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A MULTI-STEP ITERATIVE METHOD FOR APPROXIMATING COMMON FIXED POINTS OF PRESIĆ-RUS TYPE OPERATORS ON METRIC SPACES

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Abstract. The existence of coincidence points and common fixed points for mappings satisfying a Presić type condition in metric spaces is proved. A multi-step iterative method for constructing the common fixed points and its rate of convergence are also provided. This is a generalization of several fixed point and common fixed point results in literature.

1. Introduction

One of the most important results in fixed point theory is the contraction mapping principle of Banach, not only for its applications but also due to its position as a central point for a remarkable number of generalizations that appeared along the time, on various and sometimes very different directions (see for example [8], [11]). The present paper deals with two of these directions, aiming to establish a new general result at their point of intersection.

One direction, the first under our attention, was opened in 1965 by S. Presić [7] who proved the existence and uniqueness of fixed points for operators satisfying a special type of contraction condition, also providing a so-called multi-step iteration method for approximating the fixed points. In the sequel we shall consider (X, d) a metric space. Presić' condition generalizes Banach's contraction condition, namely

$$d(f(x), f(y)) \leq \alpha d(x, , y),$$

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for any $x, y \in X$, where $f : X \to X$ an operator and $\alpha \in [0,1)$ a constant, by considering instead

$$d(f(x_0, x_1, \dots, x_{k-1}), f(x_1, x_2, \dots, x_k)) \leq \sum_{i=1}^{k} \alpha_i \cdot d(x_{i-1}, x_i),$$

for any $x_0, x_1, \dots, x_k \in X$, where k a positive integer, $f : X^k \to X$ an operator and $\alpha_1, \dots, \alpha_k \in \mathbb{R}_+$ constants such that $\sum\limits_{i=1}^{k} \alpha_i < 1$.

Several general Presić type results followed in literature, see for example the papers due to M.R. Taskovic [13], M. Şerban [12], our paper [6] and I.A. Rus [10, 9], in the latter of which the following result is proved:

Theorem 1.1 (I.A. Rus [9], 1981). *Let (X, d) be a complete metric space, k a positive integer, $\varphi : \mathbb{R}_+^k \to \mathbb{R}_+$ with the properties:*

a) $\varphi(r) \leq \varphi(s)$, *for* $r, s \in \mathbb{R}_+^k,\ r \leq s$;

b) $\varphi(r, r, \dots, r) < r$, *for* $r \in \mathbb{R}_+, r > 0$;

c) φ *continuous;*

d) $\sum\limits_{i=0}^{\infty} \varphi^i(r) < \infty$;

e) $\varphi(r, 0, \dots, 0) + \varphi(0, r, 0, \dots, 0) + \dots + \varphi(0, \dots, 0, r) \leq \varphi(r, r \dots, r)$, *for any* $r \in \mathbb{R}_+$,

and $f : X^k \to X$ an operator such that:

$$d(f(x_0, \dots, x_{k-1}), f(x_1, \dots, x_k)) \leq \varphi(d(x_0, x_1), d(x_1, x_2), \dots, d(x_{k-1}, x_k)),$$

for any $x_0, x_1, \dots, x_k \in X$.

Then:

ι) *there exists a unique $x^* \in X$ solution of the equation*

$$x = f(x, x, \dots, x);$$

$\iota\iota$) *the sequence $\{x_n\}_{n \geq 0}$, with $x_0, x_1, \dots, x_{k-1} \in X$ and*

$$x_n = f(x_{n-k}, x_{n-k+1}, \dots, x_{n-1}), \text{for } n \geq k,$$

converges to x^*;

ιιι) the rate of convergence for $\{x_n\}_{n\geq 0}$ *is given by*

$$d(x_n, x^*) \leq k \sum_{i=0}^{\infty} \varphi^{\left[\frac{n+i}{k}\right]}(d_0, \ldots, d_0),$$

where $d_0 = \max\{d(x_0, x_1), d(x_1, x_2), \ldots, d(x_{k-1}, x_k)\}$.

A generalization of the contraction mapping principle on a different direction was given in 1976 by G. Jungck [4], regarding common fixed points of commuting mappings. A recent result on the existence of common fixed points has been proved by M. Abbas and G. Jungck [1] in a cone metric space setting.

Theorem 1.2 (M. Abbas, G. Jungck [1], 2008). *Let* (X, d) *be a cone metric space,* P *a normal cone with normal constant* K *and* $f, g : X \to X$ *two operators satisfying*

$$d(f(x), f(y)) \leq kd(g(x), g(y)), \text{ for all } x, y \in X, \tag{1.1}$$

where $k \in [0, 1)$ *is a constant. If the range of* g *contains the range of* f *and* $g(X)$ *is a complete subspace of* X*, then* f *and* g *have a unique coincidence point in* X*. If in addition* f *and* g *are weakly compatible, then they have a unique common fixed point in* X*.*

Remark 1.1. For the condition (1.1) see also K. Goebel [3].

Remark 1.2. In practical situations, the condition "$g(X)$ complete metric subspace" is too restrictive. It suffices to make sure there exists a complete metric subspace $Y \subset X$ such that $f(X) \subset Y \subset g(X)$, which also implies $f(X) \subseteq g(X)$. This is what we shall require in this paper.

The main aim of this paper is to establish new common fixed point results situated at the intersection of the two aforementioned directions, thus generalizing Theorems 1.1, 1.2 and several other subsequent results.

2. **Preliminaries**

We begin by recalling some concepts used in [4, 5, 1] and several related papers.

Definition 2.1 ([4])**.** *Let X be nonempty set and $f, g : X \to X$ two operators.*

If

$$f(p) = g(p)$$

for some $p \in X$, then p is called a ***coincidence point*** *of f and g, while $s = f(p) = g(p)$ is a* ***coincidence value*** *for them.*

If

$$f(p) = g(p) = p$$

for some $p \in X$, then p is called a ***common fixed point*** *of f and g.*

Remark 2.1. We shall denote by

$$C(f, g) = \{p \in X \,|\, f(p) = g(p)\}$$

the set of all coincidence points of f and g.

Obviously, the following hold:

a) $F_f \cap F_g \subset C(f, g)$;

b) $F_f \cap C(f, g) = F_g \cap C(f, g) = F_f \cap F_g$.

Definition 2.2 ([5])**.** *Let X be a nonempty set and $f, g : X \to X$. The operators f and g are said to be* ***weakly compatible*** *if they commute at their coincidence points, namely if*

$$f(g(p)) = g(f(p)),$$

for any p a coincidence point of f and g.

Lemma 2.1. *Let X be a nonempty set and $f, g : X \to X$. If f and g are weakly compatible, then*

$$C(f, g) \in I(f) \cap I(g),$$

i.e., $C(f, g)$ is an invariant set for both f and g.

Proof. Let $p \in C(f, g)$. We shall prove that $f(p), g(p) \in C(f, g)$, as well.

By definition,

$$f(p) = g(p) = q \in X. \tag{2.1}$$

As f and g are weakly compatible, we have:

$$f(g(p)) = g(f(p)),$$

which by (2.1) yields

$$f(q) = g(q),$$

so $q = f(p) = g(p) \in C(f, g)$. Thus $C(f, g) \in I(f) \cap I(g)$. □

Using this Lemma, the proof of the following Lemma is immediate.

Lemma 2.2 ([1]). *Let X be a nonempty set and $f, g : X \to X$ two weakly compatible operators.*

If they have a unique coincidence value $x^* = f(p) = g(p)$, for some $p \in X$, then x^* is their unique common fixed point.

Remark 2.2. For any operator $f : X^n \to X$, n a positive integer, we can define its **associate operator** $F : X \to X$ by

$$F(x) = f(x, \ldots, x), x \in X.$$

Obviously, $x \in X$ is a fixed point of $f : X^k \to X$, i.e., $x = f(x, \ldots, x)$, if and only if it is a fixed point of its associate operator F, in the sense of the classical definition. For details see for example [10].

Based on this remark, we can extend the previous definitions for the case $f : X^k \to X$, k a positive integer.

Definition 2.3. *Let X be a nonempty set, k a positive integer and $f : X^k \to X$, $g : X \to X$ two operators.*

*An element $p \in X$ is called a **coincidence point** of f and g if it is a coincidence point for F and g.*

*Similarly, $s \in X$ is a **coincidence value** of f and g if it is a coincidence value for F and g.*

*An element $p \in X$ is a **common fixed point** of f and g if it is a common fixed point of F and g.*

Definition 2.4. *Let X be a nonempty set, k a positive integer and $f : X^k \to X$, $g : X \to X$. The operators f and g are said to be **weakly compatible** if F and g are weakly compatible.*

The following result is a generalization of Lemma 1.4 in [1], included above as Lemma 2.2.

Lemma 2.3. *Let X be a nonempty set, k a positive integer and $f : X^k \to X, g : X \to X$ two weakly compatible operators.*

If f and g have a unique coincidence value $x^* = f(p, \dots, p) = g(p)$, then x^* is the unique common fixed point of f and g.

Proof. As f and g are weakly compatible, F and g are also weakly compatible. The proof follows by Lemma 2.2. □

3. **The main result**

The main result of the paper is the following theorem which unifies two generalizations of the contraction mapping principle of Banach, namely the Presić type result due to I.A. Rus [9] and the common fixed point result due to M. Abbas and G. Jungck [1], in metric spaces. It also provides a *multi-step iteration method* for effectively determining the coincidence values/common fixed points of the operators referred.

Theorem 3.1. *Let (X, d) be a metric space, k a positive integer, $\varphi : \mathbb{R}_+^k \to \mathbb{R}_+$ with the properties $a) - e)$ in Theorem 1.1 and $f : X^k \to X$, $g : X \to X$ two operators for which there exists a complete subspace $Y \subseteq X$ such that $f(X^k) \subseteq Y \subseteq g(X)$ and*

$$d(f(x_0, \dots, x_{k-1}), f(x_1, \dots, x_k)) \leq \tag{cPR}$$

$$\leq \varphi(d(g(x_0), g(x_1)), \dots, d(g(x_{k-1}), g(x_k))),$$

for any $x_0, \dots, x_k \in X$.

Then:

1) *f and g have a unique coincidence value, say x^*, in X;*

2) *the sequence* $\{g(x_n)\}_{n\geq 0}$ *defined by* $x_0, \ldots, x_{k-1} \in X$ *and*

$$g(x_n) = f(x_{n-k}, \ldots, x_{n-1}), n \geq k, \tag{3.1}$$

converges to x^**, with a rate estimated by*

$$d(g(x_n), x^*) \leq k \sum_{i=0}^{\infty} \varphi^{\left[\frac{n+i}{k}\right]}(d_0, \ldots, d_0), \tag{3.2}$$

where $d_0 = \max\{d(g(x_0), g(x_1)), \ldots, d(g(x_{k-1}), g(x_k))\}$*;*

3) *if in addition* f *and* g *are weakly compatible, then* x^* *is their unique common fixed point.*

Proof. Let $x_0, x_1, \ldots, x_{k-1} \in X$.
Then $f(x_0, \ldots, x_{k-1}) \in f(X^k) \subset g(X)$, so there exists $x_k \in X$ such that

$$f(x_0, \ldots, x_{k-1}) = g(x_k).$$

Further on, $f(x_1, \ldots, x_k) \in f(X^k) \subset g(X)$, so there exists $x_{k+1} \in X$ such that

$$f(x_1, \ldots, x_k) = g(x_{k+1}).$$

In this manner we construct the sequence $\{g(x_n)\}_{n\geq 0}$ such that

$$g(x_n) = f(x_{n-k}, \ldots, x_{n-1}), n \geq k. \tag{3.3}$$

We should remark now that, due to construction,

$$\{g(x_n)\}_{n\geq 0} \subseteq f(X^k) \subseteq Y \subseteq g(X). \tag{3.4}$$

We denote

$$d_0 = \max\{d(g(x_0), g(x_1)), \ldots, d(g(x_{k-1}), g(x_k))\} \tag{3.5}$$

and this is positive, assuming $g(x_0), \ldots, g(x_k)$ are not all equal (otherwise one can easily choose another convenient initial point x_j, $j \in \{0, \ldots, k-1\}$).

The following estimations hold then, by hypothesis:

$$d(g(x_k), g(x_{k+1})) = d(f(x_0, \dots, x_{k-1}), f(x_1, \dots, x_k)) \leq$$
$$\leq \varphi(d(g(x_0), g(x_1)), \dots, d(g(x_{k-1}), g(x_k))) \leq$$
$$\leq \varphi(d_0, \dots, d_0) < d_0$$
$$d(g(x_{k+1}), g(x_{k+2})) = d(f(x_1, \dots, x_k), f(x_2, \dots, x_{k+1})) \leq$$
$$\leq \varphi(d(g(x_1), g(x_2)), \dots, d(g(x_k), g(x_{k+1}))) \leq$$
$$\leq \varphi(d_0, \dots, d_0, \varphi(d_0, \dots, d_0)) \leq \varphi(d_0, \dots, d_0) < d_0$$
$$\vdots$$
$$d(g(x_{2k-1}), g(x_{2k})) = d(f(x_{k-1}, \dots, x_{2k-2}), f(x_k, \dots, x_{2k-1})) \leq$$
$$\leq \varphi(d(g(x_{k-1}), g(x_k)), \dots, d(g(x_{2k-2}), g(x_{2k-1}))) \leq$$
$$\leq \varphi(d_0, \varphi(d_0, \dots, d_0), \dots, \varphi(d_0, \dots, d_0)) \leq \varphi(d_0, \dots, d_0) < d_0$$
$$d(g(x_{2k}), g(x_{2k+1})) = d(f(x_k, \dots, x_{2k-1}), f(x_{k+1}, \dots, x_{2k})) \leq$$
$$\leq \varphi(d(g(x_k), g(x_{k+1})), \dots, d(g(x_{2k-1}), g(x_{2k}))) \leq$$
$$\leq \varphi(\varphi(d_0, \dots, d_0), \varphi(d_0, \dots, d_0), \dots, \varphi(d_0, \dots, d_0)) =$$
$$= \varphi^2(d_0, \dots, d_0) \leq \varphi(d_0, \dots, d_0) < d_0$$
$$d(g(x_{2k+1}), g(x_{2k+2})) \leq \cdots \leq \varphi^2(d_0, \dots, d_0, \varphi(d_0, \dots, d_0)) \leq$$
$$\leq \varphi^2(d_0, \dots, d_0) < d_0$$

and so on

$$d(g(x_n), g(x_{n+1})) \leq \varphi^{\left[\frac{n}{k}\right]}(d_0, \dots, d_0), n \geq k. \tag{3.6}$$

Thus, for some integer $p \geq 1$, we obtain:

$$d(g(x_n), g(x_{n+p})) \leq \varphi^{\left[\frac{n}{k}\right]}(d_0, \dots, d_0) + \cdots + \varphi^{\left[\frac{n+p-1}{k}\right]}(d_0, \dots, d_0), n \geq 0. \tag{3.7}$$

By denoting

$$l = \left[\frac{n}{k}\right] \text{ and } m = \left[\frac{n+p-1}{k}\right], \tag{3.8}$$

we have that $m \geq l$. Besides, the above relation (3.7) implies further estimation

$$d(g(x_n), g(x_{n+p})) \leq \underbrace{\varphi^l(d_0, \ldots, d_0) + \cdots + \varphi^l(d_0, \ldots, d_0)}_{k \text{ times}} +$$
$$+ \underbrace{\varphi^{l+1}(d_0, \ldots, d_0) + \cdots + \varphi^{l+1}(d_0, \ldots, d_0)}_{k \text{ times}} +$$
$$+ \cdots + \underbrace{\varphi^m(d_0, \ldots, d_0) + \cdots + \varphi^m(d_0, \ldots, d_0)}_{k \text{ times}},$$

so

$$d(g(x_n), g(x_{n+p})) \leq k\sum_{i=l}^{m} \varphi^i(d_0, \ldots, d_0), n \geq 0, p \geq 1. \tag{3.9}$$

Denoting $S_n = \sum\limits_{i=0}^{m} \varphi^i(d_0, \ldots, d_0)$ we have that

$$\sum_{i=l}^{m} \varphi^i(d_0, \ldots, d_0) = S_m - S_{l-1}, m \geq l.$$

As, for $s \in \mathbb{R}_+^k$, $\sum\limits_{i=0}^{\infty} \varphi^i(s) < +\infty$ from assumption $d)$ upon φ, there exists

$$S = \lim_{n\to\infty} S_n.$$

Considering (3.8) it follows that

$$\lim_{l\to\infty} \sum_{i=l}^{m} \varphi^i(d_0, \ldots, d_0) = S - S = 0,$$

and, in view of (3.9), $d(g(x_n), g(x_{n+p})) \to 0$, as $n \to \infty$. This means that $\{g(x_n)\}_{n\geq 0}$ is a Cauchy sequence contained, by (3.4), in the complete metric subspace Y, so there exists $x^* \in g(X)$ such that

$$x^* = \lim_{n\to\infty} g(x_n).$$

Consequently, since by (3.4) $Y \subseteq g(X)$, there exists $r \in X$ such that

$$g(r) = x^* = \lim_{n\to\infty} g(x_n). \tag{3.10}$$

Next we shall prove that $f(r,\dots,r)=x^*$ as well. In this respect we estimate

$$\begin{aligned}
&d(g(x_{n+1}),f(r,\dots,r))=d(f(x_{n-k+1},\dots,x_n),f(r,\dots,r))\le\\
&\le d(f(x_{n-k+1},\dots,x_n),f(x_{n-k+2},\dots,x_n,r)+\\
&\quad +d(f(x_{n-k+2},\dots,x_n,r),f(x_{n-k+3},\dots,x_n,r,r))+\\
&\quad +\dots+\\
&\quad +d(f(x_n,r,\dots,r),f(r,\dots,r)),
\end{aligned}$$

which by (cPR) becomes

$$\begin{aligned}
&d(g(x_{n+1}),f(r,\dots,r))\le \\
&\le \varphi(d(g(x_{n-k+1}),g(x_{n-k+2})),\dots,d(g(x_n),g(r)))+\\
&\quad +\varphi(d(g(x_{n-k+2}),g(x_{n-k+3})),\dots,d(g(x_n),g(r)),d(g(r),g(r)))+\\
&\quad +\dots+\varphi(d(g(x_n),g(r)),d(g(r),g(r)),\dots,d(g(r),g(r))).
\end{aligned}\tag{3.11}$$

Now by assumption $d)$ on φ and (3.6), it follows that

$$d(g(x_n),g(x_{n+1}))\to 0 \text{ as } n\to\infty,$$

while, by the continuity of the distance and (3.10),

$$d(g(x_n),g(r))=d(g(x_n),x^*)\to 0 \text{ as } n\to\infty.$$

Using the previous results, by letting $n\to\infty$ in (3.11), we get that $d(f(r,\dots,r),x^*)\le 0$, i.e.,

$$d(f(r,\dots,r),x^*)=0$$

So, in view of (3.10,) $f(r,\dots,r)=x^*=g(r)$ holds, i.e., r is a coincidence point of f and g, while x^* is a coincidence value for them.
In order to prove the uniqueness of x^*, we suppose there would be $q\in X$ such that

$$f(q,\dots,q)=g(q)\neq x^*.$$

Then

$$\begin{aligned} d(g(r),g(q)) &= d(f(r,\dots,r),f(q,\dots,q)) \leq \\ &\leq d(f(r,\dots,r),f(r,\dots,r,q)) + \dots + \\ &\quad + d(f(r,q,\dots,q),f(q,\dots,q)) \leq \\ &\leq \varphi(d(g(r),g(r)),\dots,d(g(r),g(r)),d(g(r),g(q))) + \dots + \\ &\quad + \varphi(d(g(r),g(q)),d(g(q),g(q)),\dots,d(g(q),g(q))) = \\ &= \varphi(0,\dots,0,d(g(r),g(q))) + \dots + \varphi(d(g(r),g(q)),0,\dots,0) \leq \\ &\leq \varphi(d(g(r),g(q)),\dots,d(g(r),g(q))). \end{aligned}$$

Supposing $g(r) \neq g(q)$, by hypothesis $e)$ on φ it would follow that

$$d(g(r),g(q)) < d(g(r),g(q)), \tag{3.12}$$

which is obviously a contradiction. This proves the uniqueness of the coincidence value x^*. In case f and g are also weakly compatible, by Lemma 2.3 this guarantees the existence and uniqueness of their common fixed point, which is actually the coincidence value here denoted by x^*.

The estimation (3.2) follows immediately from (3.7), by letting $p \to \infty$. $\square$

Remark 3.1. 1) Theorem 3.1 above reduces to the result due to I.A. Rus [9] for $g = 1_X$.

2) For $g = 1_X$ and $\varphi : \mathbb{R}_+^k \to \mathbb{R}_+$, $\varphi(r_1,\dots,r_k) = \sum\limits_{i=1}^{k} \alpha_i r_i$, with constants $\alpha_1,\dots,\alpha_k \in \mathbb{R}_+$ and $\sum\limits_{i=1}^{k} \alpha_i < 1$, the result due to S. Presić [7] is obtained.

3) For $k = 1$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(r) = r$, the result of M. Abbas and G. Jungck [1] in metric spaces is obtained, in a slightly more general version, as we replaced the condition "$g(X)$ a complete metric subspace" by the less restrictive and more practical one "there exists a complete subspace $Y \subseteq X$ such that $f(X^k) \subseteq Y \subseteq g(X)$".

4. **An extension of the main result**

While the great majority of the common fixed point results in literature deal with the case when both f and g are self-operators on X, the above result offers information about coincidence and common fixed points of two operators, one of them defined on the Cartesian product X^k, $f : X^k \to X$, where k is a positive integer, and the second one a self-operator on X, $g : X \to X$.

Our aim in this section is to establish common fixed point theorems for $f : X^k \to X$ and $g : X^l \to X$, with k and l positive integers. In this respect it is necessary to start with some definitions, which extend the corresponding ones in the previous section.

Definition 4.1. *Let X be a metric space, k, l positive integers and $f : X^k \to X$, $g : X^l \to X$ two operators.*

An element $p \in X$ is called a ***coincidence point*** *of f and g if it is a coincidence point of F and G, where $F, G : X \to X$ are their associate operators, see Remark 2.2.*

An element $s \in X$ is a ***coincidence value*** *of f and g if it is a coincidence value of F and G.*

An element $p \in X$ is a ***common fixed point*** *of f and g if it is a common fixed point of F and G.*

Definition 4.2. *Let (X, d) be a metric space, k, l positive integers and $f : X^k \to X$, $g : X^l \to X$. The operators f and g are said to be* ***weakly compatible*** *if F and G are weakly compatible.*

In these terms we state now the following result, which extends Theorem 3.1.

Theorem 4.1. *Let (X, d) be a metric space, k and l positive integers, $\varphi : \mathbb{R}_+^k \to \mathbb{R}_+$ with the properties $a) - e)$ in Theorem 1.1 and $f : X^k \to X$, $g : X^l \to X$ two operators such that f and G satisfy the conditions in Theorem 3.1, where G is the operator associated to g.*

Then:

1) *f and g have a unique coincidence value, say x^*, in X;*

2) *the sequence* $\{G(z_n)\}_{n\geq 0}$ *defined by* $z_0 \in X$ *and*

$$G(z_n) = f(z_{n-1}, \dots, z_{n-1}), n \geq 1, \tag{4.1}$$

converges to x^**;*

3) *the sequence* $\{G(x_n)\}_{n\geq 0}$ *defined by* $x_0, \dots, x_{k-1} \in X$ *and*

$$G(x_n) = f(x_{n-k}, \dots, x_{n-1}), n \geq k, \tag{4.2}$$

converges to x^* *as well, with a rate estimated by*

$$d(G(x_n), x^*) \leq k \sum_{i=0}^{\infty} \varphi^{\left[\frac{n+i}{k}\right]}(D_0, \dots, D_0), \tag{4.3}$$

where $D_0 = \max\{d(G(x_0), G(x_1)), \dots, d(G(x_{k-1}), G(x_k))\}$*;*

4) *if in addition* f *and* g *are weakly compatible, then* x^* *is their unique common fixed point.*

Proof. Considering the definitions given in the current section of this paper, all the conclusions follow by applying Theorem 3.1 for $f : X^k \to X$ and $G : X \to X$. $\square$

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