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COVERING SUBGROUPS IN FINITE PRIMITIVE π-SOLVABLE GROUPS

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Abstract. Let π be an arbitrary set of primes and let X be a π-closed Schunck class. The paper deals with the study of X-covering subgroups in finite primitive π-solvable groups, connecting them with complements, stabilizers and X-maximal subgroups. Some characterization theorems for X-covering subgroups in finite primitive π-solvable groups by means of complements of appropriate minimal normal subgroups, by means of stabilizers and by means of some X-maximal subgroups are given.

1. **Preliminaries**

All groups considered in this paper are finite. Let π be a set of primes and π' the complement to π in the set of all primes.

We first remind some definitions and theorems which will be useful for our considerations.

Definition 1.1. a) Let G be a group, M and N two normal subgroups of G such that $N \subseteq M$. The factor M/N is called a *chief factor* of G if M/N is a minimal normal subgroup of G/N.

b) A group G is said to be *π-solvable* if every chief factor of G is either a solvable π-group or a π'-group. In particular, for π the set of all primes we obtain the notion of solvable group.

Definition 1.2. a) Let G be a group and W a subgroup of G. We define

$$core_G W = \cap\{W^g \mid g \in G\},$$

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where $W^g = g^{-1}Wg$.

b) W is a *stabilizer* of G if W is a maximal subgroup of G and $core_G W = 1$.

c) A group G is said to be *primitive* if there exists a stabilizer W of G.

In the formation theory are well-known the following notions:

Definition 1.3. a) A class X of groups is a *homomorph* if X is closed under homomorphisms, i.e. if $G \in X$ and N is a normal subgroup of G, then $G/N \in X$.

b) A homomorph X is a *Schunck class* if X is primitively closed, i.e. if any group G, all of whose primitive factor groups are in X, is itself in X.

Definition 1.4. a) A class X of groups is called *π-closed* if

$$G/O_{\pi'}(G) \in X \;\Rightarrow\; G \in X,$$

where $O_{\pi'}(G)$ denotes the largest normal π'-subgroup of G.

b) We shall call *π-homomorph*, respectively *π-Schunck class*, a π-closed homomorph, respectively a π-closed Schunck class.

Definition 1.5. Let X be a class of groups, G a group and H a subgroup of G.

a) H is an *X-maximal subgroup* of G if:

(i) $H \in X$;

(ii) $H \leq H^* \leq G,\ H^* \in X \;\Rightarrow\; H = H^*$.

b) H is an *X-covering subgroup* of G if:

(i) $H \in X$;

(ii) $H \leq K \leq G,\ K_0 \trianglelefteq K,\ K/K_0 \in X \;\Rightarrow\; K = HK_0$.

Remark 1.6. If X is a class of groups, G is a group and H is an *X-covering subgroup* of G, then H is X-maximal in G.

The following results will be used in the paper:

Theorem 1.7. ([1]) *A solvable minimal normal subgroup of a finite group is abelian.*

Theorem 1.8. ([2], [3]) *Let G be a primitive π-solvable group. If G has a minimal normal subgroup which is a solvable π-group, then G has one and only one minimal normal subgroup.*

Theorem 1.9. ([3]) *If G is a primitive π-solvable group, $V < G$, such that there exists a minimal normal subgroup M of G which is a solvable π-group and $MV = G$, then V is a stabilizer of G.*

Theorem 1.10. ([5]) *Let X be a π-homomorph. The following conditions are equivalent:*

(1) X is a Schunck class;

(2) if G is a π-solvable group, $G \notin X$ and N is a minimal normal subgroup of G such that $G/N \in X$, then N has a complement in G;

(3) any π-solvable group G has X-covering subgroups.

Theorem 1.11. ([5]) *Let X be a π-Schunck class, G a π-solvable group, $G \notin X$, N a minimal normal subgroup of G such that $G/N \in X$ and H an X-covering subgroup of G. Then H is a complement of N in G, i.e. $G = HN$ and $H \cap N = 1$.*

Theorem 1.12. ([5]) *If X is a π-Schunck class, G is a π-solvable group, $G \notin X$ and N is a minimal normal subgroup of G such that $G/N \in X$, then:*

a) N has a complement H in G;

b) H is X-maximal in G;

c) H is conjugate to any X-maximal subgroup S of G with $NS = G$.

2. On stabilizers in finite primitive π-solvable groups

Lemma 2.1. *If G is a group and W a stabilizer of G, then:*

a) for any normal subgroup $K \neq 1$ of G, we have $KW = G$;

b) for any minimal normal subgroup M of G, we have $MW = G$.

Proof. a) Let $K \neq 1$ be a normal subgroup of G. Since W is maximal in G and $W \leq KW \leq G$, we have $KW = W$ or $KW = G$. Suppose that $KW = W$. It follows that $K \leq W$ and so $K^g \leq W^g$ for any $g \in G$. But K being normal in G, $K^g = K$ for any $g \in G$. Then $K \leq W^g$ for any $g \in G$, hence $K \leq core_G W = 1$. So $K = 1$, in contradiction to our hypothesis. So $KW = G$.

b) Follows immediately from a). □

Theorem 2.2. *Let G be a π-solvable group, W a stabilizer of G and M a minimal normal subgroup of G such that M is a solvable π-group. Then W is a complement of M in G, i.e. $MW = G$ and $M \cap W = 1$.*

Proof. $MW = G$ follows from Lemma 2.1. Let us now prove that $M \cap W = 1$. Since M is normal in G and $W \leq G$, we have that $M \cap W$ is normal in W. By 1.7, M is abelian. In order to prove that $M \cap W$ is normal in G, consider $g \in G$ and $m \in M \cap W$. Since $G = MW$, we have $g = nw$, where $n \in M$ and $w \in W$. So

$$g^{-1}mg = (nw)^{-1}m(nw) = w^{-1}n^{-1}mnw = w^{-1}n^{-1}nmw = w^{-1}mw \in M \cap W,$$

where we used that M is abelian and that $M \cap W \trianglelefteq W$. It follows that $M \cap W$ is normal in G. From this and from the fact that M is a minimal normal subgroup of G, we deduce that $M \cap W = 1$ or $M \cap W = M$. But $M \cap W = M$ leads to $M \subseteq W$, hence $G = MW = W$, in contradiction with the hypothesis that W is a stabilizer of G. So $M \cap W = 1$. □

Theorem 2.3. *Let G be a primitive π-solvable group such that there exists a minimal normal subgroup M of G, M solvable π-group. Let $W < G$. The following two conditions are equivalent:*

(1) W is a stabilizer of G;

(2) $MW = G$.

Proof. By 1.8, M is the unique minimal normal subgroup of G.

(1) $\Rightarrow$ (2): Let W be a stabilizer of G. Applying 2.2, we obtain that $MW = G$.

(2) $\Rightarrow$ (1): Let $MW = G$. Then, by 1.9, W is a stabilizer of G. □

3. Covering subgroups and complements in finite primitive π-solvable groups

In preparation for the main result of this section, we first prove a lemma.

Lemma 3.1. *Let X be a π-homomorph, G a π-solvable group, $G \notin X$ and N a minimal normal subgroup of G such that $G/N \in X$. Then:*

a) N is a solvable π-group;

b) N is abelian.

Proof. a) Since G is a π-solvable group and N is a minimal normal subgroup of G, we conclude that N is either a solvable π-group or a π'-group. Suppose that N is a π'-group. Then $N \leq O_{\pi'}(G) \leq G$, hence

$$G/O_{\pi'}(G) \simeq (G/N)/(O_{\pi'}(G)/N).$$

But $G/N \in X$. Then by the above isomorphism and X being a homomorph, $G/O_{\pi'}(G) \in X$. It follows by the π-closure of X that $G \in X$, a contradiction. This shows that N is a solvable π-group.

b) We apply 1.7 and a) and obtain that N is abelian. □

Theorem 3.2. *Let X be a π-Schunck class, G a finite primitive π-solvable group, $G \notin X$, N a minimal normal subgroup of G such that $G/N \in X$ and let $H \leq G$. The following two conditions are equivalent:*

(1) H is an X-covering subgroup of G;

(2) H is a complement of N in G, i.e. $HN = G$ and $H \cap N = 1$.

Proof. (1) $\Rightarrow$ (2): Let H be an X-covering subgroup of G. By applying 1.11, H is a complement of N in G.

(2) $\Rightarrow$ (1): Let H be a complement of N in G (according to 1.10, H exists), i.e. we have $HN = G$ and $H \cap N = 1$. By lemma 3.1, N is a solvable π-group, hence N is abelian. We will prove that H is an X-covering subgroup of G by verifying conditions (i) and (ii) from 1.5.b).

(i) $H \in X$. Indeed, we have:

$$H \simeq H/1 = H/H \cap N \simeq HN/N = G/N \in X.$$

(ii) Let $H \leq K \leq G$, $K_0 \trianglelefteq K$, $K/K_0 \in X$. We prove that $K = HK_0$. For this, we first prove that H is a maximal subgroup of G. Indeed, $H \neq G$ (since $H \in X$ and $G \notin X$) and let now $H \leq H^* < G$. In order to show that $H = H^*$, suppose $H < H^*$. Then there exists an element $h^* \in H^* \setminus H \subset G = HN$ and so

$$h^* = hn, \text{ with } h \in H, \ n \in N$$

hence

$$n = h^{-1}h^* \in H^* \cap N.$$

Let us show that $H^* \cap N = 1$. For this, we notice that from $N \trianglelefteq G$ and $H^* \leq G$ follows that $H^* \cap N \trianglelefteq H^*$. Furthermore, $H^* \cap N \trianglelefteq G$, since for any $g \in G$ and any $n \in H^* \cap N$, we have that $g^{-1}ng \in H^* \cap N$, as we show below:

$$g \in G = HN = H^*N = NH^* \;\Rightarrow\; g = mh^*,\; m \in N,\; h^* \in H^*$$

$$\Rightarrow\; g^{-1}ng = (mh^*)^{-1}n(mh^*) = (h^*)^{-1}m^{-1}nmh^*$$

$$= (h^*)^{-1}m^{-1}mnh^* = (h^*)^{-1}nh^* \in H^* \cap N,$$

where we used that N is abelian and that $H^* \cap N \trianglelefteq H^*$. So $H^* \cap N \trianglelefteq G$. But N is a minimal normal subgroup of G, hence $H^* \cap N = 1$ or $H^* \cap N = N$. Suppose $H^* \cap N = N$. Then $N \subseteq H^*$, hence $G = H^*N = H^*$, a contradiction. It follows that $H^* \cap N = 1$. Then

$$n = h^{-1}h^* \in H^* \cap N = 1 \;\Rightarrow\; n = 1 \;\Rightarrow\; h^{-1}h^* = 1 \;\Rightarrow\; h = h^* \in H^* \setminus H,$$

in contradiction with $h \in H$. It follows that H is a maximal subgroup of G. Hence from $H \leq K \leq G$, we have only two possibilities: $K = H$ or $K = G$.

If $K = H$, the hypotheses of (ii) become $H \leq H \leq G$, $K_0 \trianglelefteq H$, $H/K_0 \in X$ and clearly $K = H = HK_0$.

If $K = G$, the hypotheses of (ii) become $H \leq G \leq G$, $K_0 \trianglelefteq G$, $G/K_0 \in X$. We have to prove that $G = HK_0$. Observe that $K_0 \neq 1$. Indeed, supposing that $K_0 = 1$, we have $G \simeq G/1 = G/K_0 \in X$, a contradiction with $G \notin X$. Furthermore, by 1.8, N is the unique minimal normal subgroup of G. Hence for $K_0 \trianglelefteq G$, $K_0 \neq 1$ follows that $N \subseteq K_0$. So $G = HN \subseteq HK_0$, which leads to $K = G = HK_0$. □

Theorems 1.12 and 3.2 have the following consequence:

Corollary 3.3. *Let X be a π-Schunck class, G a finite primitive π-solvable group, $G \notin X$ and N a minimal normal subgroup of G such that $G/N \in X$. Then:*

a) N has a complement H in G;

b) H is an X-covering subgroup of G;

c) H is X-maximal in G;

d) H is conjugate to any X-maximal subgroup S of G with $SN = G$;

e) conditions a) and b) are equivalent.

4. Covering subgroups and stabilizers in finite primitive π-solvable groups

In this section we will establish a connection between covering subgroups and stabilizers in finite primitive π-solvable groups.

Theorem 4.1. *Let X be a π-Schunck class, G a finite primitive π-solvable group, $G \notin X$, N a minimal normal subgroup of G such that $G/N \in X$ and let $H \leq G$. The following two conditions are equivalent:*

(1) H is an X-covering subgroup of G;

(2) H is a stabilizer of G.

Proof. By lemma 3.1, N is a solvable π-group.

(1) $\Rightarrow$ (2): Let H be an X-covering subgroup of G. Then $H \in X$. This implies $H \neq G$, since $G \notin X$. Applying Theorem 3.2, we obtain that $HN = G$. This and $H < G$ show that we are in the hypotheses of Theorem 1.9. It follows that H is a stabilizer of G.

(2) $\Rightarrow$ (1): Let H be a stabilizer of G. Then, by 2.2, H is a complement of N in G. Now by applying Theorem 3.2, we conclude that H is an X-covering subgroup of G. □

Theorems 3.2 and 4.1 have the following corollary:

Corollary 4.2. *Let X be a π-Schunck class, G a finite primitive π-solvable group, $G \notin X$, N a minimal normal subgroup of G such that $G/N \in X$ and let $H \leq G$. The following three conditions are equivalent:*

(1) H is an X-covering subgroup of G;

(2) H is a complement of N in G;

(3) H is a stabilizer of G.

5. X-maximal subgroups and complements in finite π-solvable groups

In this last section of the paper, we show that there is a connection between some particular X-maximal subgroups and the complements of some special minimal normal subgroups in finite π-solvable groups. This connection allows us to characterize the X-covering subgroups in finite primitive π-solvable groups by means of these particular X-maximal subgroups.

Theorem 5.1. *Let X be a π-Schunck class, G a finite π-solvable group, $G \notin X$ and let N be a minimal normal subgroup of G such that $G/N \in X$. Then:*

a) N has a complement H in G; furthermore, H is X-maximal in G and H is conjugate to any X-maximal subgroup S of G with $SN = G$;

b) the following two conditions on $H \leq G$ are equivalent:

(i) H is an X-maximal subgroup of G such that $HN = G$;

(ii) H is a complement of N in G;

c) any two complements H_1 and H_2 of N in G are conjugate in G.

Proof. a) Immediately follows from Theorem 1.12.

b) (i) $\Rightarrow$ (ii): Let H be X-maximal in G such that $HN = G$. We have to prove that $H \cap N = 1$. Observe first that $H \neq G$, since $H \in X$ and $G \notin X$. From $H \leq G$ and $N \trianglelefteq G$ follows that $H \cap N \trianglelefteq H$. Lemma 3.1 implies that N is abelian. Let us now prove that $H \cap N$ is normal in G. Let $g \in G$ and $n \in H \cap N$. Then:

$$g \in G = HN = NH \;\Rightarrow\; g = mh, \text{ where } m \in N,\ h \in H,$$

hence

$$g^{-1}ng = (mh)^{-1}n(mh) = h^{-1}m^{-1}nmh$$

$$= h^{-1}m^{-1}mnh = h^{-1}nh \in H \cap N,$$

where we used that $H \cap N \trianglelefteq H$. In order to prove that $H \cap N = 1$, we consider the normal subgroup $H \cap N$ of G and observe that $H \cap N \subseteq N$, where N is a minimal normal subgroup of G. It follows that $H \cap N = 1$ or $H \cap N = N$. If we suppose that $H \cap N = N$, we obtain $N \subseteq H$, hence $G = HN = H$, in contradiction with $H \neq G$. So $H \cap N = 1$.

(ii) $\Rightarrow$ (i): Let H be a complement of N in G. Hence, by 1.12, H is X-maximal in G. Obviously $HN = G$, H being a complement of N in G.

c) Let H_1 and H_2 be two complements of N in G. Applying b) to H_2, we obtain that H_2 is X-maximal in G and $H_2N = G$. But H_1 is a complement of N in G. Now applying Theorem 1.12.c) it follows that H_1 is conjugate with H_2 in G. □

From Theorem 5.1.b) and Corollary 4.2 follows:

Corollary 5.2. *Let X be a π-Schunck class, G a finite primitive π-solvable group, $G \notin X$, N a minimal normal subgroup of G such that $G/N \in X$ and let $H \leq G$. The following four conditions are equivalent:*

(1) H is a complement of N in G;

(2) H is X-maximal in G and $HN = G$;

(3) H is an X-covering subgroup of G;

(4) H is a stabilizer of G.

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