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SET-VALUED APPROXIMATION OF MULTIFUNCTIONS

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Abstract. This survey paper introduces several results on approximation of multifunctions with convex and non-convex values. We consider multifunctions having at least nonempty and compact values in $\mathbb{R}^n$. The convex case (when the multifunctions have convex values) is closer to the point-to-point case. The non-convex case (the values of the multifunctions are not longer assumed to be convex) is more challenging. In the convex case we present results on the Bernstein approximation, the Stone-Weierstrass approximation theorem, and the Korovkin-type approximation. In the non-convex case we present results on linear operators on multifunctions based on a metric linear combination of ordered sets, metric piecewise linear approximations of multifunctions, and approximation by metric Bernstein, Schoenberg, and interpolation operators. The present survey paper was introduced at University of Duisburg–Essen located in Duisburg while the author was a visiting scientist under a grant of "Center of Excellence for Applications of Mathematics" supported by DAAD. The author expresses his gratitude to Prof. H. Gonska for his invitation and warm hospitality in Duisburg. The author also appreciates the valuable comments and remarks of Mr. Michael Wozniczka from the same University.

1. Introduction

The aim of the paper is to introduce some older and newer results on approximation of multifunctions with convex and non-convex values.

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Let X, Y be nonempty sets and $\mathrm{P}(Y) = \mathcal{P}(Y) \setminus \{\emptyset\}$ be the collection of nonempty subsets (parts) of Y. A *multifunction, set-valued function*, or *correspondence* is an ordinary map $F : X \to \mathrm{P}(Y)$ (sometimes denoted as $F : X \rightrightarrows Y$), see [2], [19].

Suppose that Y is a real vector space. The so-called "convex case" refers to the case when the images $F(x)$ are convex, for all $x \in X$. Otherwise we say that the "non-convex case" is in force. We just mention that in the cases connected to mathematical economics (Arrow-Debreau economical equilibrium, etc.) for some multifunctions the empty set belongs to the range of F.

We will immediately see why convexity plays such a crucial role here.

The tentative plan consists

- of substituting numbers by sets (which seems to be all right, although the point corresponding to a real number is at least convex and compact);
- and consequently of substituting the operations on numbers by some operations on sets.

The *Minkowski sum* of two non-empty sets (in $\mathbb{R}^n$ or in a vector space) is defined by

$$K + L = \{x + y \mid x \in K, \quad y \in L\}.$$

Nice property: if K and L are singletons, their Minkowski sum is exactly the usual addition of numbers or vectors. Although $\{0\}$ is the identity for addition of sets, i.e., $K + \{0\} = K$, generally no additive inverse exists $(K + X = \{0\}$ cannot be solved for X unless K is reduced to a point). Moreover,

$$K + X = K + Y \not\Longrightarrow X = Y.$$

Multiplication of a set by a scalar is defined by

$$\alpha K = \{\alpha x \mid x \in K\}.$$

For $K = \{0, 1\}$, we have $K + K = \{0, 1, 2\}$ whereas $2K = \{0, 2\}$. It is hard to accept that $K + K \neq 2K$. Thus the distributive law $\alpha K + \beta K = (\alpha + \beta)K$ generally fails to hold. However, if K is convex, then

$$\alpha K + \beta K = (\alpha + \beta)K, \text{ for } \alpha, \beta \geq 0. \tag{1.1}$$

A generalization of (1.1) can be proved by induction, namely, a set K is said to be convex if and only if

$$\alpha_1, \ldots, \alpha_N \geq 0, \quad N \geq 2 \implies \alpha_1 K + \cdots + \alpha_N K = (\alpha_1 + \cdots + \alpha_N)K. \tag{1.2}$$

Equality (1.1) suggests that the class of convex-valued multifunctions might be an appropriate setting in which we might begin considering set-valued approximation problems.

Subtraction is not well defined and is generally impossible.

$X_1 + \cdots + X_N$ generally "gets bigger" as N increases.

Let $\mathbb{K}$ be the *collection of nonempty and compact subsets* of $\mathbb{R}^n$.

We begin by posing the question: Is it possible to *approximate* a multifunction $F : [0,1] \rightrightarrows \mathbb{R}^n$ by a "simpler" one? More concrete, by a *linear* approximant of the form

$$\sum_{j=0}^{N} \varphi_j K_j = \varphi_0 K_0 + \cdots + \varphi_N K_N,$$

where the K_j 's are fixed elements in $\mathbb{K}$ and the φ_j 's are scalar-valued maps defined on $[0,1]$.

Recall that

- the *Bernstein operator* for $f \in C[0,1]$ is

$$\mathrm{B}_N(f,x) = \sum_{k=0}^{N} p_{N,k}(x) f\left(k/N\right), \tag{1.3}$$

where

$$p_{N,k}(x) = \binom{N}{k} x^k (1-x)^{N-k}, \quad x \in [0,1], \tag{1.4}$$

- the *Lagrange interpolation* formula for $f : [a,b] \to \mathbb{R}$ is

$$a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b, \quad L_N(x) = \sum_{k=0}^{N} l_k(x) f(x_k),$$

where

$$l_k(x) = \frac{(x-x_0)\dots(x-x_{k-1})(x-x_{k+1})\dots(x-x_N)}{(x_k-x_0)\dots(x_k-x_{k-1})(x_k-x_{k+1})\dots(x_k-x_N)}\,, \tag{1.5}$$

• the *Hermite-Fejér polynomial* $H_N(f,x)$ of a function $f:[-1,1]\to\mathbb{R}$, based on the zeros

$$x_k = x_k^{(N)} = \cos\frac{(2k-1)\pi}{2N}, \quad k=1,2,\dots,N, \tag{1.6}$$

of the Chebyshev polynomial $T_N(x)=\cos(N\arccos x)$, is

$$q_k(x) = \left(\frac{T_N(x)}{N(x-x_k)}\right)^2(1-xx_k), \quad H_N(x) = \sum_{k=1}^{N} q_k(x)f(x_k), \quad x\in[-1,1],$$

• and a (univariate, polynomial) *spline* $S:[a,b]\to\mathbb{R}$ is a piecewise polynomial function, that is, it consists of polynomial pieces $P_i:[x_i,x_{i+1}[\to\mathbb{R}$, where $a=x_0<x_1<\dots<x_N=b$, $i=0,1,\dots,N-1$, such that

$$\begin{cases} S(x)=P_0(x), & x\in[x_0,x_1[\,,\\ S(x)=P_1(x), & x\in[x_1,x_2[\,,\\ \dots & \\ S(x)=P_{N-1}(x), & x\in[x_{N-1},x_N], \end{cases}$$

and at x_i the two pieces P_{i-1} and P_i share common derivative values.

"*Polynomials are wonderful even after they are cut into pieces, but the cutting must be done with care. One way of doing the cutting leads to the so-called* spline *functions.*" Isaac Jacob Schoenberg (Galatzi 1903 - Madison, WI, 1990) penned these prophetic words in 1964.

Along this paper by $\square$ we denote the end of a proof and by $\triangle$ the end of a remark or an example. We mention some notations that appear along our paper. $\mathbb{B}$ denotes the closed unit ball in $\mathbb{R}^n$, $\mathbb{K}$ the family of nonempty and compact subsets in $\mathbb{R}^n$, $\mathbb{K}_c$ the collection of elements of $\mathbb{K}$ that are also convex, H the Hausdorff-Pompeiu metric on $\mathbb{K}$, $\langle\cdot,\cdot\rangle$ is the inner product on $\mathbb{R}^n$, $\mathbb{S}$ is the unit sphere in $\mathbb{R}^n$, $\sigma(\cdot,\cdot)$ the support function, $\mathbb{C}[\mathbb{K}]$ and $\mathbb{C}[\mathbb{K}_c]$ the spaces of continuous functions on

$[0,1]$ into $\mathbb{K}$ and $\mathbb{K}_c$, respectively, B_n the Banach space of continuous functions defined on the unit sphere in $\mathbb{R}^n$, and $\mathrm{B}_N(\cdot,\cdot)$ the Bernstein operator.

Some results of R. Vitale in [28] are introduced in the sequel.

The *Hausdorff-Pompeiu metric* can be introduced on $\mathbb{K}$ in several ways, one of these being as follows

$$\mathrm{H}(K_1,K_2)=\min\{\varepsilon>0 \mid K_1\subset K_2+\varepsilon\mathbb{B},\quad K_2\subset K_1+\varepsilon\mathbb{B}\} \tag{1.7}$$

where $\mathbb{B}$ is the closed unit ball in $\mathbb{R}^n$. We note that

$$\mathrm{H}(K_1,K_2)<+\infty,\quad \forall\, K_1,K_2\in\mathbb{K}. \tag{1.8}$$

Thus $(\mathbb{K},\mathrm{H})$ is a complete, separable, and locally compact metric space. Define

$$\|K\|=\mathrm{H}(\{0\},K),$$

the "norm" of $K\in\mathbb{K}$.

Proposition 1.1. *Let $A,B\in\mathbb{K}$ and α be a real number. Then*

$$\mathrm{H}(\alpha A,\alpha B)=|\alpha|\,\mathrm{H}(A,B). \tag{1.9}$$

Proof. If $\alpha=0$, we have $\mathrm{H}(\alpha A,\alpha B)=\mathrm{H}(\{0\},\{0\})=0=0\cdot\mathrm{H}(A,B)$. If $\alpha>0$, we successively have

$$\begin{aligned}\mathrm{H}(\alpha A,\alpha B)&=\min\{\varepsilon>0 \mid \alpha A\subset\alpha B+\varepsilon\mathbb{B},\ \alpha B\subset\alpha A+\varepsilon\mathbb{B}\}\\ &=\min\{\varepsilon>0 \mid A\subset B+(\varepsilon/\alpha)\mathbb{B},\ B\subset A+(\varepsilon/\alpha)\mathbb{B}\}\\ &=\alpha\,\min\{\varepsilon/\alpha>0 \mid A\subset B+(\varepsilon/\alpha)\mathbb{B},\ B\subset A+(\varepsilon/\alpha)\mathbb{B}\}=\alpha\,\mathrm{H}(A,B).\end{aligned}$$

If $\alpha<0$, then consider $\beta=-\alpha$ and it follows

$$\begin{aligned}\mathrm{H}(\alpha A,\alpha B)&=\min\{\varepsilon>0 \mid -\beta A\subset-\beta B+\varepsilon\mathbb{B},\ -\beta B\subset-\beta A+\varepsilon\mathbb{B}\}\\ &=\min\{\varepsilon>0 \mid \beta A\subset\beta B+\varepsilon\mathbb{B},\ \beta B\subset\beta A+\varepsilon\mathbb{B}\}=\beta\,\mathrm{H}(A,B)=|\alpha|\,\mathrm{H}(A,B).\end{aligned}$$

□

2. **The convex case**

2.1. $\mathbb{K}_c$ **.** We denote by $\mathbb{K}_c$ the collection of elements of $\mathbb{K}$ which are also convex. Then $\mathbb{K}_c$ is closed under

- Minkowski addition,
- Minkowski multiplication with scalars,
- the distributive property (1.1),
- $\mathbb{K}_c$ inherits its metric from $\mathbb{K}$ as a closed, separable and locally compact subspace.

Given an element $K \in \mathbb{K}$, we often form its *convex hull* denoted as $\mathrm{conv}K$, which obviously lies in $\mathbb{K}_c$. The map

$$\mathbb{K} \ni K \mapsto \mathrm{conv}K \in \mathbb{K}_c$$

is continuous in respect to the Hausdorff-Pompeiu metric since

$$\mathrm{H}(\mathrm{conv}A, \mathrm{conv}B) \leq \mathrm{H}(A, B), \quad \forall\, A, B \in \mathbb{K}, \tag{2.1}$$

and additionally satisfies $\mathrm{conv}(\alpha K_1 + \beta K_2) = \alpha\, \mathrm{conv}K_1 + \beta\, \mathrm{conv}K_2$, for all $\alpha, \beta \geq 0$.

2.1.1. *Support function.* To each $K \in \mathbb{K}$ we associate its *support function* given by

$$\sigma(p, K) = \max\{\langle p, k\rangle \mid k \in K\}, \quad p \in \mathbb{S}, \tag{2.2}$$

where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathbb{R}^n$ and $\mathbb{S}$ is the unit sphere in $\mathbb{R}^n$.

A set $K \in \mathbb{K}_c$ and a point not in K can always be separated by some hyperplane and this leads to the useful equivalence

$$K_1 \subset K_2 \Longleftrightarrow \sigma(p, K_1) \leq \sigma(p, K_2,), \quad \forall\, p \tag{2.3}$$

and the consequent uniqueness of support functions, namely

$$K_1 = K_2 \Longleftrightarrow \sigma(p, K_1) = \sigma(p, K_2), \quad \forall\, p.$$

Proposition 2.1. *As a function of* p, *the support function is Lipschitz (and thus continuous), that is*

$$|\sigma(p_1, K) - \sigma(p_2, K)| \leq \|p_1 - p_2\|\, \|K\|.$$

Obviously, we may use and we will do so, the map

$$\mathbb{K}_c \ni K \mapsto \sigma(\cdot, K)$$

to embed $\mathbb{K}_c$ in the Banach space B_n of continuous functions defined on the unit sphere in $\mathbb{R}^n$.

Proposition 2.2. ([17]) *The following properties hold:*

$$\sigma(p, \mathbb{B}) = 1, \quad \forall\, p \in \mathbb{S}, \tag{2.4}$$

$$\sigma(\cdot, \alpha K) = \alpha \sigma(\cdot, K), \quad \alpha \geq 0, \tag{2.5}$$

$$\sigma(\cdot, K_1 + K_2) = \sigma(\cdot, K_1) + \sigma(\cdot, K_2), \tag{2.6}$$

$$\mathrm{H}(K_1, K_2) = \|\sigma_1 - \sigma_2\|, \text{ (uniform norm)}, \tag{2.7}$$

$$\|K\| = \|\sigma(\cdot, K)\|, \tag{2.8}$$

where at (2.7) *we mean* $\sigma_1 = \sigma_1(\cdot, K_1)$ *and* $\sigma_2 = \sigma_2(\cdot, K_2)$.

Proof. The support function of the closed unit ball $\mathbb{B}$ is identically 1 since by the Cauchy inequality we have that $\langle p, b\rangle \leq 1$, for all $p \in \mathbb{S}$, and $b \in \mathbb{B}$ and on the other hand each $p \in \mathbb{R}^n$ with $\|p\| = 1$ also belongs to $\mathbb{B}$, so $\langle p, p\rangle = 1$. Thus (2.4) follows.

Let us see how (2.7) comes about. Since the support function of the closed unit ball $\mathbb{B}$ is identically 1, so that (2.5) and (2.6) imply $\sigma(p, K_2 + \varepsilon\mathbb{B}) = \sigma(p, K_2) + \varepsilon$. Together with (2.3) this yields, for all p,

$$K_1 \subset K_2 + \varepsilon\mathbb{B} \Longleftrightarrow \sigma(p, K_1) \leq \sigma(p, K_2) + \varepsilon \Longleftrightarrow \sigma(p, K_1) - \sigma(p, K_2) \leq \varepsilon.$$

Similarly, for all p,

$$K_2 \subset K_1 + \varepsilon\mathbb{B} \Longleftrightarrow \sigma(p, K_2) \leq \sigma(p, K_1) + \varepsilon \Longleftrightarrow \sigma(p, K_2) - \sigma(p, K_1) \leq \varepsilon.$$

For both inclusions to hold, we have to have

$$|\sigma(p, K_1) - \sigma(p, K_2)| \leq \varepsilon, \quad \forall\, p. \tag{2.9}$$

The infimum of all $\varepsilon > 0$ satisfying (2.9) is at once $\mathrm{H}(K_1, K_2)$ and $\|\sigma_1 - \sigma_2\|$. In particular, $K_2 = \{0\}$ implies $\|K\| = \|\sigma(\cdot, K)\|$. $\square$

Corollary 2.3. *For any* $p_1, p_2 \in \mathbb{S}$ *and* $K_1, K_2 \in \mathbb{K}$, *we have that*

$$|\sigma(p_1, K_1) - \sigma(p_2, K_2)| \leq \|p_1 - p_2\| \, \|K_1\| + \mathrm{H}(K_1, K_2).$$

$\mathbb{C}[\mathbb{K}]$ and $\mathbb{C}[\mathbb{K}_c]$ denote the spaces of continuous functions on $[0,1]$ into $\mathbb{K}$ and $\mathbb{K}_c$, respectively. Given a map $F \in \mathbb{C}[\mathbb{K}]$, we denote its *norm* by

$$\mathrm{H}(F) = \sup_{x \in [0,1]} \{\|F(x)\|\}$$

and define the related metric by

$$\mathrm{H}(F, G) = \sup_{x \in [0,1]} \{\mathrm{H}(F(x), G(x))\}. \tag{2.10}$$

2.2. **Bernstein approximation.** Given a multifunction F defined on $[0,1]$, we define the Nth *Bernstein approximant* to be

$$\mathrm{B}_N(F, x) = \sum_{k=0}^{N} p_{N,k}(x) F\left(k/N\right), \tag{2.11}$$

where the $p_{N,k}(\cdot)$ polynomials are given by (1.4). The addition and multiplication in the right-hand of (2.11) are understood in the Minkowski sense. It is clear that this map necessarily lies in $\mathbb{C}[\mathbb{K}]$ and, indeed, in $\mathbb{C}[\mathbb{K}_c]$ if $F \in \mathbb{C}[\mathbb{K}_c]$.

Theorem 2.4. *Let* $F \in \mathbb{C}[\mathbb{K}_c]$. *Then* $\mathrm{B}_N(F, \cdot)$ *converges uniformly to* F, *i. e.,* $\mathrm{H}(F, \mathrm{B}_N(F, \cdot)) \xrightarrow[N \to \infty]{u} 0$, *where* $\xrightarrow{u}$ *denotes the uniform convergence.*

Proof. We use the Banach space embedding

$$\begin{array}{ccc}
\mathrm{B}_N(F, \cdot) & \xrightarrow[\mathrm{H}(\cdot,\cdot)]{u\,?} & F \\
\big\downarrow & & \big\uparrow \\
\sigma(\cdot, \mathrm{B}_N(F, \cdot)) & \xrightarrow[\|\cdot\|_\infty]{u} & \sigma(\cdot, F).
\end{array}$$

The above diagram has to be read as follows. We ask if the sequence $(\mathrm{B}_N(F, \cdot))_N$ converges uniformly to F in respect to the Hausdorff-Pompeiu metric. The answer is given by embedding the sequence $(\mathrm{B}_N(F, \cdot))_N$ into the Banach space of continuous functions on $\mathbb{S}$ by the support function, then checking the uniform convergence of the sequence $(\sigma(\mathrm{B}_N(F, \cdot)))_N$ toward $\sigma(\cdot, F)$, and then returning to F.

Then $F \in \mathbb{C}[\mathbb{K}_c]$ is equivalent to the continuity of the map from $[0,1]$ into B_n given by $x \mapsto \sigma(\cdot, F(x))$.

A Bernstein approximant of F corresponds to the map

$$[0,1] \ni x \mapsto \sum_{k=0}^{N} p_{N,k}(x)\sigma(\cdot, F(k/N)).$$

Hence it is enough to show the uniform convergence (in B_n) of the latter maps to $x \mapsto \sigma(\cdot, F(x))$. Indeed

$$\begin{aligned}
\mathrm{H}(F(\cdot), \mathrm{B}_N(F,\cdot)) &\overset{(2.10)}{=} \sup_{x\in[0,1]} \{\mathrm{H}(F(x), \mathrm{B}_N(F,x))\} \\
&\overset{(2.7)}{=} \sup_{x\in[0,1]} \{\|\sigma(\cdot, F(x)) - \sigma(\cdot, \mathrm{B}_N(F,x))\|\} \\
&\overset{(2.5)}{=} \sup_{x\in[0,1]} \{\|\sigma(\cdot, F(x)) - \mathrm{B}_N\sigma(\cdot, F(x))\|\} \xrightarrow[N\to\infty]{\text{uniformly in } x} 0,
\end{aligned}$$

by a result of T. Popoviciu, [6, pp. 109–111], or [20, pp. 155–160]. □

We turn to the case when $F \in \mathbb{C}[\mathbb{K}]$ does not necessarily have convex values. This does not preclude forming $\mathrm{B}_N(F,\cdot)$.

Remark 2.5. If $K = \{0,1\}$, then $\mathrm{conv}K = [0,1]$,

$$(1/N)\sum_{k=0}^{N} K = (1/N)(K + K + \cdots + K) = \{0, 1/N, 2/N, \ldots, 1\}$$

and

$$\mathrm{H}((1/N)\sum_{k=0}^{N} K, \mathrm{conv}K) \to 0 \text{ as } N \to \infty,$$

since if $K_1 = \{0, 1/N, 2/N, \ldots, 1\}$ and $K_2 = \mathrm{conv}K$, then

$$\begin{aligned}
\mathrm{H}(K_1, K_2) &= \min\{\varepsilon > 0 \mid K_1 \subset K_2 + \varepsilon[-1,1], \quad K_2 \subset K_1 + \varepsilon[-1,1]\} \\
&= \min\{\varepsilon > 0 \mid K_2 \subset K_1 + \varepsilon[-1,1]\} = 1/(2N). \quad \triangle
\end{aligned}$$

An uncomfortable situation is revealed by the following result and its consequences.

Theorem 2.6. (Shapley-Folkman-Starr, [25]) *If* $K_1, \ldots, K_N \in \mathbb{K}$, *then*

$$\mathrm{H}(K_1 + \cdots + K_N, \mathrm{conv}(K_1 + \cdots + K_N)) \le \sqrt{n} \max_{1\le i\le N} \|K_i\|. \tag{2.12}$$

The special case $K_1 = \cdots = K_N = K$ leads to

Corollary 2.7. *If* $K_1 = K_2 = \cdots = K_N = K \in \mathbb{K}$, *then*

$$\mathrm{H}(K + \cdots + K, \mathrm{conv}(K + \cdots + K)) \overset{(2.12)}{\leq} \sqrt{n}\,\|K\|. \tag{2.13}$$

Corollary 2.8.

$$\mathrm{H}\left((K + \cdots + K)/N,\, (1/N)\mathrm{conv}(K + \cdots + K)\right) \overset{(1.9)}{\leq} (\sqrt{n}/N)\|K\| \xrightarrow[N\to\infty]{} 0.$$

It comes another uncomfortable case.

Proposition 2.9. *A set* $K \in \mathbb{K}$ *is infinitely divisible for Minkowski sums, i. e., admits the following representation*

$$K = L_N + \cdots + L_N, \ \text{for all } N \geq 2, \tag{2.14}$$

if and only if K *is convex.*

Proof. *Sufficiency.* Suppose that K is a convex set. Then a representation of the form (2.14) exists by taking $L_N = (1/N)K$ and applying (1.2).
Necessity. From (2.14) it follows that $\|L_N\| \leq \|K\|/N$. Then Corollary 2.7 and the previous inequality imply that

$$\mathrm{H}(K, \mathrm{conv}K) \leq \sqrt{n}\,\|L_N\| \leq \sqrt{n}\,\|K\|/N.$$

□

A more exact variant of the Shapley-Folkman-Starr theorem is valid. For a set $K \in \mathbb{K}$, define its *diameter* respectively, *radius* by $\mathrm{diam}\, K = \max_{x,y\in K} \|x - y\|$ and $\mathrm{rad}\, K = (1/2)\mathrm{diam}\, K$.

Theorem 2.10. (Shapley-Folkman-Starr, [18, p. 407]) *If* $K_1, \ldots, K_N$ *are compact subsets of* $\mathbb{R}^n$, *then*

$$\mathrm{H}(K_1 + \cdots + K_N,\, \mathrm{conv}(K_1 + \cdots + K_N)) \leq \sqrt{n} \max_{1\leq i\leq N} \mathrm{rad}\, K_i.$$

Another result of the same sort is mentioned below.

Lemma 2.11. (Shapley-Folkman-Starr, [18, p. 407]) *If* $K_1, \dots, K_N$ *are compact subsets of* $\mathbb{R}^n$, *then*

$$\mathrm{H}(K_1 + \cdots + K_N,\, \mathrm{conv}(K_1 + \cdots + K_N))^2 \le \sum_{i=1}^{N} (\mathrm{rad}\, K_i)^2.$$

Theorem 2.12. *Let* $F \in \mathbb{C}[\mathbb{K}]$. *Then in any interval* $[\varepsilon, 1-\varepsilon]$, $0 < \varepsilon < 1/2$, $\mathrm{B}_N(F,\cdot)$ *converges uniformly to* $\mathrm{conv}F$ *(here* $(\mathrm{conv}F)(t) = \mathrm{conv}F(t)$, $t \in [0,1]$).

Proof. With

$$\mathrm{B}_N(F,x) = \sum_{k=0}^{N} p_{N,k}(x) F\left(k/N\right),$$

we identify $K_k = p_{N,k}(x) F(k/N)$ in (2.12). Now

$$\|K_k\| \le \|F(k/N)\|\, |p_{N,k}(x)| \le \mathrm{H}(F) \sup\{\, p_{N,k}(x) \mid \varepsilon \le x \le 1-\varepsilon,\ k = 0,1,\dots,N\}.$$

The indicated supremum is shown to be $O(N^{-1/2})$ [1], so by the Shapley-Folkman-Starr theorem one has that

$$\mathrm{H}(\mathrm{B}_N(F,x), \mathrm{B}_N(\mathrm{conv}F,x)) \le \mathrm{H}(F)\, O(N^{-1/2})\, n^{1/2}.$$

Theorem 2.4 and the triangle inequality yields

$$\begin{aligned} \mathrm{H}(\mathrm{conv}F, \mathrm{B}_N(F,\cdot)) &\le \mathrm{H}(\mathrm{B}_N(F,\cdot), \mathrm{B}_N(\mathrm{conv}F,\cdot)) \\ &+ \mathrm{H}(\mathrm{B}_N(\mathrm{conv}F,\cdot), \mathrm{conv}F) \xrightarrow[N\to\infty]{u} 0. \end{aligned}$$

□

Remark 2.13. The result cannot be extended to the full interval since

- at each endpoint $x = 0,\ 1$, $\mathrm{B}_N(F,x) = F(x)$ independent of N;
- the $O(N^{-1/2})$ bound breaks down at the endpoints. △

Below there are some properties which follow directly from the support function embedding and properties of Bernstein approximant in the real-valued case.

Proposition 2.14. *Given* $F : [0,1] \to \mathbb{K}_c$ *a set-valued mapping.*

[1] It follows from the limit $\lim_{N\to\infty} \sqrt{Nx(1-x)} \max_{0\le k\le N} p_{N,k} = 1/\sqrt{2\pi}$, [13, Secs. 11, 12]

(a) *Suppose* $K_1, K_2 \in \mathbb{K}_c$. *Then*

$$K_1 \subset F(x) \subset K_2, \text{ for all } x \implies K_1 \subset \mathrm{B}_N(F,x) \subset K_2, \text{ for all } x.$$

In particular, $\cap_{t\in[0,1]} F(t) \subset \mathrm{B}_N(F,x) \subset \operatorname{conv}(\cup_{t\in[0,1]} F(t))$, *for all* x.

(b) $F(s) \subset (\supseteq) F(t)$, *for all* s,t *with* $0 \le s \le t \le 1 \implies \mathrm{B}_N(F,s) \subset (\supseteq) \mathrm{B}_N(F,t)$, *for all* s,t *with* $0 \le s \le t \le 1$.

(c) $F((s+t)/2) \subset (\supseteq)(1/2)(F(s)+F(t))$, *for all* s,t, *implies that*

$$\mathrm{B}_N(F,(s+t)/2) \subset (\supseteq)(1/2)(\mathrm{B}_N(F,s)+\mathrm{B}_N(F,t)), \text{ for all } s,t.$$

(d) *Let* $G:[0,1]\to \mathbb{K}_c$ *be a set-valued mapping. Suppose that* $F(x)\cap G(x) = \emptyset$, *for all* x. *Then, for* N *sufficiently large,*

$$\mathrm{B}_N(F,x)\cap \mathrm{B}_N(G,x) = \emptyset, \text{ for all } x.$$

Proof. (a) For the first claim we successively have

$$\begin{aligned}
K_1 \subset F(x) \subset K_2,\ \forall x &\implies \sigma(p,K_1) \le \sigma(p,F(x)) \le \sigma(p,K_2),\ \forall p,x \\
&\implies \sigma(p,K_1) \le \sigma(p,F(k/N)) \le \sigma(p,K_2), \quad \forall p,k \\
&\implies \binom{N}{k} x^k (1-x)^{N-k}\sigma(p,K_1) \le \binom{N}{k} x^k(1-x)^{N-k}\sigma(p,F(k/N)) \\
&\le \binom{N}{k} x^k (1-x)^{N-k}\sigma(p,K_2),\ \forall p,x,k \\
&\overset{\sum_k}{\Longrightarrow} \sigma(p,K_1) \le \sigma(p, \sum_{k=0}^{N}\binom{N}{k}x^k(1-x)^{N-k}F(k/N)) \le \sigma(p,K_2),\ \forall p,x \\
&\implies K_1 \subset \mathrm{B}_N(F,x) \subset K_2, \quad \forall x \in [0,1].
\end{aligned}$$

For the second claim we successively have

$$\begin{aligned}
&F(k/N) = F(k/n) = F(k/N) \\
&\implies \cap_{t\in[0,1]}F(t) \subset F(k/N) \subset \cup_{t\in[0,1]}F(t) \subset \operatorname{conv}(\cup_t F(t)) \\
&\implies \cap_{t\in[0,1]}F(t) \subset \mathrm{B}_N(F,x) \subset \operatorname{conv}\left(\cup_{t\in[0,1]}F(t)\right), \quad \forall x\in[0,1].
\end{aligned}$$

(d) Choose an arbitrary but fixed $x \in [0,1]$. Since F and G are of compact values, there exists an ε so that

$$0 < 3\varepsilon = \min_{u \in F(x),\, v \in G(x)} \|u - v\|.$$

Denote $A = \mathrm{cl}\,(F(x) + \varepsilon\mathbb{B})$ and $B = \mathrm{cl}\,(G(x) + \varepsilon\mathbb{B})$, where cl stands for the closure of a set. We have that $A \cap B = \emptyset$. Since $\mathrm{B}_N(F,x)$ and $\mathrm{B}_N(G,x)$ converge uniformly to $F(x)$, respectively $G(x)$, from a given rank N_0 we have that $\mathrm{B}_N(F,x) \subset A$ and $\mathrm{B}_N(G,x) \subset B$, for all $N > N_0$. Therefore $\mathrm{B}_N(F,x)$ and $\mathrm{B}_N(G,x)$ are disjoint sets for all $N > N_0$. $\square$

Remark 2.15. (a), (b), and (c) in Proposition 2.14 can be obtained from a more general result introduced in [31]. These considerations are presented in the sequel. $\triangle$

Definition 2.16. *Let L be an operator defined on the linear space $\mathbb{R}^{[0,1]}$ (of real-valued functions defined on $[0,1]$ with the usual operations) having values in $\mathbb{R}^{[0,1]}$. An operator $\mathcal{L}$ defined on the set $\mathbb{K}_c^{[0,1]}$ (of functions defined on $[0,1]$ with values in $\mathbb{K}_c$) having values in $\mathbb{K}_c^{[0,1]}$ and satisfying*

$$L(\sigma(p, F(\cdot)), x) = \sigma(p, \mathcal{L}(F, x)) \tag{2.15}$$

for all $F \in \mathbb{K}_c^{[0,1]}$, $x \in [0,1]$, and $p \in \mathbb{S}$, where $\mathbb{S}$ denotes the unit sphere in $\mathbb{R}^n$, is said to be a set-valued equivalent *of L.*

Example 2.17. Let $L : \mathbb{R}^{[0,1]} \to \mathbb{R}^{[0,1]}$ be an operator of the form

$$L : \mathbb{R}^{[0,1]} \ni f \mapsto \sum_{i=0}^{N} f(\xi_i)\alpha_i \in \mathbb{R}^{[0,1]} \tag{2.16}$$

with abscissae $\xi_i \in [0,1]$ and fundamental functions $\alpha_i \in \mathbb{R}_{\geq 0}^{[0,1]}$ such that $\sum_{i=0}^{N} \alpha_i = 1$. By definition, L is discretely defined, positive, linear, and exact for constant functions. The operator $\mathcal{L} : \mathbb{K}_c^{[0,1]} \to \mathbb{K}_c^{[0,1]}$, specified by

$$\mathcal{L} : \mathbb{K}_c^{[0,1]} \ni F \mapsto \sum_{i=0}^{N} F(\xi_i)\alpha_i \in \mathbb{K}_c^{[0,1]}, \tag{2.17}$$

is a set-valued equivalent of L reproducing constant functions in $\mathbb{K}_c^{[0,1]}$. $\triangle$

Definition 2.18. *Let* $L : \mathbb{R}^{[0,1]} \to \mathbb{R}^{[0,1]}$ *,* $f \in \mathbb{R}^{[0,1]}$ *, and* $k_1, k_2 \in \mathbb{R}$ *. The operator* L *is said to preserve*

(a) lower bounds *if* $k_1 \le f \Longrightarrow k_1 \le Lf$ *, i.e., for all* $x \in [0,1]$*,* $k_1 \le f(x)$ *implies* $k_1 \le L(f,x)$*, where* k_1 *is independent of* x *,*

(b) upper bounds *if* $f \le k_2 \Longrightarrow Lf \le k_2$ *,*

(c) bounds *if it preserves lower and upper bounds,*

(d) monotonicity *if* $f(x) \le f(y)$*, for all* $0 \le x \le y \le 1$*, implies*

$$L(f,x) \le L(f,y), \quad \forall\, 0 \le x \le y \le 1,$$

and

(e) midconvexity *if for all* $x, y \in [0,1]$ $f\left((x+y)/2\right) \le (f(x)+f(y))/2$ *implies*

$$L\left(f,(x+y)/2\right) \le (L(f,x)+L(f,y))/2.$$

Similarly, we agree upon

Definition 2.19. *Let* $\mathcal{L} : \mathbb{K}_c^{[0,1]} \to \mathbb{K}_c^{[0,1]}$ *,* $F \in \mathbb{K}_c^{[0,1]}$ *, and* $K_1, K_2 \in \mathbb{K}_c$ *. The operator* $\mathcal{L}$ *is said to preserve*

(a) lower bounds *if* $K_1 \subseteq F \Longrightarrow K_1 \subseteq \mathcal{L}F$ *, i.e., for all* $x \in [0,1]$*,* $K_1 \subseteq F(x) \Longrightarrow K_1 \subseteq \mathcal{L}(F,x)$*, where* K_1 *is independent of* x *,*

(b) upper bounds *if* $F \subseteq K_2 \Longrightarrow \mathcal{L}F \subseteq K_2$ *,*

(c) bounds *if it preserves lower and upper bounds,*

(d) monotonicity *if for all* $0 \le x \le y \le 1$*,* $F(x) \subseteq F(y) \Longrightarrow \mathcal{L}(F,x) \subseteq \mathcal{L}(F,y)$*, and*

(e) midconvexity *if for all* $x, y \in [0,1]$*,*

$$F\left((x+y)/2\right) \subseteq (F(x)+F(y))/2 \Longrightarrow \mathcal{L}\left(F,(x+y)/2\right) \subseteq (\mathcal{L}(F,x)+\mathcal{L}(F,y))/2.$$

Proposition 2.20. (Inheritance, [31]) *A set-valued equivalent* $\mathcal{L} : \mathbb{K}_c^{[0,1]} \to \mathbb{K}_c^{[0,1]}$ *of an operator* $L : \mathbb{R}^{[0,1]} \to \mathbb{R}^{[0,1]}$ *preserves*

(a) *(lower, upper) bounds,*

(b) *monotonicity, and*

(c) *midconvexity,*

respectively, if L *possesses the corresponding property.*

Proof. Let $F, G \in \mathbb{K}_c^{[0,1]}$ and $K_1 \in \mathbb{K}_c$.

(a) If L preserves lower bounds, for all $p \in \mathbb{S}$, $x \in [0,1]$, we successively have

$$K_1 \subseteq F(x) \Longrightarrow \sigma(p, K_1) \le \sigma(p, F(x)) \Longrightarrow \sigma(p, K_1) \le L(\sigma(p, F(\cdot)), x)$$
$$\Longrightarrow \sigma(p, K_1) \le \sigma(p, \mathcal{L}(F, x)) \Longrightarrow K_1 \subseteq \mathcal{L}(F, x).$$

The preservation of upper bounds by $\mathcal{L}$ follows analogously if L preserves upper bounds.

(b) If L is monotonicity preserving, for all $p \in \mathbb{S}$, $0 \le x \le y \le 1$, it holds

$$F(x) \subseteq F(y) \Longrightarrow \sigma(p, F(x)) \le \sigma(p, F(y)) \Longrightarrow L(\sigma(p, F(\cdot)), x) \le L(\sigma(p, F(\cdot)), y)$$
$$\Longrightarrow \sigma(p, \mathcal{L}(F, x)) \le \sigma(p, \mathcal{L}(F, y)) \Longrightarrow \mathcal{L}(F, x) \subseteq \mathcal{L}(F, y)1.$$

(c) For midconvex L and for all $p \in \mathbb{S}$, $x, y \in [0,1]$ we successively have

$$F\left(\frac{x+y}{2}\right) \subseteq \frac{F(x)+F(y)}{2} \Longrightarrow \sigma\left(p, F\left(\frac{x+y}{2}\right)\right) \le \sigma\left(p, \frac{F(x)+F(y)}{2}\right)$$
$$\Longrightarrow \sigma\left(p, F\left(\frac{x+y}{2}\right)\right) \le \frac{\sigma(p, F(x)) + \sigma(p, F(y))}{2}$$
$$\Longrightarrow L\left(\sigma\left(p, F(\cdot)\right), \frac{x+y}{2}\right) \le \frac{L(\sigma(p, F(\cdot)), x) + L(\sigma(p, F(\cdot)), y)}{2}$$
$$\Longrightarrow \sigma\left(p, \mathcal{L}\left(F, \frac{x+y}{2}\right)\right) \le \frac{\sigma(p, \mathcal{L}(F, x)) + \sigma(p, \mathcal{L}(F, y))}{2}$$
$$\Longrightarrow \sigma\left(p, \mathcal{L}\left(F, \frac{x+y}{2}\right)\right) \le \sigma\left(p, \frac{\mathcal{L}(F, x) + \mathcal{L}(F, y)}{2}\right)$$
$$\Longrightarrow \mathcal{L}\left(F, \frac{x+y}{2}\right) \subseteq \frac{\mathcal{L}(F, x) + \mathcal{L}(F, y)}{2}.$$

□

Remark 2.21. For bounds preserving operators $\mathcal{L} : \mathbb{K}_c^{[0,1]} \to \mathbb{K}_c^{[0,1]}$ and arbitrary $F \in \mathbb{K}_c^{[0,1]}$, we have

$$\bigcap_{t \in [0,1]} F(t) \subseteq \mathcal{L}F \subseteq \operatorname{conv}\left(\bigcup_{t \in [0,1]} F(t)\right)$$

since

$$\bigcap_{t\in[0,1]} F(t) \subseteq F \subseteq \bigcup_{t\in[0,1]} F(t) \subseteq \operatorname{conv}\left(\bigcup_{t\in[0,1]} F(t)\right). \quad \triangle$$

Example 2.22. The the Nth Bernstein operator $\mathrm{B}_N : \mathbb{K}_c^{[0,1]} \to \mathbb{K}_c^{[0,1]}$ as given in (2.11) preserves bounds, monotonicity, and convexity. $\triangle$

2.3. **Stone-Weierstrass approximation theorem.** The approximation theorem as originally discovered by K. Weierstrass is as follows:

Theorem 2.23. (Weierstrass) *Suppose f is a continuous complex-valued function defined on the real interval $[a,b]$. For every $\varepsilon > 0$, there exists a polynomial function p over $\mathbb{C}$ such that for all $x \in [a,b]$, we have $|f(x) - p(x)| < \varepsilon$, or equivalently, the supremum norm $\|f - p\| < \varepsilon$.*

If f is real-valued, the polynomial function can be taken over $\mathbb{R}$.

A constructive proof of this theorem for f real-valued using Bernstein polynomials can be found in many books, see [6], [20].

An *associative algebra* A over a field F is defined to be a vector space A over F together with an F-bilinear multiplication $A \times A \to A$ (where the image of (x,y) is written as xy) such that the associative law holds:

- $(xy)z = x(yz)$ for all x, y and $z \in A$.

The bilinearity of the multiplication is expressed as

- $(x+y)z = xz + yz$ for all x, y and $z \in A$,
- $x(y+z) = xy + xz$ for all x, y and $z \in A$,
- $\alpha(xy) = (\alpha x)y = x(\alpha y)$ for all $x, y \in A$ and $\alpha \in F$.

The set $C[a,b]$ of continuous real-valued functions on $[a,b]$, together with the supremum norm $\|f\| = \sup_{x\in[a,b]} |f(x)|$, is a Banach algebra, (i. e., an associative algebra and a Banach space such that $\|fg\| \leq \|f\| \cdot \|g\|$ for all f, g, [24, Chapter I]). The set of all polynomial functions forms a subalgebra of $C[a,b]$ (i. e., a vector subspace of C[a,b] that is closed under multiplication of functions), and the content of the Weierstrass approximation theorem is that this subalgebra is dense in C[a,b].

Theorem 2.24. (Stone-Weierstrass, the $\mathbb{R}$ version) *Suppose X is a compact Hausdorff space and A is a subalgebra of $C(X,\mathbb{R})$ which contains a non-zero constant function. Then A is dense in $C(X,\mathbb{R})$ if and only if it separates points.*

The Stone-Weierstrass Theorem 2.24 implies the Weierstrass Theorem 2.23 since the polynomials on $[a,b]$ form a subalgebra of $C[a,b]$ which contains the constants and separates points.

2.4. **Korovkin-type approximation results.** Recall that $\mathbb{C}[\mathbb{K}]$ and $\mathbb{C}[\mathbb{K}_c]$ denote the spaces of continuous functions on $[0,1]$ into $\mathbb{K}$ and $\mathbb{K}_c$, respectively.

Proposition 2.25. *For F and G multifunctions defined on $[0,1]$ with values in $\mathbb{K}_c$, we have*

$$F \subset G \quad (F(x) \subset G(x),\ \forall\, x) \implies \mathrm{B}_N(F,\cdot) \subset \mathrm{B}_N(G,\cdot).$$

Proof. Successively we have

$$\begin{aligned} F(k/N) \subset G(k/N) &\overset{(2.3)}{\Longrightarrow} \sigma(p, F(k/N)) \le \sigma(p, G(k/N)), \quad \forall\, p \in \mathbb{S} \\ &\overset{(2.5)}{\Longrightarrow} \sigma(p, p_{N,k}(x)F(k/N)) \le \sigma(p, p_{N,k}(x)G(k/N)), \\ &\qquad \forall\, p \in \mathbb{S},\ x \in [0,1],\ k = 0,1,\ldots,N, \\ &\overset{(2.6)}{\Longrightarrow} \sigma(p, \mathrm{B}_N(F,x)) \le \sigma(p, \mathrm{B}_N(G,x)), \quad \forall\, p \in \mathbb{S},\ x \in [0,1], \\ &\overset{(2.3)}{\Longrightarrow} \mathrm{B}_N(F,x) \subset \mathrm{B}_N(G,x), \quad \forall\, x \in [0,1]. \end{aligned}$$

□

A map $T : \mathbb{C}[\mathbb{K}_c] \to \mathbb{C}[\mathbb{K}_c]$ is said to be $\mathbb{K}_c$-*linear* if

$$T(\alpha F + \beta G) = \alpha\, TF + \beta\, TG, \quad \forall\, \alpha, \beta \ge 0,\ F, G \in \mathbb{C}[\mathbb{K}_c],$$

and $\mathbb{K}_c$-*positive (monotone)* if

$$F \subset G \implies TF \subset TG, \quad \forall\, F, G \in \mathbb{C}[\mathbb{K}_c].$$

Remark 2.26. The Bernstein polynomial $\mathrm{B}_N(\cdot,\cdot)$ is an example of such a map (operator) T. △

Theorem 2.27. *Let* $(T_\nu)_\nu$ *be a sequence of* $\mathbb{K}_c$ *-linear and* $\mathbb{K}_c$ *-positive maps. In order to have*

$$T_\nu F \to F \text{ for each } F \in \mathbb{C}[\mathbb{K}_c],$$

it is necessary and sufficient that

(a) $T_\nu F^{(i)} \to F^{(i)}$, $i = 0, 1, 2$ *where* $F^{(i)}(x) = x^i \mathbb{B}$,

(b) $\sup\{\mathrm{H}(T_\nu F, F) \mid F(x) = K, \ \|K\| = 1\} \to 0$.

Proof. *Necessity.* (a) is obvious. Suppose that (b) does not hold. Then there exists an $\varepsilon > 0$ and a subsequence (K_{ν_j}) of (K_ν) such that $\mathrm{H}(T_{\nu_j} K_{\nu_j}, K_{\nu_j}) \geq \varepsilon$ $(F_{\nu_j}(x) = K_{\nu_j})$. Local compactness of $\mathbb{K}_c$ and the uniform normalization $\|K_{\nu_j}\| = 1$ assure the existence of a convergent subsequence of the (K_{ν_j}). Without loss of generality denote this new subsequence again as (K_ν) and suppose that $K_\nu \to K_\infty$. Then by the triangle inequality

$$\mathrm{H}(T_\nu K_\nu, K_\nu) \leq \mathrm{H}(T_\nu K_\nu, T_\nu K_\infty) + \mathrm{H}(T_\nu K_\infty, K_\infty) + \mathrm{H}(K_\infty, K_\nu).$$

Now $\varepsilon_\nu = \mathrm{H}(K_\infty, K_\nu) \to 0$. The twin inclusions $K_\nu \subset K_\infty + \varepsilon_\nu \mathbb{B}$ and $K_\infty \subset K_\nu + \varepsilon_\nu \mathbb{B}$ together with the properties of T_ν, imply

$$T_\nu K_\nu \subset T_\nu K_\infty + \varepsilon_\nu T_\nu \mathbb{B}, \quad T_\nu K_\infty \subset T_\nu K_\nu + \varepsilon_\nu T_\nu \mathbb{B},$$

so that $\mathrm{H}(T_\nu K_\nu, T_\nu K_\infty) \leq \varepsilon_\nu \mathrm{H}(T_\nu \mathbb{B}) \to 0$. Hence $\lim \mathrm{H}(T_\nu K_\infty, K_\infty) \geq \varepsilon$, but this violates our assumption.

Sufficiency. It is rather long in [28]. An easier path is supplied by Theorem 2.32 that follows. $\square$

Remark 2.28. Theorem 2.27 and Remark 2.26 imply Theorem 2.4. $\triangle$

Now we introduce a result in [15] that

- generalizes Theorem 2.27,
- allows transferring a Korovkin system from the single-valued to the multivalued case.

Mathematically the growth function is modeled by a multifunction F associating a compact convex subset of $\mathbb{R}^n$ for every value $x \in [0, 1]$. We need a couple of special functions: for a given $K \in \mathbb{K}_c$, K will denote the constant function $F(x) = K$, while $x\mathbb{B}$ and $x^2\mathbb{B}$ denote the multifunctions $F(x) = x\mathbb{B}$ and $F(x) = x^2\mathbb{B}$, respectively.

Let X and Y be metric spaces and $\mathcal{F}$ a family of functions from X into Y. The family $\mathcal{F}$ is said to be *equicontinuous* at a point $x_0 \in X$, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\rho(f(x_0), f(x)) < \varepsilon$ for all $f \in \mathcal{F}$ and $x \in X$ such that $\rho(x_0, x) < \delta$.

Let X be a compact Hausdorff space, $C(X)$ the Banach space of real-valued continuous functions on X. We consider a set $M \subset C(X)$ of "test functions", and we denote by $\operatorname{span}(M)$ the linear subspace of $C(X)$ spanned by M. The *Korovkin closure* $\mathcal{K}(M)$ is the set of all functions $f \in C(X)$ which satisfies the following property:

For every equicontinuous net (T_α) of positive linear operators on $C(X)$ one has:

If $T_\alpha(g) \to g$ for all $g \in M$, then $T_\alpha(f) \to f$.
One says that M is a *Korovkin system* for $C(X)$ if $\mathcal{K}(M) = C(X)$.

Remark 2.29. (a) If the constant function 1 belongs to M, the hypothesis of equicontinuity is superfluous. Indeed,

$$|T_\alpha(f) - T_\alpha(g)| = |fT_\alpha(1) - gT_\alpha(1)| = |f - g| \cdot T_\alpha(1) \to |f - g|.$$

(b) For compact metric spaces X, the net (T_α) is equivalent to a sequence (T_n).

(c) On the unit interval $X = [0,1]$, the polynomials $p_0 = 1$, $p_1 = x$, and $p_2 = x^2$ form a Korovkin system. $\triangle$

Let X and Y be compact Hausdorff spaces. There are natural embeddings of $C(X)$ and $C(Y)$ into $C(X \times Y)$. Indeed, every function $f : X \to \mathbb{R}$ may be considered as a function from $X \times Y$ into $\mathbb{R}$ not depending on the second variable and, likewise, for functions on Y.

Lemma 2.30. *If M_1 is a Korovkin system for $C(X)$ and M_2 is a Korovkin system for $C(Y)$, then $M = M_1 \cup M_2$ is a Korovkin system for $C(X \times Y)$.*

For an arbitrary compact Hausdorff space X we denote

$$\mathcal{C} = C(X, \mathbb{K}_c)$$

the set of all continuous multifunctions F defined on X with values $F(x)$ in the set $\mathbb{K}_c$ of nonempty convex, and compact subsets of $\mathbb{R}^n$.

An operator $T : \mathcal{C} \to \mathcal{C}$ is called *linear* if

$$T(F+G) = TF + TG, \quad T(\alpha F) = \alpha TF, \quad \forall\, F, G \in \mathcal{C},\ \alpha \geq 0.$$

It is said to be *monotone* if

$$F \subset G \implies TF \subset TG.$$

As in the real case, we say that a set $\mathcal{M} \subset \mathcal{C}$ of test functions is a *Korovkin closure* for $\mathcal{C}$ if

For every equicontinuous net (T_α) of monotone linear operators on $\mathcal{C}$ one has:

If $T_\alpha(G) \to G$ for all $G \in \mathcal{M}$, then $T_\alpha(F) \to F$ for all $F \in \mathcal{K}(\mathcal{M})$.

One says that $\mathcal{M}$ is a *Korovkin system* for $\mathcal{C}$ if $\mathcal{K}(\mathcal{M}) = \mathcal{C}$.

Remark 2.31. (a) If the constant function $\mathbb{B}$ belongs to $\mathcal{M}$, the hypothesis of equicontinuity of the net (T_α) is superfluous since it follows from $T_\alpha(\mathbb{B}) \to \mathbb{B}$.

(b) For compact metric spaces X, the net (T_α) is equivalent to a sequence (T_n). $\triangle$

Theorem 2.32. *Let X be a compact Hausdorff space, and $\mathbb{B}$ the unit ball for an arbitrary norm in $\mathbb{R}^n$. If M is a Korovkin system of nonnegative functions for $C(X)$, then*

$$\mathcal{M} = \{x \mapsto f(x)\mathbb{B} \mid f \in M\} \cup \{\textit{all constant functions}\}$$

is a Korovkin system for $\mathcal{C}$.

Proof. Let Y be the dual unit sphere of $\mathbb{B} = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$, that is, the set of linear functionals y on $\mathbb{R}^n$ such that

$$\|y\| = \sup\{y(x) \mid x \in \mathbb{B}\} = 1.$$

Topologically Y is homeomorphic to the Euclidean sphere $\mathbb{S}$.

With every $K \in \mathbb{K}_c$ we associate the support function

$$\sigma(\cdot, K) : Y \to \mathbb{R}, \quad \sigma(p, K) = \max\{\langle p, x\rangle \mid x \in K\}.$$

We summarize several properties of the support function.

(a) Since $\sigma(\cdot, K)$ is sublinear on $\mathbb{R}^n$, it is continuous, Proposition 2.1.

(b) $\sigma(y, \mathbb{B}) = 1$ for all $y \in Y$, this is (2.4).

(c) $\sigma(\cdot, K_1 + K_2) = \sigma(\cdot, K_1) + \sigma(\cdot, K_2)$, $\sigma(\cdot, \alpha K) = \alpha\sigma(\cdot, K)$, $\alpha \geq 0$, these equalities are (2.6) and (2.5), respectively.

(d) $\sup\{\sigma(\cdot, K_1), \sigma(\cdot, K_2)\} = \sigma(\cdot, K)$, where $K = \operatorname{conv}\{K_1 \cup K_2\}$.

(e) $K_1 \subset K_2 + \varepsilon\mathbb{B} \Longleftrightarrow \sigma(\cdot, K_1) \leq \sigma(\cdot, K_2) + \varepsilon$. In particular $K_1 \subset K_2 \Longleftrightarrow \sigma(\cdot, K_1) \leq \sigma(\cdot, K_2)$. See (2.3).

(f) Equality (2.7) takes place, i. e., $\mathrm{H}(K_1, K_2) = \|\sigma_1 - \sigma_2\|$, (uniform norm).

Thus the mapping

$$\mathbb{K}_c \ni K \;\mapsto\; \sigma(\cdot, K)$$

is a linear isometric order embedding of $\mathbb{K}_c$ into $C(Y)$. The linear subspace

$$L = \{\sigma(\cdot, K_1) - \sigma(\cdot, K_2) \mid K_1, K_2 \in \mathbb{K}_c\}$$

is a vector lattice by (d), containing 1 by (b). As clearly L separates the points, L is dense in $C(Y)$ by the Stone-Weierstrass theorem.

The embedding $\sigma : \mathbb{K}_c \to C(Y)$ yields a linear isometric order embedding

$$i : \mathcal{C}\,(= C(X, \mathbb{K}_c)) \;\to\; C(X, C(Y))$$

given by $i(F)(x) = \sigma(\cdot, F(x))$ for all $F \in \mathcal{C}$ and $x \in X$. Combining with the isomorphism

$$j : C(X, C(Y)) \to C(X \times Y)$$

given by $j(f)(x, y) = (f(x))(y)$ for all $f \in C(X, C(Y))$ and all $(x, y) \in X \times Y$, we obtain a linear order embedding

$$\kappa : \mathcal{C} \to C(X \times Y).$$

The image of κ generates a dense vector sublattice of $C(X \times Y)$ and contains the constant function 1. Thus, every monotone linear operator T on $\mathcal{C}$ extends uniquely to a positive linear operator $\overline{T}$ on $C(X \times Y)$. For an equicontinuous family (T_α) of monotone linear operators on $\mathcal{C}$, the family $(\overline{T}_\alpha)$ of extensions is equicontinuous on $C(X \times Y)$.

It remains to prove that $\kappa(\mathcal{M})$ is a Korovkin system for $C(X \times Y)$. One easily checks that under κ

$$x \to f(x)\mathbb{B} \text{ goes to } (x, y) \to f(x),$$

the constant function

$$x \to K \text{ goes to } (x, y) \to \sigma(y, K).$$

As M is a Korovkin system for X and the functions $\sigma(\cdot, C)$ for $C \in \mathbb{K}_c$ generate a dense linear subspace of $C(Y)$, Lemma 2.30 allows one to conclude that $\kappa(\mathcal{M})$ is a Korovkin system for $C(X \times Y)$. □

Theorem 2.33. *Let X be a compact Hausdorff space, and $\mathbb{B}$ the unit ball for an arbitrary norm in $\mathbb{R}^n$. If M is a Korovkin system of nonnegative functions for $C(X)$, then*

$$\mathcal{M} = \{x \mapsto f(x)\mathbb{B} \mid f \in M\} \cup \{\mathbb{B}, e_1, \ldots, e_n\},$$

where $\{e_1, \ldots, e_n\}$ is the canonical basis in $\mathbb{R}^n$, is a Korovkin system for $\mathcal{C} = C(X, \mathbb{K}_c)$.

We recall just [11] and [23] (the latter is not new but useful) for the convex case.

It seems that the work with Minkowski operations on sets pushes us considering only convex sets in order to get substantial results. Naturally arises the question: what is happening in the non-convex case, i. e., do exist methods allowing us to get satisfactory deep results in the non-convex case? We will see that the answer is positive.

Hereafter we review some results on the non-convex case, mainly in [10] but also [9] and [11].

3. The non-convex case

3.1. Preliminaries. Recall that $\mathbb{K}$ is the collection of all compact nonempty subsets of $\mathbb{R}^n$. This section follows [10].

We introduce the following notions.

- The *linear Minkowski combination* of two nonempty sets A and B in $\mathbb{R}^n$ is defined (as we already saw) as $\lambda A + \mu B = \{\lambda a + \mu b \mid a \in A,\ b \in B\}$, with $\lambda, \mu \in \mathbb{R}$.
- The *Euclidean distance from a point* $a \in \mathbb{R}^n$ *to a set* $B \in \mathbb{K}$ is defined as

$$\mathrm{d}(a, B) = \inf_{b \in B} \|a - b\| \overset{\mathbb{K}}{=} \min_{b \in B} \|a - b\|,$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$.

- The *Hausdorff(-Pompeiu) distance between two sets* $A, B \in \mathbb{K}$ is defined by

$$\begin{aligned} \mathrm{H}(A, B) &= \max\left\{\max_{a \in A} \mathrm{d}(a, B),\ \max_{b \in B} \mathrm{d}(b, A)\right\} \\ &= \max\left\{\max_{a \in A} \min_{b \in B} \|a - b\|,\ \max_{b \in B} \min_{a \in A} \|a - b\|\right\} \\ &\overset{(1.7)}{=} \min\{\varepsilon > 0 \mid A \subset B + \varepsilon\mathbb{B}, \quad B \subset A + \varepsilon\mathbb{B}\}. \end{aligned} \tag{3.1}$$

We already saw that $\|A\| = \mathrm{H}(\{0\}, A)$.

- The *set of all projections of* $a \in \mathbb{R}^n$ *into a set* $B \in \mathbb{K}$ is

$$\Pi_B(a) = \{b \in B \mid \|a - b\| = \mathrm{d}(a, B)\}.$$

- For $A, B \in \mathbb{K}$ and $0 \le t \le 1$, the *one sided* t*-weighted metric average of* A *and* B (in this is order) is

$$M(A, t, B) = \bigcup_{a \in A} \{ta + (1 - t)\Pi(A, B)\} \tag{3.2}$$

and the t*-weighted metric average of* A *and* B is

$$A \oplus_t B = \{ta + (1 - t)b \mid (a, b) \in \Pi(A, B)\} \tag{3.3}$$

with

$$\Pi(A, B) = \{(a, b) \in A \times B \mid a \in \Pi_A(b) \text{ or } b \in \Pi_B(a)\}.$$

Proposition 3.1. *The graph of the mapping* $[0, 1] \ni t \mapsto C(t) = A \oplus_t B$ *is the union of the graphs of* $(1 - t)a + tb$ *for* $(a, b) \in \Pi(A, B)$.

Proposition 3.2. *The one sided metric average and the metric average, for all* $A, B \in \mathbb{K}$ *and* $0 \leq s, t \leq 1$, *have the following metric properties:*

$$A \oplus_t B = M(A, t, B) \cup M(B, 1 - t, A), \tag{3.4}$$

$$M(M(A, t, B), s, B) = M(A, ts, B), \tag{3.5}$$

$$M(A \cap B, t, B) = A \cap B \subset M(B, s, A), \tag{3.6}$$

$$A \oplus_t B = (A \cap B) \cup M(A \setminus B, t, B) \cup M(B \setminus A, 1 - t, A), \tag{3.7}$$

$$A \oplus_1 B = C(1) = A, \tag{3.8}$$

$$A \oplus_0 B = C(0) = B, \tag{3.9}$$

$$A \oplus_t B = C(t) \in \mathbb{K}, \tag{3.10}$$

$$A \oplus_t B = B \oplus_{1-t} A, \tag{3.11}$$

$$A \oplus_t A = A, \tag{3.12}$$

$$A \cap B \subset A \oplus_t B \subset tA + (1 - t)B \subset \operatorname{conv}(A \cup B), \tag{3.13}$$

$$\mathrm{H}(A \oplus_t B,\ A \oplus_s B) = \mathrm{H}(C(t), C(s)) = |t - s|\mathrm{H}(A, B), \tag{3.14}$$

$$\mathrm{H}(A \oplus_t B,\ A) = (1 - t)\mathrm{H}(A, B), \tag{3.15}$$

$$\mathrm{H}(A \oplus_t B,\ B) = t \cdot \mathrm{H}(A, B), \tag{3.16}$$

$$\textit{if } B \textit{ is a convex superset of } A, \textit{ then for } 0 \leq t \leq s \leq 1, \tag{3.17}$$

$$A \subset A \oplus_s B \subset A \oplus_t B \subset B.$$

Proof. Let us see where these relations come from.

Equality (3.4) follows from (3.2) and (3.3).

From

$$M(M(A,t,B),s,B) = \{s(ta + (1-t)b) + (1-s)b \mid a \in A, \quad b \in \Pi_B(a)\}$$
$$= \{(ts)a + (1-(ts))b \mid a \in A, \quad b \in \Pi_B(s)\} = M(A,ts,B),$$

(3.5) follows.

Equalities (3.6) and (3.7) follow from the definitions (3.2) and (3.3).

Since

$$A \oplus_1 B = \{1 \cdot a + 0 \cdot b \mid (a,b) \in \Pi(A,B)\} = \{a \mid b \in \Pi_B(a),\ a \in A\},$$

(3.8) follows.

Equality (3.9) follows similarly.

For $A, B \in \mathbb{K}$, $A \oplus_t B$ is nonempty and compact, that is (3.10).

Equalities (3.11) and (3.12) are obvious.

Inclusions in (3.13) immediately follow.

We follow [1] and consider an arbitrary element $x = ta+(1-t)b \in C(t)$ with $a \in \Pi_A(b)$ or $b \in \Pi_B(a)$, then $y = sa+(1-s)b \in C(s)$. Since $\|x-y\| = |t-s| \cdot \|a-b\|$, we have that $\mathrm{H}(C(t), C(s)) \le |t-s| \mathrm{H}(A,B)$. To prove the reverse inequality without loss of generality we admit that $0 \le s < t \le 1$. Then

$$\mathrm{H}(A,B) = \mathrm{H}(C(0), C(1)) \le \mathrm{H}(C(0), C(s)) + \mathrm{H}(C(s), C(t)) + \mathrm{H}(C(t), C(1))$$
$$\le s\mathrm{H}(A,B) + (t-s)\mathrm{H}(A,B) + (1-t)\mathrm{H}(A,B) = H(A,B).$$

Now we conclude that equality (3.14) is true.

We have

$$(1-t)\mathrm{H}(A,B) \overset{(3.14)}{=} \mathrm{H}(A \oplus_t B,\ A \oplus_1 B) \overset{(3.8)}{=} \mathrm{H}(A \oplus_t B,\ A)$$

and (3.15) follows.

In order to get (3.16) we have

$$t \cdot \mathrm{H}(A,B) \overset{(3.14)}{=} \mathrm{H}(A \oplus_t B,\ A \oplus_0 B) \overset{(3.9)}{=} \mathrm{H}(A \oplus_t B,\ B).$$

We recall the proof of (3.17) in [8]. Obviously, we have that since $A \subset B$, $A = A \cap B \subset B$. By (3.4),

$$M(A,t,B) = M(A \cap B, t, B) \subset M(B, 1-t, A)$$

and therefore $A \oplus_t B = M(B, 1-t, A)$. Hence, by (3.13) and the convexity of B,

$$A = A \cap B \subset M(B, 1-t, A) = A \oplus_t B \subset \operatorname{conv}(A \cup B) = B.$$

it remains to prove that $M(B, 1-s, A) \subset M(B, 1-t, A)$. By the convexity of B, for each $b \in B$ and $a \in \Pi_A(b)$, the whole segment $[a, ta + (1-t)b]$ is a subset of $M(B, 1-t, A)$. Since, for $s \geq t$, $[a, sa + (1-s)b] \subset [a, ta + (1-t)b]$, the conclusion follows. $\square$

• The *modulus of continuity of* $f : [a,b] \to X$ with images in a metric space (X, ρ) is

$$\omega_{[a,b]}(f, \delta) = \sup\{\rho(f(x), f(y)) \mid |x - y| \leq \delta,\ x, y \in [a,b]\}, \quad \delta > 0. \tag{3.18}$$

Hereafter X is either $\mathbb{R}^n$ or $\mathbb{K}$, and ρ is either the Euclidean distance or the Hausdorff-Pompeiu distance, respectively. A property of the modulus is

$$\omega_{[a,b]}(f, \lambda\delta) \leq \lceil \lambda \rceil \omega_{[a,b]}(f, \delta), \tag{3.19}$$

[5, Problem 6. II, p. 38], where $\lceil \cdot \rceil$ is the ceiling function.

• By $Lip([a,b], \mathcal{L})$ we denote the set of all Lipschitz functions $f : [a,b] \to X$ satisfying $\rho(f(x), f(y)) \leq \mathcal{L}|x - y|, \quad x, y \in [a,b]$, where $\mathcal{L}$ is a constant independent of x and y.

• The *variation* of a function $f : [a,b] \to X$ on a partition $\chi = \{a = x_0 < \cdots < x_N = b \mid x_i \in [a,b], i = 0, \ldots, N\}$ is defined by $V(f, \chi) = \sum_{k=1}^N \rho(f(x_k), f(x_{k-1}))$. The *total variation* of f on $[a,b]$ is $\overset{b}{\underset{a}{\vee}}(f) = \sup_\chi V(f, \chi)$. It is said that f is of *bounded variation* if $\overset{b}{\underset{a}{\vee}}(f) < \infty$. In this case we define the function

$$v_f(x) = \overset{x}{\underset{a}{\vee}}(f), \quad x \in [a,b]. \tag{3.20}$$

Obviously, v_f is nondecreasing. If f is also continuous, then v_f is continuous as well. For the sake of completeness we recall the next statement.

Proposition 3.3. *A function* $f : [a, b] \to X$ *is continuous and of bounded variation on* $[a, b]$ *if and only if* v_f *is a continuous function on* $[a, b]$.

It holds

$$\rho(f(x), f(y)) \le \bigvee_x^y (f) = v_f(y) - v_f(x), \text{ for } x < y. \tag{3.21}$$

From (3.21) it follows that

$$\omega_{[a,b]}(f, \delta) \le \omega_{[a,b]}(v_f, \delta). \tag{3.22}$$

• By $CBV[a, b]$ is denoted the set of all functions which are continuous and of bounded variation on $[a, b]$.

• For a multifunction $F : [a, b] \to \mathbb{K}$ any single-valued function $f : [a, b] \to \mathbb{R}^n$ with $f(x) \in F(x)$, for all $x \in [a, b]$ is said to be a *selection* of F, e. g. [29], [14], [30], [19, Chapter 2], and [21, Chapter 2].

A set of selections of F, let it be $\{f^\alpha \mid \alpha \in \mathcal{A}\}$, is said to be a *representation* of F if $F(x) = \{f^\alpha(x) \mid \alpha \in \mathcal{A}\}, \quad \forall x \in [a, b]$. This is expressed by writing $F = \{f^\alpha \mid \alpha \in \mathcal{A}\}$. Note that such a representation always exists thanks to the axiom of choice.

A concrete **motivation** for the study of metric average is the reconstruction of a 3D object from a set of its 2D cross-section, [26], with applications in tomography, microscopy, and computer vision. An algorithm for the computation of the metric average of two simple polygons is introduced in [16].

3.2. **Linear operators on multifunctions based on a metric linear combination of ordered sets.** A new operation on a finite number of ordered sets is introduced. Using this operation a new adaptation of linear operators to multifunctions is presented.

Definition 3.4. *Let* $(A_0, A_1, \ldots, A_N)$ *be a finite sequence of nonempty compact sets. A vector* $(a_0, a_1, \ldots, a_N)$ *with* $a_i \in A_i$, $i = 0, \ldots, N$, *for which there exists* j, $0 \le j \le N$ *such that*

$$a_{i-1} \in \Pi_{A_{i-1}}(a_i), \quad 1 \le i \le j \text{ and } a_{i+1} \in \Pi_{A_{i+1}}(a_i), \quad j \le i \le N-1$$

is called a ***metric chain*** *of* $(A_0, \dots, A_N)$.

Thus each element of each set A_i, $i = 0, \dots, N$ generates at least one metric chain. The collection of all metric chains of $(A_0, \dots, A_N)$ is denoted by $CH(A_0, \dots, A_N)$. The set $CH(A_0, \dots, A_N)$ depends on the order of the sets A_i, $i = 0, \dots, N$.

Definition 3.5. *A* ***metric linear combination*** *of a sequence of sets* $A_0, \dots, A_N$ *with coefficients* $\lambda_0, \dots, \lambda_N \in \mathbb{R}$, *is*

$$\bigoplus_{k=0}^{N} \lambda_k A_k = \left\{ \sum_{k=0}^{N} \lambda_k a_k \mid (a_0, \dots, a_N) \in CH(A_0, \dots, A_N) \right\}. \tag{3.23}$$

Since for two sets $CH(A, B) = \Pi(A, B)$, in the special case $N = 1$ and $\lambda_0, \lambda_1 \in [0, 1]$, $\lambda_0 + \lambda_1 = 1$, the metric linear combination is the metric average, [1].

Proposition 3.6. *Several properties of the metric linear combinations are introduced below.*

(i) $\bigoplus_{k=0}^{N} \lambda_k A_k = \bigoplus_{k=0}^{N} \lambda_{N-k} A_{N-k}$,

(ii) $\bigoplus_{k=0}^{N} \lambda_k A = \left(\sum_{k=0}^{N} \lambda_k \right) A$,

(iii) $\bigoplus_{k=0}^{N} \lambda A_k = \lambda \left(\bigoplus_{k=0}^{N} 1 \cdot A_k \right)$,

(iv) *For* $\lambda_0, \dots, \lambda_N$ *such that* $\sum_{k=0}^{N} \lambda_k = 1$, $\bigoplus_{k=0}^{N} \lambda_k A = A$.

Proof. (i) We remark that

$$(a_0, \dots, a_N) \in CH(A_0, \dots, A_N) \iff (a_N, \dots, a_0) \in CH(A_N, \dots, A_0).$$

Then

$$\begin{aligned} \bigoplus_{k=0}^{N} \lambda_k A_k &= \left\{ \sum_{k=0}^{N} \lambda_k a_k \mid (a_0, \dots, a_N) \in CH(A_0, \dots, A_N) \right\} \\ &= \left\{ \sum_{k=0}^{N} \lambda_{N-k} a_{N-k} \mid (a_N, \dots, a_0) \in CH(A_N, \dots, A_0) \right\} = \bigoplus_{k=0}^{N} \lambda_{N-k} A_{N-k}. \end{aligned}$$

(ii) Since $CH(A, \dots, A) = A$, the property follows.

(iii) is obvious.

(iv) follows from (ii). $\square$

Remarks 3.7. • Now we can define the *metric sum* of two sets by

$$A_0 \oplus A_1 = \oplus_{k=0}^{1} 1 \cdot A_k.$$

This operation is commutative by property (i) in Proposition 3.6 and it is not associative.

• Similarly we can define *metric subtraction* between two sets by

$$A_0 \ominus A_1 = \oplus_{k=0}^{1} \lambda_k A_k$$

with $\lambda_0 = 1$ and $\lambda_1 = -1$. Then from (ii) in Proposition 3.6 it follows that

$$A \ominus A = \{0\}. \tag{3.24}$$

In spite of the previous result, the operation $A \ominus B$ does not have the usual properties of subtraction as follows from the example

$$A = [0,1], \quad B = \{0,1\} \implies A \ominus B = B \ominus A = \{-1, 0, 1\}.$$

• With the operation defined by (3.23), the class of sample based linear operators for real-valued functions, namely such defined by

$$A_\chi(f, x) = \sum_{k=0}^{N} c_k(x) f(x_k) \tag{3.25}$$

can be adapted to set-valued functions. $\triangle$

Definition 3.8. *Let* $F : [a,b] \to \mathbb{K}$, $\{a = x_0, x_1, \ldots, x_N = b\} \subset [a,b]$ *and let* $\{F(x_k),\, k = 0, \ldots, N\}$ *be samples of* F *at* χ*. For* A_χ *of the form* (3.25) *it is defined a* ***metric linear operator*** A_χ^M *on* F *by*

$$(A_\chi^M F)(x) = A_\chi^M(F, x) = \bigoplus_{k=0}^{N} c_k(x) F(x_k). \tag{3.26}$$

This operator is said to be the ***metric analogue*** *of* (3.25).

Remark 3.9. Due to property (ii) in Proposition 3.6, the metric analogue of a linear operator which preserves constants, preserves constant multifunctions, too. Indeed, for a nonzero constant c we have

$$c = \sum_{k=0}^{N} c_k(x)\, c \implies \sum_{k=0}^{N} c_k(x) = 1 \overset{(3.25)}{\implies} K = \bigoplus_{k=0}^{N} c_k(x)\, K.$$

The analogue of (ii) in Proposition 3.6 does not hold for Minkowski linear combinations with some negative coefficients, even for convex sets. This is one reason why only positive operators, based on Minkowski sum, were applied to multifunctions, [23]. $\triangle$

The analysis of the approximation properties of $A_\chi^M F$ is based on properties of the metric piecewise linear approximation operator. These are studied in the next subsection.

3.3. **Metric piecewise linear approximations of multifunctions.** From now on

- $F : [a, b] \to \mathbb{K}$, $\{F_k = F(x_k)\}_{k=0}^N$, where $a = x_0 < x_1 < \cdots < x_N = b$ and $\chi = (x_0, \ldots, x_N)$ denotes a partition of $[a, b]$,
- $CH = CH(F_0, \ldots, F_N)$, $\delta_k = x_{k+1} - x_k$, $k = 0, \ldots, N-1$,
- $\delta_{max} = \max\{\delta_k \mid 0 \le k \le N-1\}$, $\delta_{min} = \min\{\delta_k \mid 0 \le k \le N-1\}$.

In case of a uniform partition, we have $\delta_{max} = \delta_{min} = h = (b-a)/N$ and denote such a partition by χ_N.

Definition 3.10. *The* ***metric piecewise linear approximation*** *to a multifunction F at a partition χ is*

$$S_\chi^M(F, x) = \{\lambda_k(x) f_k + (1 - \lambda_k(x)) f_{k+1} \mid (f_0, \ldots, f_N) \in CH\},\ x \in [x_k, x_{k+1}],$$

where

$$\lambda_k(x) = (x_{k+1} - x)/(x_{k+1} - x_k). \tag{3.27}$$

$\lambda_k(\cdot)$ in (3.27) was proposed in [1].

By construction, the set valued function $S_\chi^M F$ has a representation in terms of selections

$$S_\chi^M F = \{s(\chi, \varphi) \mid \varphi \in CH(F_0, \ldots, F_N)\}, \tag{3.28}$$

where $s(\chi, \varphi)$ is a piecewise linear single-valued function interpolating the data (x_k, f_k), $k = 0, \ldots, N$ with $\varphi = (f_0, \ldots, f_N)$.

Recall the *piecewise linear interpolant based on metric average*, introduced in [1], is

$$S_\chi^{MA}(F,x) = F_k \oplus_{\lambda_k(x)} F_{k+1}, \quad x \in [x_k, x_{k+1}]$$

with $\lambda_k(x)$ defined by (3.27).

Lemma 3.11. *For a multifunction* $F : [a,b] \to \mathbb{K}$ *the metric piecewise linear approximation and the piecewise linear interpolant based on the metric average coincide, that is*

$$S_\chi^{MA} F = S_\chi^{M} F \tag{3.29}$$

and

$$\mathrm{H}(F(x), S_\chi^{MA}(F,x)) \le 2\omega_{[a,b]}(F, \delta_{max}), \quad x \in [a,b]. \tag{3.30}$$

Proof. To prove (3.29) we first show that $(S_\chi^{MA} F)(x) \subset (S_\chi^{M} F)(x)$ for any $x \in [a,b]$, and then show the reverse inclusion.

For a fixed $x \in [x_k, x_{k+1}]$ and any $y \in S_\chi^{MA}(F,x)$, one has $y = \lambda_k(x) f_k + (1-\lambda_k(x)) f_{k+1}$ for some $(f_k, f_{k+1}) \in \Pi(F_k, F_{k+1})$. Thus there exists a metric chain $\varphi = (f_0, \ldots, f_k, \;\; f_{k+1}, \ldots, f_N)$, $\varphi \in CH$, such that $y = s(\chi, \varphi)(x)$.

We now show the reverse inclusion, namely $(S_\chi^{M} F)(x) \subset (S_\chi^{MA} F)(x)$. It is obvious that, for any $x \in [a,b]$ and any $\varphi \in CH$, $s(\chi,\varphi)(x) \in (S_\chi^{MA} F)(x)$.

To prove (3.30) we use (3.29), (3.15), and the triangle inequality for the Hausdorff-Pompeiu metric, and obtain for $x \in [x_k, x_{k+1}]$,

$$\mathrm{H}(F(x), S_\chi^{M}(F,x)) \overset{(3.29)}{=} \mathrm{H}(F(x), S_\chi^{MA}(F,x))$$
$$\le \mathrm{H}(F(x), F(x_k)) + \mathrm{H}(F(x_k), S_\chi^{MA}(F,x)) = \mathrm{H}(F(x), F(x_k)) + \mathrm{H}(F_k, F_k \oplus_{\lambda_k(x)} F_{k+1})$$
$$\overset{(3.15)}{=} \mathrm{H}(F(x), F(x_k)) + (1-\lambda_k)\mathrm{H}(F_k, F_{k+1}) \le 2\omega_{[a,b]}(F, \delta_k).$$

□

Next we show that $S_\chi^{M} F$ and its piecewise linear selections (3.28) "inherit" some continuity properties of a continuous multifunction F.

Lemma 3.12. *Let* $F \in Lip([a,b]),\mathcal{L})$ *and let* χ *be a partition of* $[a,b]$. *Then the metric piecewise linear approximation satisfies*

$$S_\chi^M F \in Lip([a,b],\mathcal{L}).$$

Proof. Since by (3.29) we have that $S_\chi^{MA} F = S_\chi^M F$, we use the piecewise linear interpolant based on metric average instead of the metric piecewise linear approximation.

Suppose that $x, y \in [x_k, x_{k+1}]$. Then

$$\mathrm{H}(S_\chi^{MA}(F,x), S_\chi^{MA}(F,y)) = \mathrm{H}(F_k \oplus_{\lambda_k(x)} F_{k+1}, F_k \oplus_{\lambda_i(y)} F_{k+1})$$
$$\overset{(3.14)}{=} |\lambda_k(x) - \lambda_k(y)|\mathrm{H}(F_k, F_{k+1}) \leq |\lambda_k(x) - \lambda_k(y)|\mathcal{L}(x_{k+1} - x_k) = \mathcal{L}|x-y|.$$

Now let $x \in [x_j, x_{j+1}]$ and $y \in [x_k, x_{k+1}]$, where $0 \leq j < k \leq N-1$. Using the triangle inequality, (3.14), and the Lipschitz continuity of F, we get

$$\mathrm{H}(S_\chi^{MA}(F,x), S_\chi^{MA}(F,y)) \leq \frac{x_{j+1}-x}{x_{j+1}-x_j}\mathrm{H}(F_j, F_{j+1}) + \mathrm{H}(F_{j+1}, F_k)$$
$$+\frac{y-x_k}{x_{k+1}-x_k}\mathrm{H}(F_k, F_{k+1}) \leq \mathcal{L}(x_{j+1} - x + x_k - x_{j+1} + y - x_k) \leq \mathcal{L}|y-x|.$$

□

Corollary 3.13. *Under the conditions of Lemma 3.12 and for any* $s(\chi,\varphi)$ *in* (3.28),

$$s(\chi,\varphi) \in Lip([a,b],\mathcal{L}).$$

Proof. The proof of this corollary is similar to the proof of the previous lemma and uses the observation that

$$|s(\chi,\varphi)(x_k) - s(\chi,\varphi)(x_{k+1})| \tag{3.31}$$
$$\leq \mathrm{H}(S_\chi^M(F,x_k), S_\chi^M(F,x_{k+1})), \quad k = 0,\ldots,N-1.$$

□

Now we consider the case when F is a general continuous multifunction. It follows a statement concerning the so-called "global smoothness preservation".

Lemma 3.14. *Let* $F : [a, b] \to \mathbb{K}$ *be a continuous multifunction. Then for any partition* χ *of* $[a, b]$ *the modulus of continuity to the metric piecewise linear approximation satisfies*

$$\omega_{[a,b]}(S_\chi^M F, \delta) \le 4\omega_{[a,b]}(F, \delta). \tag{3.32}$$

Proof. By definition, for any $\delta > 0$,

$$\omega_{[a,b]}(S_\chi^M F, \delta) = \sup\{\mathrm{H}(S_\chi^M(F, x), S_\chi^M(F, y)) \mid |x - y| \le \delta,\ x, y \in [a, b]\}$$
$$\overset{(3.29)}{=} \omega_{[a,b]}(S_\chi^{MA} F, \delta) = \sup\{\mathrm{H}(S_\chi^{MA}(F, x), S_\chi^{MA}(F, y))|\ |x - y| \le \delta,\ x, y \in [a, b]\}.$$

In this case $x, y \in [x_j, x_{j+1}]$, $|x - y| \le \delta$, the claim of the lemma is obtained using (3.14) and (3.19). Indeed

$$\mathrm{H}(S_\chi^{MA}(F, x), S_\chi^{MA}(F, y)) = \mathrm{H}(F_j \oplus_{\lambda_j(x)} F_{j+1}, F_j \oplus_{\lambda_j(y)} F_{j+1})$$
$$= (|x - y|/\delta_j)\mathrm{H}(F_j, F_{j+1}) \le (|x - y|)/\delta_j)\omega_{[a,b]}(F, \delta_j) \tag{3.33}$$
$$\le (|x - y|)/\delta_j)(1 + \delta_j/\delta)\omega_{[a,b]}(F, \delta) \le ((|x - y|)/\delta_j + (|x - y|)/\delta)\,\omega_{[a,b]}(F, \delta),$$

which implies (3.32).

Now, let $x \in [x_j, x_{j+1}]$, $y \in [x_k, x_{k+1}]$, $0 \le j < k \le N - 1$, and $|x - y| \le \delta$. By the triangle inequality,

$$\mathrm{H}(S_\chi^{MA}(F, x), S_\chi^{MA}(F, y)) \le \mathrm{H}(S_\chi^{MA}(F, x), S_\chi^{MA}(F, x_{j+1})) \tag{3.34}$$
$$+\mathrm{H}(S_\chi^{MA}(F, x_{j+1}), S_\chi^{MA}(F, x_k)) + \mathrm{H}(S_\chi^{MA}(F, x_k), S_\chi^{MA}(F, y)),$$

while by the interpolation property of $S_\chi^{MA} F$ and since $|x_k - x_{j+1}| \le \delta$, we have

$$\mathrm{H}(S_\chi^{MA}(F, x_{j+1}), S_\chi^{MA}(F, x_k)) \le \omega_{[a,b]}(F, \delta). \tag{3.35}$$

Applying (3.33) and (3.35) to (3.34) we obtain

$$\mathrm{H}(S_\chi^{MA}(F, x), S_\chi^{MA}(F, y)) \tag{3.36}$$
$$= ((x_{j+1} - x)/\delta_j + (x_{j+1} - x)/\delta + 1 + (y - x_k)/\delta_k + (y - x_k)/\delta)\,\omega_{[a,b]}(F, \delta)$$
$$\le (3 + (x_{j+1} - x + y - x_k)/\delta_j)\,\omega_{[a,b]}(F, \delta) \le 4\omega_{[a,b]}(F, \delta).$$

Hence we also have (3.32) in the second situation. $\square$

Corollary 3.15. *For any* $s(\chi,\varphi)$ *in* (3.28) *and any* $x, y \in [x_j, x_{j+1}]$, $0 \le j \le N-1$, $|x-y| \le \delta$,

$$|s(\chi,\varphi)(x) - s(\chi,\varphi)(y)| \le (|x-y|/\delta_j + |x-y|/\delta)\,\omega_{[a,b]}(F,\delta). \tag{3.37}$$

For $|x-y| \le \delta \le \delta_{\min}$ *and* $x, y \in [x_j, x_{j+2}]$, $j = 0, \dots, N-2$,

$$\omega_{[a,b]}(s(\chi,\varphi),\delta) \le (2/\delta)|x-y|\,\omega_{[a,b]}(F,\delta) \le 2\omega_{[a,b]}(F,\delta). \tag{3.38}$$

Proof. One has

$$\begin{aligned}|s(\chi,\varphi)(x) - s(\chi,\varphi)(y)| &\le \mathrm{H}(S^M(F,x), S^M(F,y))\\ &\overset{(3.33)}{\le} (|x-y|/\delta_j + |x-y|/\delta)\,\omega_{[a,b]}(F,\delta)\end{aligned}$$

and so (3.37) follows.

We establish inequality (3.38). Then

$$\begin{aligned}&\omega_{[a,b]}(s(\chi,\varphi),\delta)\\ &= \sup\{|(s(\chi,\varphi),x) - (s(\chi,\varphi),y)| \mid |x-y| \le \delta \le \delta_{\min},\ x,y \in [x_j, x_{j+2}]\}\\ &\le \sup\{\mathrm{H}(S^M(F,x), S^M(F,y)) \mid |x-y| \le \delta \le \delta_{\min},\ x,y \in [x_j, x_{j+2}]\}\\ &\le 2\omega_{[a,b]}(F,\delta).\end{aligned}$$

□

Lemma 3.16. *Let* $F \in CBV([a,b])$. *Then for any* $s(\chi,\varphi)$ *in* (3.28),

$$\omega_{[a,b]}(s(\chi,\varphi),\delta) \le 3\omega_{[a,b]}(F,\delta) + \omega_{[a,b]}(v_F,\delta) \le 4\,\omega_{[a,b]}(v_F,\delta).$$

Proof. Denote $s = s(\chi,\varphi)$. For a given $\delta > 0$, let $x \in [x_j, x_{j+1}]$, $y \in [x_k, x_{k+1}]$, $0 \le j \le k \le N-1$, such that $|x-y| \le \delta$. Then

$$|s(x) - s(y)| \le |s(x) - s(x_{j+1})| + \sum_{l=j+1}^{k-1} |s(x_{l+1}) - s(x_l)| + |s(y) - s(x_k)|.$$

By (3.37), (3.28), (3.31), and by the definition of $S_\chi^M F$, we get

$$|s(x) - s(y)| \le ((x_{j+1} - x)/\delta_j + (x_{j+1} - x)/\delta)\,\omega_{[a,b]}(F, \delta)$$
$$+ \sum_{l=j+1}^{k-1} \mathrm{H}(F(x_{l+1}), F(x_l)) + ((y - x_k)/\delta_k + (y - x_k)/\delta)\,\omega_{[a,b]}(F, \delta).$$

Since $(x_{j+1} - x)/\delta + (y - x_k)/\delta < 1$, by the definition of the bounded variation of F and by (3.22), we obtain

$$|s(x) - s(y)| \le 3\omega_{[a,b]}(F, \delta) + \bigvee_{x_{j+1}}^{x_k}(F) \le 3\omega_{[a,b]}(F, \delta) + \omega_{[a,b]}(v_F, \delta) \le 4\omega_{[a,b]}(F, \delta).$$

Taking the supremum over $|x - y| \le \delta$, the proof ends. $\square$

3.4. **Approximation by metric linear operators.** We will use the metric piecewise approximation to obtain error estimates for metric linear operators.

Let $A_\chi^M F$ be defined by (3.26), namely

$$A_\chi^M F(x) = A_\chi^M(F, x) = \bigoplus_{k=0}^{N} c_k(x) F(x_k)$$

and $S_\chi^M F$ be a metric piecewise linear approximation as defined in Subsection 3.3. By Definition 3.10, namely by

$$S_\chi^M(F, x) = \{\lambda_k(x) f_k + (1 - \lambda_k(x)) f_{k+1} \mid (f_0, \dots, f_N) \in CH\}, \ x \in [x_k, x_{k+1}],$$

we get that

$$A_\chi^M F \equiv A_\chi^M(S_\chi^M F). \tag{3.39}$$

Moreover, by (3.25), (3.26), and (3.28),

$$A_\chi^M(S_\chi^M F) = \{A_\chi s(\chi, \varphi) \mid \varphi \in CH(F_0, \dots, F_N)\}. \tag{3.40}$$

The metric analogues of linear operators of the form (3.25), approximate certain classes of set-valued functions. By (3.39) and (3.40) the approximation results depend on the way A_χ approximates piecewise linear real-valued functions.

Theorem 3.17. *Let* A_χ *be of the form* (3.25)*. Then for a continuous multifunction* $F : [a, b] \to \mathbb{K}$ *one has*

$$\mathrm{H}(A_\chi^M(F,x),F(x)) \leq 2\omega_{[a,b]}(F,\delta_{\max}) + \sup_{\varphi\in CH} |A_\chi(s(\chi,\varphi),x) - s(\chi,\varphi)(x)|. \quad (3.41)$$

Proof. By the triangle inequality and by (3.39),

$$\mathrm{H}(A_\chi^M(F,x),F(x)) \leq \mathrm{H}(A_\chi^M(S_\chi^M F,x),S_\chi^M(F,x)) + \mathrm{H}(S_\chi^M(F,x),F(x)),$$

while by (3.40)

$$\mathrm{H}(A_\chi^M(S_\chi^M(F,x)),S_\chi^M(F,x)) \leq \sup_{\varphi\in CH} |A_\chi(s(\chi,\varphi),x) - s(\chi,\varphi)(x)|.$$

This inequality together with (3.30), that is,

$$\mathrm{H}(F(x),S_\chi^{MA}(F,x)) \leq 2\omega_{[a,b]}(F,\delta_{\max}), \quad x\in[a,b],$$

completes the proof. □

Assume that $g : [a, b] \times [0,\infty[\to [0,\infty[$ is a continuous real-valued function, nondecreasing in the second argument, satisfying $g(x, 0) = 0$, and $\mathcal{S}_\chi$ denotes the set of piecewise linear continuous single-valued functions, with values in $\mathbb{R}^n$ and knots at χ.

Corollary 3.18. *Let* $F \in Lip([a, b],\mathcal{L})$ *and let* A_χ *be of the form* (3.25)*, satisfying*

$$|A_\chi(s,x) - s(x)| \leq C\cdot\mathcal{L}\cdot g(x,\delta_{\max}), \quad s\in\mathcal{S}_\chi\cap Lip([a,b],\mathcal{L}).$$

Then

$$\mathrm{H}(A_\chi^M(F,x),F(x)) \leq 2\mathcal{L}\delta_{\max} + C\cdot\mathcal{L}\cdot g(x,\delta_{\max}). \quad (3.42)$$

Corollary 3.19. *Let* $F \in CBV[a, b]$ *and let* A_χ *be of the form* (3.25)*, satisfying*

$$|A_\chi(s,x) - s(x)| \leq C\omega_{[a,b]}(s,g(x,\delta_{\max})), \quad s\in\mathcal{S}_\chi. \quad (3.43)$$

Then

$$\mathrm{H}(A_\chi^M(F,x),F(x)) \leq 2\,\omega_{[a,b]}(F,\delta_{\max}) + 4C\omega_{[a,b]}(v_F,g(x,\delta_{\max})). \quad (3.44)$$

For continuous set-valued functions which are not of bounded variation there are some limited results only for uniform partitions, see [10].

Corollary 3.20. *Let* $F : [a,b] \to \mathbb{K}(\mathbb{R}^m)$ *be continuous, and let* A_N *be a linear operator of the form* (3.25) *defined on a uniform partition* χ_N *with* $h = (b-a)/N$, *satisfying*

$$|A_N(s,x) - s(x)| \le Cg(x, \omega_{[a,b]}(s,h)), \quad s \in \mathcal{S}_\chi. \tag{3.45}$$

Then

$$\mathrm{H}(A_\chi^M(F,x), F(x)) \le 2\omega_{[a,b]}(F,h) + Cg(x, 2\omega_{[a,b]}(F,h)). \tag{3.46}$$

3.5. **Examples.**

3.5.1. *Metric Bernstein operators.* We recall the Bernstein operator $\mathrm{B}_N(f,x)$ in (1.3). It is known [7, Chapter 10] that there is a constant C independent of f such that

$$|f(x) - \mathrm{B}_N(f,x)| \le C \cdot \omega_{[0,1]}(f, \sqrt{x(1-x)/N}\,). \tag{3.47}$$

The classical Bernstein operator for $F : [0,1] \to \mathbb{K}$ with sums of numbers replaced by Minkowski sums of sets is given by (2.11). We have shown by Theorem 2.12 that for $x \in]0,1[$ the limit of $\mathrm{B}_N(F)(x)$ when $N \to \infty$ is $\mathrm{conv}F(x)$, therefore these operators cannot approximate multifunctions with general images.

Definition 3.21. *For* $F : [0,1] \to \mathbb{K}$ *the* ***metric Bernstein operator*** *is*

$$\begin{aligned} \mathrm{B}_N^M(F,x) &= \bigoplus_{k=0}^{N} \binom{N}{k} x^k (1-x)^{N-k} F\left(\frac{k}{N}\right) \\ &= \left\{ \sum_{k=0}^{N} \binom{N}{k} x^k (1-x)^{N-k} f_k \mid (f_0, \ldots, f_N) \in CH \right\} \end{aligned}$$

where $CH = CH(F(0), F(1/N), \ldots, F(1))$.

Corollary 3.22. *Let* $F \in Lip([0,1], \mathcal{L})$, *then*

$$\mathrm{H}(\mathrm{B}_N^M(F,x), F(x)) \le 2\mathcal{L}/N + C\mathcal{L}\sqrt{x(1-x)/N}.$$

Proof. Apply Corollary 3.18 with $A_\chi^M(F) = \mathrm{B}_N^M(F)$ and (3.47) and the conclusion follows. □

Corollary 3.23. *Let* $F \in CBV[0,1]$, *then*

$$\mathrm{H}(\mathrm{B}_N^M(F,x), F(x)) \le 2\omega_{[0,1]}(F, 1/N) + 4C\omega_{[0,1]}(v_F, \sqrt{x(1-x)/N}\,).$$

Proof. Apply Corollary 3.19 with $A_{\chi}^{M}(F) = \mathrm{B}_{N}^{M}(F)$ and (3.47). $\square$

Since (3.45) does not hold for these operators, Corollary 3.20 cannot be applied.

3.5.2. *Metric Schoenberg operators.* For a uniform partition χ_N, the "classical" set-valued analogues of the Schoenberg spline operators for $F : [0,1] \to \mathbb{K}$ is

$$S_{m,N}(F,x) = \sum_{k=0}^{N} F(k/N) b_m(Nx-k), \tag{3.48}$$

where $b_m(x)$ is the (normalized) B-spline of order m (degree $m-1$) with integer knots and support $[0,m]$, and where the linear combination is in Minkowski sense. In [28] by an example it is shown that operators (3.48) with $m=2$ and $N \to \infty$ cannot approximate F with general compact images in any point of $[0,1] \setminus \chi_N$.

Definition 3.24. *The* ***metric Schoenberg operator*** *of order* m *for a multifunction* $F : [0,1] \to \mathbb{K}$ *and a uniform partition* χ_N *is defined by*

$$S_{m,N}^{M}(F,x) = \bigoplus_{k=0}^{N} b_m(Nx-k) F(k/N) = \left\{ \sum_{k=0}^{N} b_m(Nx-k) f_k \mid (f_0,\dots,f_N) \in CH \right\},$$

where $CH = CH(F(0), F(1/N), \dots, F(1))$.

The estimate below for the single valued case may be found in [5, p. 167]

$$|S_{m,N} f - f| \le \lfloor (m+1)/2 \rfloor \, \omega_{[0,1]}(f, 1/N) \ \text{ on } \ [(m-1)/N\,, 1] \tag{3.49}$$

where $\lfloor \cdot \rfloor$ is the floor function.

Corollary 3.25. *Let* F *be a continuous multifunction defined on* $[0,1]$. *Then*

$$\mathrm{H}(S_{m,N}^{M}(F,x), F(x)) \le 2\,(1 + \lfloor (m+1)/2 \rfloor)\, \omega_{[0,1]}(F, 1/N), \ x \in [(m-1)/N, 1]\,.$$

Proof. By Corollary 3.20 and by (3.49) we have

$$\begin{aligned} \mathrm{H}(S_{m,N}^{M}(F,x), F(x)) &\le 2\omega_{[0,1]}(F,1/N) + \lfloor (m+1)/2 \rfloor \, \omega_{[0,1]}(f,1/N) \\ &\overset{(3.37)}{\le} 2\,(1 + \lfloor (m+1)/2 \rfloor)\, \omega_{[0,1]}(F,1/N), \quad x \in [(m-1)/N, 1]\,. \end{aligned}$$

$\square$

If a function f is Lipschitz on $[0,1]$ of rank $\mathcal{L}$, then from (3.49) it follows

$$|S_{m,N}f - f| \le \lfloor (m+1)/2 \rfloor \, \mathcal{L}/N. \tag{3.50}$$

Corollary 3.26. *For $F \in Lip([0,1],\mathcal{L})$,*

$$\mathrm{H}(S^M_{m,N}(F,x), F(x)) \le (2 + \lfloor (m+1)/2 \rfloor) \, \frac{\mathcal{L}}{N}\,, \quad x \in [(m-1)/N\,, 1]\,.$$

Proof. By Corollary 3.18 and by (3.50) one has

$$\mathrm{H}(S^M_{m,N}(F,x), F(x)) \le 2\mathcal{L}/N + \lfloor (m+1)/2 \rfloor \, \mathcal{L}/N,$$

from where the conclusion follows. □

3.5.3. *Metric Polynomial Interpolants.*

Definition 3.27. (i) *Let (x_k, A_k) be given, where $x_0 < x_1 < \cdots < x_N$ are real numbers and $A_k \in \mathbb{K}$, $k = 0, \dots, N$ are sets. The **metric polynomial interpolant** of these data is*

$$\bigoplus_{k=0}^{N} l_k A_k,$$

with l_k defined by (1.5).

(ii) *For $F : [a,b] \to \mathbb{K}$, the **metric polynomial interpolation operator** at the partition χ of $[a,b]$, is given by*

$$P^M_\chi(F,x) = \bigoplus_{k=0}^{N} l_k(x) F(x_k) = \left\{ \sum_{k=0}^{N} l_k(x) f_k \mid (f_0, f_1, \dots, f_N) \in CH(F_0, \dots, F_N) \right\},$$

with $F_k = F(x_k)$, $k = 0, \dots, N$.

Let the interpolation points χ be the roots of the Chebyshev polynomial of degree $N+1$ on $[-1,1]$. It is known that (see, e.g., [22]) $\sum_{k=0}^{N} |\, l_k(x)| \le C \ln N$. Here and below C stands for a generic constant.

For a real-valued function f,

$$|f - \sum_{k=0}^{N} l_k(x) f(x_i)| \le (1 + \sum_{k=0}^{N} |l_k(x)|) E_N(f),$$

with $E_N(f)$ the error of the best approximation by polynomials of degree N on $[-1,1]$. Since $E_N(f) \leq C\omega_{[-1,1]}(f,1/N)$, [7, (1.3) in Chap. 7], we obtain for a Lipschitz function f

$$\left| f - \sum_{k=0}^{N} l_k(x) f(x_k) \right| \leq C \ln N/N \xrightarrow{N\to\infty} 0. \tag{3.51}$$

When adapting these interpolation operators to Lipschitz multifunctions, by Theorem 3.17 we get

Corollary 3.28. *For $F \in Lip([0,1],\mathcal{L})$, and let the points χ be the roots of the Chebyshev polynomial of degree $N+1$ on this interval, then*

$$\mathrm{H}(P_\chi^M(F,x), F(x)) \leq 2\mathcal{L}\delta_{\max} + C\ln N/N = O\left(\ln N/N\right).$$

The last equality follows from the fact that $\delta_{\max} \leq \pi/N$ for N large enough.

Remark 3.29. In [3] and [4] the family of nonempty convex and compact subsets in $\mathbb{R}^n$ is used for similar goals but in a different framework. $\triangle$

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