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SINGULARITY OF A BOUNDARY VALUE PROBLEM OF THE ELASTICITY EQUATIONS IN A POLYHEDRON

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Abstract. In this work we study the regularity of a boundary value problem governed by the Lamé equations in a cylindrical domain. By studying the longitudinal displacement singularity along an edge and the perpendicular displacement singularity to the same edge, we arrive to describe the behavior singular of solutions of the *Lamé* equations in a polyhedron.

1. **Introduction**

Let Ω be homogeneous, elastic and isotropic medium occupying a bounded domain in $\mathbb{R}^2$, limited by straight polygonal boundary Γ which is supposed to be regular, $\Gamma = \bigcup\limits_{j=1}^{N} \Gamma_j$, $\Gamma_i \cap \Gamma_j = \varnothing, \forall i \neq j$, where $\Gamma_j =]S_j, S_{j+1}[$, and S_j are the different corners of Ω. $\omega_j, 0 < \omega_j \leq 2\pi,\ j = 0, ..., N$ represent the opening of the angle that makes Γ_j and Γ_{j+1} toward the interior of Ω, η^j and τ^j represent the unit outward normal vector and the tangent vector on Γ_j, respectively.

L is the *Lamé* operator defined by:

$$Lu = \mu\Delta u + (\lambda + \mu)\, \nabla.divu,$$

where u, f represent the displacement vector, and external forces density respectively. $\Sigma(u)$ is the stress tensor given by *Hook*'s law using *Lamé* coefficients λ and μ ($\lambda > 0$ and $\lambda + \mu \geq 0$)

$$\Sigma(u) = (\sigma_{ij}(u))_{ij},\ \text{where } \sigma_{ij}(u) = 2\mu\varepsilon_{ij}(u) + \lambda tr(\varepsilon(u))\delta_{ij},$$

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where δ_{ij} is the *Kronecker* symbol and $\varepsilon_{ij}(u) = \frac{1}{2}(\partial_i x_j + \partial_j x_i)$ is the linearized tensor of deformation. We will suppose $\nu_0 = \frac{1}{2-\nu}$, where ν designates the *Poisson* coefficient such as $0 < \nu < \frac{1}{2}$.

In the case of a polyhedron, we consider a domain Q of $\mathbb{R}^3$, limited by straight polyhedral boundary Σ. It is considered a particular edge, denoted A, of Σ. It is assumed to fix ideas that A is carried by the axis $z^{'}Oz$, the adjacent faces Γ_0 and Γ_ω are carried by the plans $\{y = 0\}$ and $\{y = ax\}$, respectively. The dihedral so definite has for measure ω toward the interior of Q.

It is indispensable to signal that the results that will be demonstrated in this work are not verified to the corners neighborhood. That's why, we fix an opened interval I, whose closure is interior to A. Besides we fix a neighborhood U of the origin O in $Q \cap \{z = 0\}$, such as $\overline{U} \times \overline{I}$ doesn't have any corners of Q. $\eta^{'} = (\eta_1, \eta_2, \eta_3)^t = (\eta, \eta_3)^t$ and $\tau^{'} = (\tau_1, \tau_2, \tau_3)^t = (\tau, \tau_3)^t$ represent the unit outward normal vector and the tangent vector on Σ respectively.

We consider the corresponding cylinder $Q = \Omega \times \mathbb{R}$ which has an edge along $z^{'}Oz$.

For $f \in L^2(Q)^3$, the problem considered here consists of finding the displacement field $u : \Omega \longrightarrow \mathbb{R}^3$, if possible in $H^2(Q)^3$, satisfying:

$$(P)\begin{cases} Lu + f = 0 \text{ in } Q \\ \left(u.\eta^{'}, \left(\Sigma(u).\eta^{'}\right).\tau^{'}\right) = 0, \text{ on } \Sigma \end{cases},$$

Or equivalent variational form:

$$(P_V)\begin{cases} \text{Find } u \in V \text{ such as} \\ a(u,v) = \ell(v), \text{ for all } v \in V \end{cases}$$

where

$$\begin{aligned} a(u,v) &= \sum_{i,j=1}^{3} \int_Q \sigma_{ij}(u)\varepsilon_{ij}(v)dx, \ \ell(v) = \sum_{i=1}^{3} \int_Q f_i v_i dx, \\ V &= \left\{ v \in H^1(Q)^3;\ u.\eta = 0, \text{ in } \Sigma \right\} \end{aligned}$$

It is assumed that u, therefore as f, is to bounded support in the direction of z.

To describe the behavior of u along an edge, it is necessary to introduce, as in P. Grisvard [5], the following three convolution kernels, in z :

$$\begin{array}{rcl} K_{\lambda,\mu,r}(r,z) & = & \dfrac{r\sqrt{1+\nu}}{\pi\left[r^2+(1+\nu)z^2\right]}, \\ K_{\lambda,\mu,\theta}(r,z) & = & \dfrac{r}{\pi\left[r^2+z^2\right]}, \\ K_{\lambda,\mu,z}(r,z) & = & \dfrac{r\sqrt{1+\nu}}{\pi\left[(1+\nu)r^2+z^2\right]} \end{array}$$

1.1. **Singular solutions.** In B. Benabderrahmane [1] and P. Grisvard [8], there was found that the solutions of the problem (P), (in the case $f=0$) are characterized by the following transcendent equation (1.1) :

$$\sin^2\alpha\omega = \sin^2\omega,\ \alpha \neq 0, \neq \pm 1 \tag{1.1}$$

where $\operatorname{Re}\alpha \in \left]0,1\right[$.

It is easy to verify that the solutions of the transcendent equation (1.1) are given by

$$\alpha_\ell = \frac{\ell\pi}{\omega} \pm 1,\ \ell \in \mathbb{N}^*.$$

Besides they are simple if $\omega \neq \frac{k\pi}{2}$, $k \in \mathbb{Z}^*$, else they are double. By the simple calculations we find that:

$*$ If $\omega < \frac{\pi}{2}$, then $u \in H^2(\Omega)^2$;

$*$ If $\omega = \frac{\pi}{2}, \pi$, it was a simple poles $\alpha = 0, \pm 1$;

$*$ If $\omega = \frac{3\pi}{2}$, then $\alpha = \frac{1}{3}$ is a double root.

In the other cases, there is only one simple real root when $\omega \in \left]\pi, \frac{3\pi}{2}\right[\bigcup \left]\frac{3\pi}{2}, 2\pi\right[$; and no solution when $\omega \in \left]\frac{\pi}{2}, \pi\right[$.

It is known in B. Benabderrahmane [2] that there are linearly independent functions S_α and $S_\alpha^{'} \in V$, such as S_α, $S_\alpha^{'} \notin H^2(\Omega)^2$ and LS_α, $LS_\alpha^{'} \in L^2(\Omega)^2$ and as the *Lamé* operator is an isomorphism of

$$Sp\left(H^2(\Omega)^2, S_\alpha, S'_\alpha\right) \cap V \text{ on } L^2(\Omega)^2,$$

where the Sp symbol designates the vector space generated by elements that are contained in parentheses that follow. These functions are given explicitly, in B. Benabderrahmane [2], by $S_\alpha(r,\theta) = r^\alpha \Psi_\alpha(\theta)$ such as

$$\Psi_\alpha(\theta) = \begin{cases} [(\rho_0 - \rho_1)\sin(\alpha+1)\omega - 2\rho_1 \sin(\alpha-1)\omega]\cos\alpha\theta + \\ \qquad (\rho_0+\rho_1)\sin(\alpha+1)\omega\cos(\alpha-2)\theta, \\ [(-\rho_1+\rho_0)\sin(\alpha+1)\omega - 2\rho_1\sin(\alpha-1)\omega]\sin\alpha\theta \\ \qquad -(\rho_0+\rho_1)\sin(\alpha+1)\omega\sin(\alpha-2)\theta \end{cases} \tag{1.2}$$

where $\rho_0 = \nu_0(\alpha-1) - 2,\ \rho_0 = \nu_0(\alpha+1) + 2.$

2. Singularity in a polyhedron

The behavior of the singular solutions of *Lamé* equations in a polyhedron is described by the following theorem:

Theorem 2.1. *Let $\omega < 2\pi,\ u \in V$. For $f \in L^2(Q)^3$, there are functions $C_\alpha, C'_\alpha, C_{\alpha'}$ and $C'_{\alpha'}$ such as $C_\alpha, C'_\alpha \in H^{1-\alpha}(\mathbb{R}),\ C_{\alpha'}, C'_{\alpha'} \in H^{1-\alpha'}(\mathbb{R})$ verifying*

$$\begin{cases} u_r - \sum\limits_{\alpha,\ 0<\operatorname{Re}\alpha<1} (K_{\lambda,\mu,r}(r,z) * C_\alpha)\, r^\alpha \Psi_{\alpha,r}(\theta) - \\ \quad - \sum\limits_{\alpha,\ 0<\operatorname{Re}\alpha<1} \left(K_{\lambda,\mu,r}(r,z) * C'_\alpha\right) r^\alpha \Phi_{\alpha,r}(\theta) \end{cases} \in H^2(U\times\mathbb{R}) \tag{1.3}$$

$$\begin{cases} u_\theta - \sum\limits_{\alpha,\ 0<\operatorname{Re}\alpha<1} (K_{\lambda,\mu,\theta}(r,z) * C_\alpha)\, r^\alpha \Psi_{\alpha,\theta}(\theta) - \\ \quad \sum\limits_{\alpha,\ 0<\operatorname{Re}\alpha<1} \left(K_{\lambda,\mu,\theta}(r,z) * C'_\alpha\right) r^\alpha \Phi_{\alpha,\theta}(\theta) \end{cases} \in H^2(U\times\mathbb{R}) \tag{1.4}$$

$$\begin{cases} u_3 - \sum\limits_{\alpha',\ 0<\operatorname{Re}\alpha'<1} (K_{\lambda,\mu,z}(r,z) * C_{\alpha'})\, r^\alpha \Psi_{\alpha'}(\theta) - \\ \quad \sum\limits_{\alpha',\ 0<\operatorname{Re}\alpha'<1} \left(K_{\lambda,\mu,z}(r,z) * C'_{\alpha'}\right) r^\alpha \Phi_{\alpha'}(\theta) \end{cases} \in H^2(U\times\mathbb{R}) \tag{1.5}$$

where the functions

$$\Psi_\alpha(\theta) = (\Psi_{\alpha,r}(\theta), \Psi_{\alpha,\theta}(\theta))$$

are given by (1.2) *and*

$$\Phi_\alpha = \frac{\partial \Psi_\alpha(\theta)}{\partial \alpha} = \left[\log r \Psi_\alpha(\theta) + \frac{\partial}{\partial r}\Psi_\alpha(\theta)\right].$$

The functions $\Psi_{\alpha,r}(\theta), \Psi_{\alpha,\theta}(\theta)$ represents the radial part, angular part of $\Psi_\alpha(\theta)$, respectively. The functions $\Psi_{\alpha'}(\theta)$ are the first singular functions of the *Laplace* operator in a polygon.

The first sums in (1.3) and (1.4) are extended to all $\alpha; \operatorname{Re}\alpha \in]0,1[$ simple roots of the equation (1.1), while the second sums are extended to all the double roots of the same equation. In (1.5), the first sums are extended to all $\alpha^{'}$ simple roots of the corresponding transcendent equation to the *Laplace* operator with the boundary conditions associated and the second sums are extended to all $\alpha^{'}$ double roots of the same equation.

The symbol $*$ represents the convolution in relation to z. The Indices r, θ and z in the relations (1.3), (1.4) and (1.5) are, respectively, the radial component, angular and longitudinal vector by using cylindrical coordinates.

For more details, we are given the similar of the Theorem 2.1, in the following cases:

- Case of simple roots such as $0 < \operatorname{Re}\alpha < 1$;
- Case of double roots such as $0 < \operatorname{Re}\alpha < 1$;
- Case of the fissure $(\omega = 2\pi)$.

Theorem 2.2. *We assume that* $\omega \in \left]\pi, \frac{3\pi}{2}\right[\bigcup \left]\frac{3\pi}{2}, 2\pi\right[$. *Let* $u \in V$ *be a variational solution, is to bounded support in the direction of* z. *For all* $f \in L^2(Q)^3$, *there are functions* C *and* C_α *such as*

$$C \in H^{1-\frac{\pi}{\omega}}(\mathbb{R}), C_\alpha \in H^{1-\alpha}(\mathbb{R}) \ \textit{and}$$

$$\begin{cases} u_r - \sum\limits_{\alpha,\ 0<\alpha<1} \left(K_{\lambda,\mu,r}(r,z) * C_\alpha\right) r^\alpha \Psi_{\alpha,r}(\theta) \in H^2(U \times \mathbb{R}) \\ u_\theta - \sum\limits_{\alpha,\ 0<\alpha<1} \left(K_{\lambda,\mu,\theta}(r,z) * C_\alpha\right) r^\alpha \Psi_{\alpha,\theta}(\theta) \in H^2(U \times \mathbb{R}) \\ \quad u_3 - \left(K_{\lambda,\mu,z}(r,z) * C\right) r^{\frac{\pi}{\omega}} \cos\left(\frac{\pi}{\omega}\theta\right) \in H^2(U \times \mathbb{R}) \end{cases}$$

where $\alpha = \frac{\ell\pi}{\omega} \pm 1$, $\ell \in \mathbb{N}^*$ *are the simple roots of the equation* (1.1).

For $\omega = \frac{3\pi}{2}$, $\alpha = \frac{1}{3}$ is a double root of the equation (1.1). Therefore, it is necessary to modify the result of the Theorem 2.2 as follows: there are two constants C and $C^{'}$ such as

$$C \in H^{\frac{2}{3}}(\mathbb{R}), C^{'} \in H^{\frac{1}{3}}(\mathbb{R}) \text{ and}$$

$$\begin{cases} u_r - (K_{\lambda,\mu,r}(r,z) * C)\, r^{\frac{1}{3}} \Phi_{\frac{1}{3},r}(\theta) \in H^2(U \times \mathbb{R}) \\ u_\theta - (K_{\lambda,\mu,\theta}(r,z) * C)\, r^{\frac{1}{3}} \Phi_{\frac{1}{3},\theta}(\theta) \in H^2(U \times \mathbb{R}) \\ u_3 - \left(K_{\lambda,\mu,z}(r,z) * C^{'}\right) r^{\frac{2}{3}} \cos\left(\frac{2\theta}{3}\right) \in H^2(U \times \mathbb{R}) \end{cases}$$

In the case $\omega = 2\pi$, we obtain the existence of the functions C and $C^{'}$ of $H^{\frac{1}{2}}(\mathbb{R})$ such as

$$\begin{cases} u_r - (K_{\lambda,\mu,r}(r,z) * C)\, \sqrt{r} \Phi_{\frac{1}{2},r}(\theta) \in H^2(U \times \mathbb{R}) \\ u_\theta - (K_{\lambda,\mu,\theta}(r,z) * C)\, \sqrt{r} \Phi_{\frac{1}{2},\theta}(\theta) \in H^2(U \times \mathbb{R}) \\ u_3 - \left(K_{\lambda,\mu,z}(r,z) * C^{'}\right) \sqrt{r} \cos\left(\frac{\theta}{2}\right) \in H^2(U \times \mathbb{R}). \end{cases}$$

The demonstration is essentially based on the study of the following points:

- Decompose every problem in plane part, u and u_θ, and in longitudinal part, u_3.
- Study of the longitudinal displacement singularity along an edge.
- Study of the perpendicular displacement singularity along an edge.

2.1. **Problem decomposition.** We start by studying the *Lamé* solutions in the tridimensional domain $Q = \Omega \times \mathbb{R}$, who present an edge along $z^{'}Oz$.

For $f \in L^2(Q)^3$, let $u \in V$ be a variational solution of (P), then we have

$$a(u,v) = \ell(v), \text{ where}$$

$$a(u,v) = \sum_{i,j=1}^{3} \int_Q \sigma_{ij}(u)\varepsilon_{ij}(v)dx_1dx_2dx_3,\ \ell(v) = \sum_{i=1}^{3} \int_Q f_i v_i dx_1dx_2dx_3.$$

The invariance of the problems in relation to z implies the following partial regularity result:

Lemma 2.1. *We have*

$$\frac{\partial^2 u}{\partial x \partial z},\ \frac{\partial^2 u}{\partial y \partial z} \text{ and } \frac{\partial^2 u}{\partial z^2} \in L^2(Q)^3.$$

Let's decompose the fields u and f to the plane components and longitudinal component by posing:

$$u = (v, u_3)^t \text{ and } f = (g, f_3)^t ,$$

where v and g are vector fields of dimension 2 (also depend of z).

We will use the following notations:

- $\triangle_2$: Laplace in dimension 2, (variables x_1, x_2).
- ∇_2 : *Gradient* in dimension 2, .(variables x_1, x_2).
- Div_2 : *Divergence* in dimension 2, (variables x_1, x_2).

Using these notations the Lamé equations in dimension 3 become

$$\begin{cases} \mu\left(\triangle_2 v + \dfrac{\partial^2 v}{\partial z^2}\right) + (\lambda+\mu)\nabla_2\left(Div_2 v + \dfrac{\partial u_3}{\partial z}\right) = g \\ \mu\left(\triangle_2 u_3 + \dfrac{\partial^2 u_3}{\partial z^2}\right) + (\lambda+\mu)\dfrac{\partial}{\partial z}\left(Div_2 v + \dfrac{\partial u_3}{\partial z}\right) = f_3. \end{cases}$$

Thanks to Lemma 2.1, it can see that

$$\begin{cases} \mu\left(\triangle_2 v + \dfrac{\partial^2 v}{\partial z^2}\right) + (\lambda+\mu)\nabla_2 Div_2 v = g - (\lambda+\mu)\nabla_2\left(\dfrac{\partial u_3}{\partial z}\right) \in L^2(Q)^3 \\ \mu\left(\triangle_2 u_3 + \dfrac{\partial^2 u_3}{\partial z^2}\right) + (\lambda+\mu)\dfrac{\partial^2 u_3}{\partial z^2} = f_3 - (\lambda+\mu)\dfrac{\partial}{\partial z}(Div_2 v) \in L^2(Q)^3 . \end{cases} \tag{1.6}$$

This formulation has the advantage to decouple v and u_3. The left member in the first equations in (1.6) concerns the plane components of u, while the right member concerns the longitudinal component.

2.2. **Study of the boundary conditions.** It is assumed that

$$\eta' = (\eta_1, \eta_2, \eta_3)^t = (\eta, \eta_3)^t \text{ and } \tau' = (\tau_1, \tau_2, \tau_3)^t = (\tau, \tau_3)^t .$$

The condition $u.\eta' = 0$ becomes $u_3\eta_3 = -v.\eta$. As $\eta_3 = 0$ and $\tau_3 = 1$ then

$$u.\eta' = 0 \Leftrightarrow v.\eta = 0 \text{ (no condition on } u_3)$$

Concerning the condition on $\left(\sum(u).\eta'\right)$, we set $u = (v, 0) + (0, 0, u_3)$. Using the relations $\sigma_{ij}(u) = 2\mu\varepsilon_{ij}(u) + \lambda tr(\varepsilon(u))\delta_{ij}$, $i, j = 1, 2, 3$, it results

$$\sigma_{11}(v, 0) = (\lambda+\mu)\frac{\partial u_1}{\partial x} + \lambda\frac{\partial u_2}{\partial y},\ \sigma_{12}(v, 0) = \sigma_{21}(v, 0) = \mu\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right),$$

$$\sigma_{13}(v,0)=\sigma_{31}(v,0)=\mu\frac{\partial u_1}{\partial z},\ \sigma_{23}(v,0)=\sigma_{32}(v,0)=\mu\frac{\partial u_2}{\partial z},$$

$$\sigma_{22}(v,0)=(\lambda+\mu)\frac{\partial u_2}{\partial y}+\lambda\frac{\partial u_1}{\partial x},\ \sigma_{33}(v,0)=\mu\left(\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y}\right),$$

$$\sigma_{11}(0,0,u_3)=\sigma_{22}(0,0,u_3)=\lambda\frac{\partial u_3}{\partial z},\sigma_{12}(0,0,u_3)=\sigma_{21}(0,0,u_3)=0,$$

$$\sigma_{13}(0,0,u_3)=\sigma_{31}(0,0,u_3)=\lambda\frac{\partial u_3}{\partial x},\ \sigma_{23}(0,0,u_3)=\sigma_{32}(0,0,u_3)=\lambda\frac{\partial u_3}{\partial y},$$

$$\sigma_{33}(0,0,u_3)=(\lambda+\mu)\frac{\partial u_3}{\partial z}.$$

Using the fact that $\eta_3=0$ and $\tau_3=1$, these last relations involve

$$\begin{cases}(\sigma(v,0).\eta).\tau=(\sigma(v).\eta).\tau+\mu\frac{\partial}{\partial z}(u_3\eta_3)=(\sigma(v).\eta).\tau\\ (\sigma(0,0,u_3).\eta).\tau=\lambda\dfrac{\partial u_3}{\partial\eta}.\end{cases}$$

Therefore, we have the conditions that must be verified by each components of $u=(v,u_3)$ for the considered boundary conditions :

$$u.\eta^{'}=0\Leftrightarrow v.\eta=0 \text{ and no condition on } u_3$$

$$\left(\sum(u).\eta^{'}\right).\tau^{'}=0\Leftrightarrow\left(\sum(v).\eta\right).\tau=-\lambda\ \frac{\partial u_3}{\partial\eta}=0.$$

2.3. **Study of the longitudinal displacement along an edge.** In (1.6) the second equation is none other than the *Laplace* equation in Q, using a change of scale in z. By posing

$$z=\sqrt{\frac{\mu}{\lambda+2\mu}}z^{'},$$

we obtain

$$\mu\bigtriangleup_2 u_3+(\lambda+2\mu)\frac{\partial}{\partial z}\left(Div_2 v+\frac{\partial u_3}{\partial z}\right)=\mu\bigtriangleup_2 u_3+\mu\frac{\partial^2 u_3}{\partial(z')^2}=\mu\bigtriangleup u_3.$$

This result attached to the results of the preceding paragraph permits us, for the longitudinal displacement part, to deduce the following problem:

$$(P_1)\begin{cases}\Delta u_3=f_3 \text{ in } Q\\ \dfrac{\partial u_3}{\partial\eta}=h \text{ on } \Sigma,\end{cases}$$

where $h\in H^{-\frac{1}{2}}(U\times\mathbb{R})$, thanks to Lemma 2.1.

The study of this problem is already made by P. Grisvard [10]. The application of results of P. Grisvard [9], concerning the *Laplace* equations, gives after change of scale in z the following decomposition of u_3 :

$$u_3 - \sum_{\alpha,\ 0<\operatorname{Re}\alpha<1} \left(K_{\lambda,\mu,z}\left(r,z\right) * C\right) r^{\alpha}\Psi_{\alpha}\left(\theta\right) \in H^2\left(Q\right),$$

where $C \in H^{1-\alpha}(\mathbb{R})$ and the functions $\Psi_{\alpha}(\theta)$ are the first singular functions of the problem (P_1), which are given, see P.Grisvard [8], by $\Psi_{\alpha}(\theta) = \cos\alpha\theta$ where $K_{\lambda,\mu,z}(r,z)$ represents the kernel of the *Laplace* operator. This establishes the part of the Theorem 2.2 that concerns the longitudinal part u_3.

2.4. **Study of the perpendicular displacement singularity along an edge.** We analyze the behavior of v from the first equation of (1.6) :

$$\mu\left(\triangle_2 v + \frac{\partial^2 v}{\partial z^2}\right) + (\lambda+\mu)\,\triangledown_2\, Div_2 v = g - (\lambda+\mu)\,\triangledown_2\left(\frac{\partial u_3}{\partial z}\right) \in L^2\left(Q\right)^3.$$

To simplify we note h the second member of this equation. Using the partial *Fourier* transformation in z, we see that the previous equation amounts to the following problem which is governed by the *Lamé* system resolving:

$$L\widehat{v} - \mu\zeta^2\widehat{v} = \widehat{h}.$$

Concerning the boundary conditions, as we can see that the conditions remain unaltered, we will be able to have the same conditions but non homogeneous. However by subtracting v to a field $u \in H^2\left(Q\right)^2$ verifying the same conditions to limits that v, consequently the field $w = v - u$ verifies the homogeneous conditions. To simplify the notations, we will note this field again by v.

The uniqueness of the variational solution implies that $\widehat{v} \in D_L$ where

$$D_L = \left\{u \in sp\left(H^2\left(\Omega\right)^2, S_\alpha, S_\alpha^{'}\right);\ \left(u.\eta^{'}, \left(\Sigma(u).\eta^{'}\right).\tau^{'}\right) = 0,\ \text{on } \Sigma\right\}.$$

Therefore

$$\widehat{v} = \widehat{v}_R + \sum_{\alpha,\ 0<\operatorname{Re}\alpha<1} \widehat{C}_\alpha \Im_\alpha$$

where $\widehat{v}_R \in H^2(Q)^2$ and $\widehat{C}_\alpha \in \mathbb{R}$, for all $\zeta \in \mathbb{R}$. Moreover, according B. Benabderrahmane [2], we have the following inequalities:

$$\begin{cases} \zeta^2 \left\|\widehat{v}_R\right\|_{L^2(Q)^2} + \zeta \left\|\widehat{v}_R\right\|_{H^1(Q)^2} + \left\|\widehat{v}_R\right\|_{H^2(Q)^2} \leq C \left\|\widehat{h}\right\|_{L^2(Q)^2} \\ \sum\limits_{\alpha,\ 0<\operatorname{Re}\alpha<1} \left|\widehat{C}_\alpha\right| |\zeta|^{1-\alpha} \leq C \left\|\widehat{h}\right\|_{L^2(Q)^2}. \end{cases}$$

From where it comes that $\widehat{v} \in H^2(Q)^2$ and $\widehat{C}_\alpha \in H^{1-\alpha}(\mathbb{R})$. Besides the following decomposition:

$$\widehat{v} = \widehat{v}_R + \sum_{\alpha,\ 0<\operatorname{Re}\alpha<1} \widehat{C}_\alpha \Im_\alpha,$$

which is equivalent by proceeding the inverse *Fourier* transformation, taking into account the fact that $\widehat{f * g} = \widehat{f}.\widehat{g}$, to

$$\begin{cases} v_r = (v_R)_r + \sum\limits_{\alpha,\ 0<\operatorname{Re}\alpha<1} \left(K_{\lambda,\mu,r}(r,z) * C_\alpha\right)(S_\alpha)_r \\ v_\theta = (v_R)_\theta + \sum\limits_{\alpha,\ 0<\operatorname{Re}\alpha<1} \left(K_{\lambda,\mu,\theta}(r,z) * C_\alpha\right)(S_\alpha)_\theta \end{cases}$$

because

$$K_{\lambda,\mu,r}(r,\zeta) = e^{\frac{-r|\zeta|}{\sqrt{1+\nu}}} \text{ and } K_{\lambda,\mu,\theta}(r,\zeta) = e^{-r|\zeta|}$$

and by definition

$$(\Im_\alpha)_r = e^{\frac{-r|\zeta|}{\sqrt{1+\nu}}} (S_\alpha)_r \text{ and } (\Im_\alpha)_\theta = e^{-r|\zeta|} (S_\alpha)_\theta.$$

This establishes the first inequality of the Theorem 2.2.

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