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SIMPLE CRITERIA FOR STARLIKENESS OF ORDER β

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Abstract. In this paper we obtain a new criterion for starlikeness of order β for an analytic function $f \in \mathcal{A}_n$. This criterion involves only the second derivative of the given function and generalizes a well-known result due to P. T. Mocanu.

1. **Introduction**

Let $\mathcal{H} = \mathcal{H}(U)$ denote the class of functions analytic in the unit disc

$$U = \{z \in \mathbb{C} : |z| < 1\}.$$

For n a positive integer and $a \in \mathbb{C}$ let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + ...\}.$$

Let $\mathcal{A}_n$ denote the class of functions

$$f(z) = z + a_{n+1} z^{n+1} + ..., n \geq 1$$

that are analytic on the unit disc and let $\mathcal{A}_1 = \mathcal{A}$.

Let $\mathcal{D}$ be a domain in $\mathbb{C}$. A function $f : \mathcal{D} \to \mathbb{C}$ is called univalent on $\mathcal{D}$ if $f \in \mathcal{H}(\mathcal{D})$ and f is injective on $\mathcal{D}$.

The analytic function f, with $f(0) = 0$ and $f'(0) \neq 0$ is starlike on U (i.e. f is univalent on U and $f(U)$ is starlike with respect to origin) if and only if $\Re\left[\frac{zf'(z)}{f(z)}\right] > 0$, for $z \in U$.

An analytic function f with $f(0) = 0$ and $f'(0) \neq 0$ is starlike of order β, $\beta \geq 0$ if and only if $\Re\left[\frac{zf'(z)}{f(z)}\right] > \beta$, for $z \in U$, $\beta \geq 0$.

Let denote S^* and $S^*(\beta)$ the subclasses of $\mathcal{A}$ consisting of functions f which are starlike and starlike of order β.

Let $\mathcal{D}$ be a domain in $\mathbb{C}$. A function $f : \mathcal{D} \to$ is convex on $\mathcal{D}$ if f is univalent on $\mathcal{D}$ and $f(\mathcal{D})$ is a convex domain in $\mathbb{C}$.

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If f and g are analytic functions in U, then we say that f is subordinate to g, written $f \prec g$,or $f(z) \prec g(z)$, if there is a function w analytic in U with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $f(z) = g[w(z)]$, for $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

We shall use the following results to prove our main results.

Lemma 1.1. [8] *Let h be a starlike function with $h(0) = 0$. If the function $p \in \mathcal{H}[a, n]$ satisfies the differential subordination*

$$zp'(z) \prec h(z) \tag{1.1}$$

then

$$p(z) \prec q(z) = a + \frac{1}{n}\int_0^z \frac{h(t)}{t}dt.$$

Function q is the best (a,n)-dominant of subordination.

Lemma 1.2. [3] *Let h be a convex function with $h(0) = a$ and let $\gamma \in \mathbb{C}^*$ with $\Re\gamma \geq 0$. If the function $p \in \mathcal{H}[a, n]$ and*

$$p(z) + \frac{1}{\gamma}zp'(z) \prec h(z) \tag{1.2}$$

then

$$p(z) \prec q(z) \prec h(z)$$

where

$$q(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}}\int_0^1 h(t)t^{\frac{\gamma}{n}-1}dt. \tag{1.3}$$

Lemma 1.3. [4] *Let be a function $p \in \mathcal{H}[a, n]$.*

1. *If $\Psi \in \Psi_n\{\Omega, a\}$ then*

$$\Psi(p(z), zp'(z), z^2p''(z); z) \in \Omega \Rightarrow \Re\ p(z) > 0, z \in U$$

2. *If $\Psi \in \Psi_n\{\Omega, a\}$ then*

$$\Re\ \Psi(p(z), zp'(z), z^2p''(z); z) > 0, z \in U \Rightarrow \Re\ p(z) > 0, z \in U$$

Lemma 1.4. [7] *Let n be a positive integer and*

$$\alpha_n = \frac{n+2}{C_n} \tag{1.4}$$

where

$$C_n = 2\Big[1 + \frac{n+2}{n}\ln 2 - \int_0^1 \frac{t^{\frac{1}{n}}}{1+t}dt\Big]. \tag{1.5}$$

If $f \in \mathcal{A}_n$ and

$$\Re\big[zf''(z)\big] > -\alpha_n, z \in U \tag{1.6}$$

then $f \in S^*$.

In this paper we obtain a new criterion for starlikeness of order β for an analytic function $f \in \mathcal{A}_n$. This criterion involves only the second derivative of the given function and generalizes a well-known result due to P.T. Mocanu.

2. **Main results**

Theorem 2.1. *Let n be a positive integer, $\beta \in [0, \frac{1}{2}]$ and*

$$\alpha_n(\beta) := \frac{(n+2)-(n+4)\beta}{2\Big[\frac{n+2}{n}\ln 2 - \frac{n+4}{n}\beta ln 2 - (1-\beta)\int_0^1 \frac{t^{\frac{1}{n}}}{1+t}dt + 1\Big]}.$$

If $f \in \mathcal{A}_n$ and

$$\Re\big[zf^{"}(z)\big] > -\alpha_n(\beta), z \in U, n \in \mathbb{N}$$

then $f \in S^*(\beta)$.

Proof. We will show first that f is univalent on U. From the definition of $\alpha_n = \alpha_n(\beta)$ we have that $\alpha_n(\beta) > 0$. If $\alpha \in [0, \alpha_n]$ the inequality $\Re[zf''(z)] > -\alpha$, $z \in U$ is equivalent with the following subordination

$$zf''(z) \prec -\frac{2\alpha z}{1+z} = h(z).$$

Since the function f is starlike and $f' \in \mathcal{H}[1, n]$ by applying Lemma 1.1 we obtain that

$$f''(z) \prec 1 + \frac{1}{n}\int_0^z \frac{h(t)}{t}dt = 1 - \frac{2\alpha}{n}log(1+z) = q(z)$$

where the function q is convex.

Due to the fact that the function q is convex and has real coefficients we get that:

$$\Re f'(z) > \gamma = \gamma(\alpha) = q(1) = 1 - \frac{2\alpha}{n}\ln 2, z \in U. \tag{2.1}$$

We prove the following inequality:

$$\alpha_n \le \frac{n}{\ln 4}. \tag{2.2}$$

We have:

$$\begin{aligned}\alpha_n &= \frac{(n+2)-(n+4)\beta}{2\Big[\frac{n+2}{n}\ln 2 - \frac{n+4}{n}\beta\ln 2 - (1-\beta)\int_0^1 \frac{t^{\frac{1}{n}}}{1+t}dt + 1\Big]} \\ &\le \frac{n+2}{2\frac{n+2}{n}\ln 2} = \frac{n}{2\ln 2} = \frac{n}{\ln 4},\end{aligned}$$

as desired.

Since $\alpha_n \le \frac{n}{\ln 4}$ then

$$\Re f'(z) > \gamma(\alpha) \ge 0, z \in U. \tag{2.3}$$

So f is univalent on U.

Next, we prove that $f \in S^*(\beta)$. If

$$P(z) := \frac{f(z)}{z} \tag{2.4}$$

then P satisfies the differential subordination

$$zP'(z) + P(z) = f'(z) \prec q(z).$$

From the previous relation, by using Lemma 1.2 for $\gamma = 1$ we obtain the exact subordination $P(z) \prec Q(z)$, where the function Q is convex and is defined by

$$Q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z q(t) t^{\frac{1}{n}-1} dt = 1 - \frac{2\alpha}{n^2 t^{\frac{1}{n}}} \int_0^1 t^{\frac{1}{n}-1} log(1+t) dt. \tag{2.5}$$

Because the function Q is convex, from differential subordination $P \prec Q$ we have that

$$\Re\, P(z) > \delta = \delta(\alpha) = Q(1) = 1 - \frac{2\alpha}{n}\Big[\ln 2 - \int_0^1 \frac{t^{\frac{1}{n}}}{1+t} dt\Big]. \tag{2.6}$$

If we denote by

$$p(z) := \frac{\frac{zf'(z)}{f(z)} - \beta}{1-\beta} \tag{2.7}$$

then

$$zf'(z) = (1-\beta)p(z)f(z) + \beta f(z).$$

By differentiating the previous equality we get that:

$$zf''(z) + (1-\beta)f'(z) = (1-\beta)p'(z)f(z) + (1-\beta)p(z)f'(z)$$

and hence

$$zf''(z) + (1-\beta)f'(z) = \frac{f(z)}{z}\Big[(1-\beta)zp'(z) + (1-\beta)p(z)\frac{zf'(z)}{f(z)}\Big].$$

The previous equality can also be written as

$$zf''(z) + (1-\beta)f'(z) = P(z)[(1-\beta)zp'(z) + (1-\beta)^2 p^2(z) + \beta(1-\beta)p(z)] \tag{2.8}$$

where by P we denoted the function $P(z) = \frac{f(z)}{z}$.

Since

$$\beta(1-\beta)p(z)P(z) = \beta f'(z) - \beta^2 P(z) \tag{2.9}$$

the equality (2.8) becomes

$$zf''(z) + (1-2\beta)f'(z) = P(z)[(1-\beta)zp'(z) + (1-\beta)^2 p^2(z) - \beta^2]. \tag{2.10}$$

It is obvious that

$$\begin{aligned}\Re[zf''(z)+(1-2\beta)f'(z)] &= \Re[zf''(z)]+(1-2\beta)\Re f'(z)\\ &> -\alpha+(1-2\beta)\gamma(\alpha). \qquad (2.11)\end{aligned}$$

By using the first part of Lemma 1.3 and the inequality (2.11) we will show that $\Re\, p(z)>0$, $z\in U$.

In order to do that it is sufficient to show that the function

$$\Psi(r,s,z)=P(z)[(1-\beta)s+(1-\beta)^2r^2-\beta^2]$$

is an admissible function.

We have that

$$\begin{aligned}\Re\,\Psi(\delta i,\sigma,z) &= \Re\{P(z)[(1-\beta)\sigma-(1-\beta)^2\delta^2-\beta^2]\}=\\ &= [(1-\beta)\sigma-(1-\beta)^2\delta^2-\beta^2]\Re\, P(z)\\ &\le -\alpha+(1-2\beta)\gamma(z) \qquad (2.12)\end{aligned}$$

By using Lemma 1.3 we want to show that $\Re\, P(z)>0, z\in U$.

Since,

$$\sigma\le-\frac{n(1+\delta^2)}{2},\delta,\sigma\in\mathbb{R}. \qquad (2.13)$$

Next, we will verify that $0\le\alpha\le\alpha_n, \Re\, P(z)>0, z\in U$. By using the relation (2.13) we obtain that

$$[(1-\beta)\sigma-(1-\beta)^2\delta^2-\beta^2]\Re\, P(z)\le\Big[-\frac{(1-\beta)n}{2}(1+\delta^2)-(1-\beta)^2\delta^2-\beta^2\Big]$$

$$\Re\, P(z)=-\frac{(1-\beta)n}{2}\Re\, P(z)-\Big[\frac{n(1-\beta)}{2}\delta^2+(1-\beta)^2\delta^2+\beta^2\Big]$$

$$\Re\, P(z)\le-\frac{(1-\beta)n}{2}\Re\, P(z)=\frac{n(1-\beta)}{2}[-\Re\, P(z)]\le-\frac{n(1-\beta)}{2}\delta(\alpha).$$

In order that the relation (2.12) to be satisfied it is sufficient that the following inequality to be true:

$$-\frac{n(1-\beta)}{2}\delta(z)\le-\alpha+(1-2\beta)\gamma(\alpha).$$

If $\alpha\le\alpha_0$ then the previous inequality is satisfied.

By using Lemma 1.3 and the relation (2.11) we obtain that $\Re\, p(z)>0, z\in U$.

Hence, applying the analytical characterization for starlike functions of order β we proved that $f\in S^*(\beta)$.

If we take $\beta=0$ in Lemma 2.1 we obtain a well-known result, due to P.T. Mocanu [7].

Next, for $n=1$, we obtain the following particular result.

Corolarry 2.2. *Let* $\beta \in [0, \frac{1}{2}]$ *and* $\alpha_1(\beta) := \frac{3-5\beta}{2[4\ln 2 - 6\beta \ln 2 + \beta]}$. *If* $f \in \mathcal{A}_1$ *and* $\Re\big[zf''(z)\big] > -\alpha_1(\beta), z \in U$ *then* $f \in S^*(\beta)$.

Proof. We put $n = 1$ in the previous result and obtain:

$$\begin{cases} \delta(1) = 1 - 2\alpha[\ln 2 - \int_0^1 \frac{t}{1+t} dt] = 1 - 2\alpha[2\ln 2 - 1] \\ \gamma(1) = 1 - 2\alpha \ln 2 \end{cases}$$

We have that:

$$\begin{cases} \delta(1) = 1 - 2\alpha[2\ln 2 - 1] \\ \gamma(1) = 1 - 2\alpha \ln 2 \end{cases}$$

By using the following equality: $-\frac{1-\beta}{2}\delta(1) = -\alpha + (1-2\beta)\gamma(1)$ we obtain

$$-\frac{1-\beta}{2} + 2\alpha(1-\beta)\ln 2 - \alpha(1-\beta) = -\alpha + 1 - 2\alpha \ln 2 - 2\beta + 4\alpha\beta \ln 2$$

and hence

$$\alpha = \frac{3-5\beta}{2[4\ln 2 - 6\beta \ln 2 + \beta]}, \alpha = \alpha_1(\beta)$$

where

$$\alpha_1(\beta) = \frac{3-5\beta}{2[4\ln 2 - 6\beta \ln 2 + \beta]} \leq \frac{1}{2\ln 2}.$$

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