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SIMPSON, NEWTON AND GAUSS TYPE INEQUALITIES

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Abstract. In this paper using the Simpson's quadrature formula, the Newton quadrature formula and the Gauss quadrature formula, we present new inequalities between means.

1. Introduction

This papers deals with the comparison of means. If s and t are two real parameters and a and b are positive numbers, then we may consider the following two families of means:

- the *Gini means*,

$$G_{s,t}(a,b)=\begin{cases}\left(\frac{a^s+b^s}{a^t+b^t}\right)^{1/(s-t)}, & \text{if } s\neq t\\ \exp\left(\frac{a^s\log a+b^s\log b}{a^s+b^s}\right), & \text{if } s=t\end{cases};$$

- the *Stolarski means*,

$$S_{s,t}(a,b)=\begin{cases}\left(\frac{t(a^s-b^s)}{s(a^t-b^t)}\right)^{1/(s-t)}, & \text{if } (s-t)\,st\neq 0,\ a\neq b\\ \exp\left(-\frac{1}{s}+\frac{a^s\log a-b^s\log b}{a^s-b^s}\right), & \text{if } s=t\neq 0,\ a\neq b\\ \left(\frac{a^s-b^s}{s(\log a-\log b)}\right)^{1/s}, & \text{if } s\neq 0,\ t=0,\ a\neq b\\ \sqrt{ab}, & \text{if } s=t=0\\ a, & \text{if } a=b.\end{cases}$$

Some particular cases are important in themselves.

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$G_{s,0}(a,b)$ coincides with the *Hölder mean* of order $s>0$,

$$A_s(a,b)=\left(\frac{a^s+b^s}{2}\right)^{1/s}=\left(\frac{s}{b^s-a^s}\int_a^b x^{2s-1}dx\right)^{1/s}$$

($A_1(a,b)$ is precisely the *arithmetic mean* of a and b, also denoted $A(a,b)$).

$G_{0,0}(a,b)$ coincides with the *geometric mean,*

$$G(a,b)=\sqrt{ab}=\left(\frac{1}{b-a}\int_a^b \frac{1}{x^2}dx\right)^{-1/2};$$

$S_{1,0}(a,b)$ coincides with the *logarithmic mean,*

$$L(a,b)=\frac{b-a}{\ln b-\ln a}=\left(\frac{1}{b-a}\int_a^b \frac{dx}{x}\right)^{-1}$$

while $S_{1,1}(a,b)$ coincides with the *identric mean,*

$$I(a,b)=\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}=\exp\left(\frac{1}{b-a}\int_a^b \ln x dx\right).$$

We will be concerned with the problem of comparing the different means. Our approach is based on certain inequalities satisfied by the 4-convex functions. Recall that in the differentiable case these are precisely those 4-time differentiable functions f such that $f^{(4)}(x)\geq 0$ for all x.

Lemma 1.1. *If $f\in C^4([a,b])$ and $f^{(4)}\geq 0$, then the mean value of f,*

$$M(f)=\frac{1}{b-a}\int_a^b f(x)\,dx,$$

does not exceed any of the following three sums:

i) $\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right];$

ii) $\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right];$

iii) $\left[f\left(\frac{a+b}{2}-\frac{b-a}{6}\sqrt{3}\right)+f\left(\frac{a+b}{2}+\frac{b-a}{6}\sqrt{3}\right)\right].$

Proof. According to Simpson's quadrature formula,

$$\frac{1}{b-a}\int_a^b f(x)\,dx=\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{(b-a)^4}{2880}f^{(4)}(\xi_1),$$

for some $\xi_1\in(a,b)$, whence *i)*. The cases *ii)* and *iii)* are motivated by the Newton quadrature formula,

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right] - \frac{(b-a)^4}{648} f^{(4)}(\xi_2),$$

and respectively by the Gauss quadrature formula

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{2}\left[f\left(\frac{a+b}{2} - \frac{b-a}{6}\sqrt{3}\right) + f\left(\frac{a+b}{2} + \frac{b-a}{6}\sqrt{3}\right)\right] + \frac{(b-a)^4}{4320} f^{(4)}(\xi_3),$$

where ξ_2 and ξ_3 are suitable points in (a, b). □

2. **Applications**

Theorem 2.1. *If $a, b > 0$ then holds the following inequality*

$$G^2(a,b) \geq \frac{6a^2b^2(a+b)^2}{(a^2+b^2)(a+b)^2 + 16a^2b^2}$$

or, in an equivalent form,

$$A\left(a^2, b^2\right) A^2(a,b) + 2G^4(a,b) \geq 3G^2(a,b)\, A^2(a,b).$$

Proof. In Lemma 1.1, we take $f(x) = \frac{1}{x^2}$, from which $f^{(4)}(x) = \frac{120}{x^6} > 0$, therefore

$$\frac{1}{G^2(a,b)} = \frac{1}{b-a}\int_a^b \frac{1}{x^2}dx \leq \frac{1}{6}\left(\frac{1}{a^2} + \frac{16}{(a+b)^2} + \frac{1}{b^2}\right).$$

After calculus we obtain:

$$G^2(a,b) \geq \frac{6a^2b^2(a+b)^2}{(a^2+b^2)(a+b)^2 + 16a^2b^2},$$

that is,

$$A\left(a^2, b^2\right) A^2(a,b) + 2G^4(a,b) \geq 3G^2(a,b)\, A^2(a,b).$$

□

Theorem 2.2. *If $a, b, t > 0$ then the following inequality holds*

$$G_t^2(a,b) \geq \frac{(b^t - a^t)(ab(a+b))^{t+1}}{t(b-a)\left((a^{t+1}+b^{t+1})(a+b)^{t+1} + 2^{t+3}(ab)^{t+1}\right)}$$

or, in an equivalent form,

$$A\left(a^{t+1}, b^{t+1}\right) A^{t+1}(a+b) + 2G^{2t+2}(a,b) \geq \frac{3(b^t - a^t)}{t(b-a)} \cdot \frac{G^{2t+2}(a,b)}{G_t^2(a,b)} \cdot A^{t+1}(a,b).$$

Proof. In Lemma 1.1, we take $f(x) = \frac{1}{x^{t+1}}$, from which $f^{(4)}(x) > 0$ and so the proof follows easily. □

Theorem 2.3. *If $a, b > 0$ then the following inequality holds*

$$I^6(a,b) \geq ab\left(\frac{a+b}{2}\right)^4$$

or, in an equivalent form,

$$I(a,b) \geq G^{1/3}(a,b)\, A^{2/3}(a,b).$$

Proof. In Lemma 1.1, we take $f(x) = \ln x$ for which $f^{(4)}(x) < 0$, therefore $I(a,b) = \exp\left(\frac{1}{b-a}\int_a^b \ln x dx\right)$

$$\geq \exp \tfrac{1}{6}\left(\ln a + 4\ln\left(\tfrac{a+b}{2}\right) + \ln b\right) = \sqrt[6]{ab\left(\tfrac{a+b}{2}\right)^4}.$$ □

Exercise 2.1. *If $a, b > 0$ then*

$$\frac{A(a,b)}{L(a,b)} \geq 1 + \frac{2}{3}\ln\frac{A(a,b)}{G(a,b)}.$$

Proof. From the definitions of identric and logarithmic mean, we have

$$\ln I(a,b) = \frac{a}{L(a,b)} + \ln b - 1$$

and

$$\ln I(a,b) = \frac{b}{L(a,b)} + \ln a - 1.$$

After addition, we obtain:

$$\frac{a+b}{L(a,b)} + \ln ab - 2 = 2\ln I(a,b)$$

or, equivalently,

$$\frac{A(a,b)}{L(a,b)} + \ln G(a,b) - 1 = \ln I(a,b). \quad (2.1)$$

Using the statement of the Theorem 2.3 we obtain:

$$\frac{A(a,b)}{L(a,b)} + \ln G(a,b) - 1 \geq \ln \left(G^2(a,b) A^4(a,b) \right)^{\frac{1}{6}}.$$

□

Theorem 2.4. *If $a, b > 0$ then the following inequality holds:*

$$L(a,b) \geq \frac{(a+b)^2 + 8ab}{6ab(a+b)}$$

or, in an equivalent form,

$$3L(a,b) \geq \frac{A(a,b)}{G^2(a,b)} + \frac{2}{A(a,b)}.$$

Proof. In Lemma 1.1 we take $f(x) = \frac{1}{x}$ for which $f^{(4)}(x) > 0$, therefore

$$\frac{1}{L(a,b)} = \frac{1}{b-a} \int_a^b \frac{dx}{x} \leq \frac{1}{6} \left(\frac{1}{a} + \frac{8}{a+b} + \frac{1}{b} \right)$$

or, equivalently,

$$L(a,b) \geq \frac{6ab(a+b)}{(a+b)^2 + 8ab}.$$

□

Theorem 2.5. *If $a, b > 0$ and $t \in \left(-\infty, \frac{1}{2}\right] \cup \left[1, \frac{3}{2}\right] \cup [2, +\infty)$, then*

$$A_t^t(a,b) \leq \frac{t(b-a)\left(2^{2t-1}\left(a^{2t-1} + b^{2t-1}\right) + 4(a+b)^{2t-1}\right)}{3 \cdot 2^{2t}\left(b^t - a^t\right)}$$

or, in an equivalent form,

$$A_t^t(a,b) \leq \frac{t(b-a)}{3(b^t - a^t)} \left(A\left(a^{2t-1}, b^{2t-1}\right) + 2A^{2t-1}(a,b) \right).$$

If $t \in \left(\frac{1}{2}, 1\right) \cup \left(\frac{3}{2}, 2\right)$, then the reverse inequality holds.

Proof. In Lemma 1.1 we take $f(x) = x^{2t-1}$ for which

$$f^{(4)}(x) = (2t-1)(2t-2)(2t-3)(2t-4)x^{2t-5}.$$

If $t \in \left(-\infty, \frac{1}{2}\right] \cup \left[1, \frac{3}{2}\right] \cup [2, +\infty)$, then

$$f^{(4)}(x) > 0$$

and

$$A_t^t(a,b)=\frac{t}{b^t-a^t}\int_a^b x^{2t-1}dx\le\frac{t(b-a)}{6(b^t-a^t)}\left(a^{2t-1}+4\left(\frac{a+b}{2}\right)^{2t-1}+b^{2t-1}\right)$$

and the proof continues in an easy manner. □

3. Newton Type Inequalities

Theorem 3.1. *If $a,b>0$ then the following inequality holds*

$$G^2(a,b)\ge\frac{8a^2b^2(2a+b)^2(a+2b)^2}{(a^2+b^2)(2a+b)^2(a+2b)^2+27a^2b^2(5a^2+8ab+5b^2)}$$

or, in an equivalent form,

$$16A\left(a^2,b^2\right)A(2a,b)A(a,2b)+27G^4(a,b)\left(5A\left(a^2,b^2\right)+4G^4(a,b)\right)$$
$$\ge 64G^2(a,b)A(2a,b)A(a,2b).$$

Proof. In Lemma 1.1 we take $f(x)=\frac{1}{x^2}$ for which $f^{(4)}(x)>0$, therefore

$$\frac{1}{G^2(a,b)}=\frac{1}{b-a}\int_a^b\frac{dx}{x^2}\le\frac{1}{8}\left(\frac{1}{a^2}+\frac{27}{(2a+b)^2}+\frac{27}{(a+2b)^2}+\frac{1}{b^2}\right).$$

□

Theorem 3.2. *If $a,b,t>0$ then $G_t^2(a,b)$ is greater or equal to*

$$\frac{8\left(b^t-a^t\right)(ab)^{t+1}(2a+b)^{t+1}(a+2b)^{t+1}}{t(b-a)\left(\left(a^{t+1}+b^{t+1}\right)(2a+b)^{t+1}(a+2b)^{t+1}+3^{t+2}(ab)^{t+1}(2a+b)^{t+1}+(a+2b)^{t+1}\right)}.$$

Proof. In Lemma 1.1 ii) we take $f(x)=\frac{1}{x^{t+1}}$ for which $f^{(4)}(x)>0$ and so on. □

Theorem 3.3. *If $a,b>0$ then the following inequality holds*

$$I^8(a,b)\ge ab\left(\frac{2a+b}{3}\right)^3\left(\frac{a+2b}{3}\right)^3.$$

Proof. In Lemma 1.1 ii), we take $f(x)=\ln x$ for where $f^{(4)}(x)<0$, therefore

$$\begin{aligned} I(a,b) &= \exp\left(\frac{1}{b-a}\int_a^b \ln x dx\right) \\ &\geq \exp\left(\frac{1}{8}\left(\ln a + 3\ln\frac{2a+b}{3} + 3\ln\frac{a+2b}{3} + \ln b\right)\right) \\ &= \left(ab\left(\frac{2a+b}{3}\right)^3\left(\frac{a+2b}{3}\right)^3\right)^{\frac{1}{8}}. \end{aligned}$$

□

Exercise 3.1. *If $a,b>0$ then*

$$\frac{A(a,b)}{L(a,b)} \geq 1 + \ln\left(\left(\frac{2}{3}\right)^6 \frac{A^{\frac{3}{8}}(2a,b)\,A^{\frac{3}{8}}(a,2b)}{G^{\frac{3}{4}}(a,b)}\right).$$

Proof. Using (2.1) and the Theorem 3.3 we obtain

$$\frac{A(a,b)}{L(a,b)} + \ln G(a,b) - 1 \geq \ln\left(ab\left(\frac{2a+b}{3}\right)^3\left(\frac{a+2b}{3}\right)^3\right)^{\frac{1}{8}}$$

and the proof follows easily. □

Theorem 3.4. *If $a,b>0$ then the following inequality holds:*

$$L(a,b) \geq \frac{4ab(2a+b)(a+2b)}{(a+b)(a^2+16ab+b^2)}.$$

Proof. In Lemma 1.1 ii) we take $f(x) = \frac{1}{x}$ for which $f^{(4)}(x) > 0$, therefore

$$\frac{1}{L(a,b)} = \frac{1}{b-a}\int_a^b \frac{dx}{x} \leq \frac{1}{8}\left(\frac{1}{a} + \frac{9}{2a+b} + \frac{9}{a+2b} + \frac{1}{b}\right)$$

and so on. □

Theorem 3.5. *If $a,b>0$ and $t \in \left(-\infty, \frac{1}{2}\right] \cup \left[1, \frac{3}{2}\right] \cup [2, +\infty)$, then*

$$A_t^t(a,b) \leq \frac{t(b-a)\left(3^{2t-1}\left(a^{2t-1}+b^{2t-1}\right) + 3(2a+b)^{2t-1} + 3(a+2b)^{2t-1}\right)}{8\cdot 3^{2t-1}\left(b^t - a^t\right)}.$$

If $t \in \left(\frac{1}{2}, 1\right) \cup \left(\frac{3}{2}, 2\right)$, then the reverse inequality holds true.

Proof. In Lemma 1.1 ii) we take $f(x) = x^{2t-1}$ for which $f^{(4)}(x) > 0$, for $t \in \left(-\infty, \frac{1}{2}\right] \cup \left[1, \frac{3}{2}\right] \cup [2, +\infty)$, therefore

$$A_t^t(a,b) = \frac{t}{b^t - a^t}\int_a^b x^{2t-1}dx$$
$$\leq \frac{t(b-a)}{8(b^t-a^t)}\left(a^{2t-1} + 3\left(\frac{2a+b}{3}\right)^{2t-1} + 3\left(\frac{a+2b}{3}\right)^{2t-1} + b^{2t-1}\right)$$

and the proof follows. □

4. Gauss Type Inequalities

Theorem 4.1. *If $a, b > 0$ then*

$$G^2(a,b) \leq \frac{\left(a^2+4ab+b^2\right)^2}{12\left(a^2+ab+b^2\right)}.$$

Proof. In Lemma 1.1 iii) we take $f(x) = \frac{1}{x^2}$ for which $f^{(4)}(x) > 0$, therefore

$$\frac{1}{G^2(a,b)} = \frac{1}{b-a}\int_a^b \frac{dx}{x^2}$$
$$\geq \frac{1}{2}\left(\frac{1}{\left(\frac{a+b}{2} - \frac{(b-a)\sqrt{3}}{6}\right)^2} + \frac{1}{\left(\frac{a+b}{2} + \frac{(b-a)\sqrt{3}}{6}\right)^2}\right)$$
$$= \frac{12\left(a^2+ab+b^2\right)}{\left(a^2+4ab+b^2\right)^2}.$$

□

Theorem 4.2. *If $a, b, t > 0$ then $G_t^2(a,b)$ does not exceeds*

$$\frac{2\left(b^t-a^t\right)\left(a^2+4ab+b^2\right)^{t+1}}{t(b-a)\left(\left(\left(3+\sqrt{3}\right)a + \left(3-\sqrt{3}\right)b\right)^{t+1} + \left(\left(3-\sqrt{3}\right)a + \left(3+\sqrt{3}\right)b\right)^{t+1}\right)}.$$

Proof. In Lemma 1.1 iii) we take $f(x) = \frac{1}{x^{t+1}}$ for which $f^{(4)}(x) > 0$, therefore

$$\frac{1}{G_t^2(a,b)} = \frac{t}{b^t-a^t}\int_a^b \frac{dx}{x^{t+1}}$$
$$\geq \frac{t(b-a)6^{t+1}}{2(b^t-a^t)}\left(\frac{1}{\left(\left(3+\sqrt{3}\right)a + \left(3-\sqrt{3}\right)b\right)^{t+1}} + \frac{1}{\left(\left(3-\sqrt{3}\right)a + \left(3+\sqrt{3}\right)b\right)^{t+1}}\right)$$

and the proof just follows. □

Theorem 4.3. *If* $a, b > 0$ *then*

$$I^2(a,b) \leq \frac{a^2 + 4ab + b^2}{6}.$$

Proof. In Lemma 1.1 iii) we take $f(x) = \ln x$ for which $f^{(4)}(x) < 0$, therefore

$$\begin{aligned} I(a,b) &= \exp\left(\frac{1}{b-a}\int_a^b \ln x dx\right) \\ &\leq \exp\left(\frac{1}{2}\left(\ln\left(\frac{a+b}{2} - \frac{(b-a)\sqrt{3}}{6}\right) + \ln\left(\frac{a+b}{2} + \frac{(b-a)\sqrt{3}}{6}\right)\right)\right) \\ &= \sqrt{\frac{a^2 + 4ab + b^2}{6}}. \end{aligned}$$

□

Exercise 4.1. *If* $a, b > 0$ *then*

$$\frac{A(a,b)}{L(a,b)} \leq 1 + \frac{1}{2}\ln\left(\frac{1}{3} + \frac{2A^2(a,b)}{3G^2(a,b)}\right).$$

Proof. Using (2.1) and Theorem 4.3 we obtain the desired result. □

Theorem 4.4. *If* $a, b > 0$ *then*

$$L(a,b) \leq \frac{2\left(a^2 + 4ab + b^2\right)}{3(a+b)}.$$

Proof. In Lemma 1.1 iii) we take $f(x) = \frac{1}{x}$ for which $f^{(4)}(x) > 0$, therefore

$$\begin{aligned} \frac{1}{L(a,b)} &= \frac{1}{b-a}\int_a^b \frac{dx}{x} \\ &\geq \frac{1}{2}\left(\frac{1}{\frac{a+b}{2} - \frac{(b-a)\sqrt{3}}{6}} + \frac{1}{\frac{a+b}{2} + \frac{(b-a)\sqrt{3}}{6}}\right) \\ &= \frac{3(a+b)}{2(a^2 + 4ab + b^2)}. \end{aligned}$$

□

Theorem 4.5. *If* $a, b > 0$ *and* $t \in \left(-\infty, \frac{1}{2}\right] \cup \left[1, \frac{3}{2}\right] \cup [2, +\infty)$, *then*

$$\frac{t(b-a)}{2 \cdot 6^{2t+1}(b^t - a^t)}\left(\left(\left(3+\sqrt{3}\right)a + \left(3-\sqrt{3}\right)b\right)^{2t+1} + \left(\left(3-\sqrt{3}\right)a + \left(3+\sqrt{3}\right)b\right)^{2t+1}\right)$$

does not exceeds $A_t^t(a,b)$.

If $t \in \left(\frac{1}{2}, 1\right) \cup \left(\frac{3}{2}, 2\right)$, *then the reverse inequality holds true.*

Proof. In Lemma 1.1 iii) we take $f(x) = x^{2t-1}$ for which $f^{(4)}(x) > 0$, if $t \in \left(-\infty, \frac{1}{2}\right] \cup \left[1, \frac{3}{2}\right] \cup [2, +\infty)$, therefore

$$A_t^t(a,b) = \frac{t}{b^t - a^t}\int_a^b x^{2t-1}dx$$
$$\geq \frac{t(b-a)}{2(b^t - a^t)}\left(\left(\frac{\left(3+\sqrt{3}\right)a + \left(3-\sqrt{3}\right)b}{6}\right)^{2t+1} + \left(\frac{\left(3-\sqrt{3}\right)a + \left(3+\sqrt{3}\right)b}{6}\right)^{2t+1}\right).$$

□

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