define the family $\mathcal{B}(\mu,\alpha),\ \mu \geq 0,\ 0 \leq \alpha < 1$ so that it consists of functions $f \in \mathcal{A}$ satisfying the condition

$$\left| f'(z)\left(\frac{z}{f(z)}\right)^{\mu} - 1\right| < 1-\alpha \qquad (z \in \mathcal{U}). \tag{1.5}$$

The family $\mathcal{B}(\mu,\alpha)$ is a comprehensive class of analytic functions which includes various new classes of analytic univalent functions as well as some very well-known ones. For example, $\mathcal{B}(1,\alpha)\equiv\mathcal{S}^*(\alpha)$, and $\mathcal{B}(0,\alpha)\equiv\mathcal{R}(\alpha)$. Another interesting subclass is the special case $\mathcal{B}(2,\alpha)\equiv\mathcal{B}(\alpha)$ which has been introduced by Frasin and Darus [3](see also [1, 2]).

In this paper, we will obtain the order of convexity of the following integral operators:

$$\int_0^z \left(\frac{f_1(t)}{t}\right)^{\frac{1}{\beta_1}} \cdots \left(\frac{f_n(t)}{t}\right)^{\frac{1}{\beta_n}} dt \tag{1.6}$$

and

$$\int_0^z \left(te^{f(t)}\right)^{\gamma} dt \tag{1.7}$$

where the functions $f_1(t), f_2(t), ..., f_n(t)$ and $f(t)$ are in $\mathcal{B}(\mu,\alpha)$.

In order to prove our main results, we recall the following lemma:

Lemma 1.1. *(Schwarz Lemma). Let the analytic function $f(z)$ be regular in the unit disc $\mathcal{U}$, with $f(0)=0$. If $|f(z)| \leq 1$, for all $z \in \mathcal{U}$, then*

$$|f(z)| \leq |z|, \qquad \text{for all } z \in \mathcal{U}$$

and equality holds only if $f(z)=\varepsilon z$, where $|\varepsilon|=1$.

2. Main results

Theorem 2.1. *Let $f_i(z) \in \mathcal{A}$ be in the class $\mathcal{B}(\mu,\alpha)$, $\mu \geq 1,\ 0 \leq \alpha < 1$ for all $i=1,2,\cdots,n$. If $|f_i(z)| \leq M \quad (M \geq 1;\, z \in \mathcal{U})$ then the integral operator*

$$\int_0^z \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{\frac{1}{\beta_i}} dt \tag{2.1}$$

is in $\mathcal{K}(\delta)$, where

$$\delta = 1 - \sum_{i=1}^{n} \frac{1}{|\beta_i|}\left((2-\alpha)\,M^{\mu-1}+1\right) \tag{2.2}$$

and $\sum_{i=1}^{n} \frac{1}{|\beta_i|}\left((2-\alpha)\,M^{\mu-1}+1\right) < 1$, $\beta_i \in \mathbb{C}-\{0\}$ for all $i=1,2,\cdots,n$.

Proof. Define the function $F(z)$ by

$$F(z) = \int_0^z \prod_{i=1}^n \left(\frac{f_i(t)}{t}\right)^{\frac{1}{\beta_i}} dt$$

for $f_i(z) \in \mathcal{B}(\mu, \alpha)$. Since

$$F'(z) = \prod_{i=1}^n \left(\frac{f_i(z)}{z}\right)^{\frac{1}{\beta_i}}$$

we see that

$$\frac{zF''(z)}{F'(z)} = \sum_{i=1}^n \frac{1}{\beta_i}\left(\frac{zf_i'(z)}{f_i(z)} - 1\right). \tag{2.3}$$

It follows from (2.3) that

$$\begin{aligned}\left|\frac{zF''(z)}{F'(z)}\right| &\le \sum_{i=1}^n \frac{1}{|\beta_i|}\left(\left|\frac{zf_i'(z)}{f_i(z)}\right| + 1\right)\\ &= \sum_{i=1}^n \frac{1}{|\beta_i|}\left(\left|f_i'(z)\left(\frac{z}{f_i(z)}\right)^{\mu}\right|\left|\left(\frac{f_i(z)}{z}\right)^{\mu-1}\right| + 1\right).\end{aligned} \tag{2.4}$$

Since $|f_i(z)| \le M \quad (z \in \mathcal{U})$, applying the Schwarz lemma, we have

$$\left|\frac{f_i(z)}{z}\right| \le M \qquad (z \in \mathcal{U}).$$

Therefore, from (2.4), we obtain

$$\left|\frac{zF''(z)}{F'(z)}\right| \le \sum_{i=1}^n \frac{1}{|\beta_i|}\left(\left|f_i'(z)\left(\frac{z}{f_i(z)}\right)^{\mu}\right| M^{\mu-1} + 1\right). \tag{2.5}$$

From (2.5) and (1.5), we see that

$$\begin{aligned}\left|\frac{zF''(z)}{F'(z)}\right| &\le \sum_{i=1}^n \frac{1}{|\beta_i|}\left(\left(\left|f_i'(z)\left(\frac{z}{f_i(z)}\right)^{\mu} - 1\right| + 1\right) M^{\mu-1} + 1\right)\\ &\le \sum_{i=1}^n \frac{1}{|\beta_i|}\left((2-\alpha) M^{\mu-1} + 1\right)\\ &= 1 - \delta.\end{aligned}$$

This completes the proof. □

Corollary 2.2. *Let $f_i(z) \in \mathcal{A}$ be in the class $\mathcal{B}(\mu, \alpha)$, $\mu \ge 1$, $0 \le \alpha < 1$ for all $i = 1, 2, \cdots, n$. If $|f_i(z)| \le M$ $(M \ge 1; z \in \mathcal{U})$ then the integral operator $\int_0^z \prod_{i=1}^n \left(\frac{f_i(t)}{t}\right)^{\beta_i} dt$ is convex function in $\mathcal{U}$, where*

$$\sum_{i=1}^n \frac{1}{|\beta_i|} = 1/\left((2-\alpha) M^{\mu-1} + 1\right), \quad \beta_i \in C - \{0\}$$

for all $i = 1, 2, \cdots, n$.

Letting $\mu = 1$ in Theorem 2.1, we have

Corollary 2.3. *Let* $f_i(z) \in \mathcal{A}$ *be in the class* $\mathcal{S}^*(\alpha)$, $0 \le \alpha < 1$ *for all* $i = 1, 2, \cdots, n$. *If* $|f_i(z)| \le M \quad (M \ge 1;\ z \in \mathcal{U})$ *then the integral operator* $\int_0^z \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{\beta_i} dt \in \mathcal{K}(\delta)$, *where*

$$\delta = 1 - \sum_{i=1}^{n} \frac{1}{|\beta_i|} (3 - \alpha) \tag{2.6}$$

where $\sum_{i=1}^{n} \frac{1}{|\beta_i|} (3 - \alpha) < 1$, $\beta_i \in \mathbb{C} - \{0\}$ *for all* $i = 1, 2, \cdots, n$.

Letting $n = 1$ and $\alpha = \delta = 0$ in Corollary 2.3, we have

Corollary 2.4. *Let* $f(z) \in \mathcal{A}$ *be starlike function in* $\mathcal{U}$. *If* $|f(z)| \le M \quad (M \ge 1;\ z \in \mathcal{U})$ *then the integral operator* $\int_0^z \left(\frac{f(t)}{t}\right)^{\frac{1}{\beta}} dt$ *is convex in* $\mathcal{U}$ *where* $|\beta| = 3$, $\beta \in \mathbb{C}$.

Theorem 2.5. *Let* $f \in \mathcal{A}$ *be in the class* $\mathcal{B}(\mu, \alpha)$, $\mu \ge 0,\ 0 \le \alpha < 1$. *If* $|f(z)| \le M \ (M \ge 1;\ z \in \mathcal{U})$ *then the integral operator*

$$G(z) = \int_0^z \left(te^{f(t)}\right)^{\gamma} dt \tag{2.7}$$

is in $\mathcal{K}(\delta)$, *where*

$$\delta = 1 - |\gamma| \left((2 - \alpha) M^{\mu} + 1\right) \tag{2.8}$$

and $|\gamma| < \frac{1}{(2-\alpha)M^{\mu}+1}$, $\gamma \in \mathbb{C}$.

Proof. Let $f \in \mathcal{A}$ be in the class $\mathcal{B}(\mu, \alpha)$, $\mu \ge 0,\ 0 \le \alpha < 1$. It follows from (2.7) that

$$\frac{G''(z)}{G'(z)} = \gamma \left(\frac{1}{z} + f'(z)\right)$$

and hence

$$\begin{aligned} \left|\frac{zG''(z)}{G'(z)}\right| &= |\gamma| \left(|1 + zf'(z)|\right) \\ &\le |\gamma| \left(1 + \left|f'(z) \left(\frac{z}{f(z)}\right)^{\mu}\right| \left|\left(\frac{f(z)}{z}\right)^{\mu}\right| |z|\right). \end{aligned} \tag{2.9}$$

Applying the Schwarz lemma once again, we have

$$\left|\frac{f(z)}{z}\right| \le M \qquad (z \in \mathcal{U}).$$

Therefore, from (2.9), we obtain

$$\left|\frac{zG''(z)}{G'(z)}\right| \le |\gamma| \left(1 + \left|f'(z) \left(\frac{z}{f(z)}\right)^{\mu}\right| M^{\mu}\right) \qquad (z \in \mathcal{U}). \tag{2.10}$$

From (2.5) and (2.10), we see that

$$\begin{aligned}\left|\frac{zG''(z)}{G'(z)}\right| &\leq |\gamma|\left((2-\alpha)M^{\mu}+1\right)\\ &= 1-\delta.\end{aligned}$$

□

Letting $\mu = 0$, in Theorem 2.5, we have

Corollary 2.6. *Let $f \in \mathcal{A}$ be in the class $\mathcal{R}(\alpha)$, $0 \leq \alpha < 1$. Then the integral operator $\int_0^z \left(te^{f(t)}\right)^{\gamma} dt \in \mathcal{K}(\delta)$, where*

$$\delta = 1 - |\gamma|(3-\alpha) \tag{2.11}$$

and $|\gamma| < \frac{1}{3-\alpha}$, $\gamma \in \mathbb{C}$.

Letting $\mu = 1$, in Theorem 2.5, we have

Corollary 2.7. *Let $f \in \mathcal{A}$ be in the class $\mathcal{S}^*(\alpha)$, $0 \leq \alpha < 1$. If $|f(z)| \leq M$ $(M \geq 1;\ z \in \mathcal{U})$ then the integral operator $\int\limits_0^z \left(te^{f(t)}\right)^{\gamma} dt \in \mathcal{K}(\delta)$, where*

$$\delta = 1 - |\gamma|\left((2-\alpha)M+1\right) \tag{2.12}$$

and $|\gamma| < \frac{1}{(2-\alpha)M+1}$, $\gamma \in \mathbb{C}$.

Letting $\alpha = \delta = 0$ in Corollary2.7, we have

Corollary 2.8. *Let $f(z) \in \mathcal{A}$ be starlike function in $\mathcal{U}$. If $|f(z)| \leq M$ $(M \geq 1;\ z \in \mathcal{U})$ then the integral operator $\int\limits_0^z \left(te^{f(t)}\right)^{\gamma} dt$ is convex in $\mathcal{U}$ where $|\gamma| = \frac{1}{2M+1}$, $\gamma \in \mathbb{C}$.*

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