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SOME APPLICATIONS OF SALAGEAN INTEGRAL OPERATOR

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Abstract. In this paper we introduce and study some new subclasses of starlike, convex, close-to-convex and quasi-convex functions defined by Salagean integral operator. Inclusion relations are established and integral operator $L_c(f)(c \in N = \{1, 2, ...\})$ is also discussed for these subclasses.

1. **Introduction**

Let A denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$. Also let S denote the subclass of A consisting of univalent functions in U. A function $f(z) \in S$ is called starlike of order $\gamma, 0 \leq \gamma < 1$, if and only if

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \gamma \ \ (z \in U)\ . \tag{1.2}$$

We denote by $S^*(\gamma)$ the class of all functions in S which are starlike of order γ in U.

A function $f(z) \in S$ is called convex of order $\gamma, 0 \leq \gamma < 1$, in U if and only if

$$\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \gamma \ \ (z \in U)\ . \tag{1.3}$$

We denote by $C(\gamma)$ the class of all functions in S which are convex of order γ in U.

It follows from (1.2) and (1.3) that:

$$f(z) \in C(\gamma) \qquad \text{if and only if} \qquad zf'(z) \in S^*(\gamma)\ . \tag{1.4}$$

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The classes $S^*(\gamma)$ and $C(\gamma)$ was introduced by Robertson [12].

Let $f(z) \in A$, and $g(z) \in S^*(\gamma)$. Then $f(z) \in K(\beta,\gamma)$ if and only if

$$\operatorname{Re}\left\{\frac{zf^{'}(z)}{g(z)}\right\} > \beta \;\; (z \in U)\,, \tag{1.5}$$

where $0 \leq \beta < 1$ and $0 \leq \gamma < 1$. Such functions are called close-to-convex functions of order β and type γ. The class $K(\beta,\gamma)$ was introduced by Libera [4].

A function $f(z) \in A$ is called quasi-convex of order β and type γ if there exists a function $g(z) \in C(\gamma)$ such that

$$\operatorname{Re}\left\{\frac{(zf^{'}(z))^{'}}{g^{'}(z)}\right\} > \beta \;\; (z \in U)\,, \tag{1.6}$$

where $0 \leq \beta < 1$ and $0 \leq \gamma < 1$. We denote this class by $K^*(\beta,\gamma)$. The class $K^*(\beta,\gamma)$ was introduced by Noor [10].

It follows from (1.5) and (1.6) that:

$$f(z) \in K^*(\beta,\gamma) \qquad \text{if and only if} \qquad zf^{'}(z) \in K(\beta,\gamma)\ . \tag{1.7}$$

For a function $f(z) \in A$, we define the integral operator $I^n f(z), n \in N_0 = N \cup \{0\}$, where $N = \{1, 2, ...\}$, by

$$I^0 f(z) = f(z)\ , \tag{1.8}$$

$$I^1 f(z) = If(z) = \int_0^z f(t)t^{-1}dt\ \ , \tag{1.9}$$

and

$$I^n f(z) = I(I^{n-1} f(z))\ \ . \tag{1.10}$$

It is easy to see that:

$$I^n f(z) = z + \sum_{k=2}^{\infty} \frac{a_k}{k^n}\, z^k \;\; (n \in N_0)\ , \tag{1.11}$$

and

$$z(I^n f(z))^{'} = I^{n-1} f(z)\ . \tag{1.12}$$

The integral operator $I^n f(z)$ $(f \in A)$ was introduced by Salagean [13] and studied by Aouf et al. [1]. We call the operator I^n by Salagean integral operator.

Using the operator I^n, we now introduce the following classes:

$$S_n^*(\gamma) = \{f \in A : I^n f \in S^*(\gamma)\} \ ,$$

$$C_n(\gamma) = \{f \in A : \ I^n f \in C(\gamma)\} \ ,$$

$$K_n(\beta,\gamma) = \{f \in A : I^n f \in K(\beta,\gamma)\} \ ,$$

and

$$K_n^*(\beta,\gamma) = \{f \in A : I^n f \in K^*(\beta,\gamma)\} \ .$$

In this paper, we shall establish inclusion relation for these classes and integral operator $L_c(f)(c \in N)$ is also discussed for these classes. In [11], Noor introduced and studied some classes defined by Ruscheweyh derivatives and in [6] Liu studied some classes defined by the one-parameter family of integral operator $I^\sigma f(z)(\sigma > 0, f \in A)$.

2. **Inclusion relations**

We shall need the following lemma.

Lemma 2.1. [8], [9] *Let $\varphi(u,v)$ be a complex function, $\phi : D \to C, D \subset C \times C$, and let $u = u_1 + iu_2, v = v_1 + iv_2$. Suppose that $\varphi(u,v)$ satisfies the following conditions:*

(i) $\varphi(u,v)$ is continuous in D;

(ii) $(1,0) \in D$ and $\mathrm{Re}\,\{\varphi(1,0)\} > 0$;

(iii) $\mathrm{Re}\,\{\varphi(iu_2,v_1)\} \le 0$ for all $(iu_2,v_1) \in D$ and such that $v_1 \le -\frac{1}{2}(1+u_2^2)$.

Let $h(z) = 1 + c_1 z + c_2 z^2 + ...$ be analytic in U, such that $(h(z), zh^{'}(z)) \in D$ for all $z \in U$. If $\mathrm{Re}\,\{\varphi(h(z), zh^{'}(z))\} > 0\,(z \in U)$, then $\mathrm{Re}\,\{h(z)\} > 0$ for $z \in U$.

Theorem 2.1. $S_n^*(\gamma) \subset S_{n+1}^*(\gamma)(0 \le \gamma < 1, n \in N_0)$.

Proof. Let $f(z) \in S_n^*(\gamma)$ and set

$$\frac{z(I^{n+1}f(z))^{'}}{I^{n+1}f(z)} = \gamma + (1-\gamma)h(z)\,, \tag{2.1}$$

where $h(z) = 1 + h_1 z + h_2 z^2 + ...$ Using the identity (1.12), we have

$$\frac{I^n f(z)}{I^{n+1}f(z)} = \gamma + (1-\gamma)h(z)\,. \tag{2.2}$$

Differentiating (2.2) with respect to z logarithmically, we obtain

$$\begin{aligned} \frac{z(I^n f(z))'}{I^n f(z)} &= \frac{z(I^{n+1} f(z))'}{I^{n+1} f(z)} + \frac{(1-\gamma)zh'(z)}{\gamma+(1-\gamma)h(z)} \\ &= \gamma+(1-\gamma)h(z) + \frac{(1-\gamma)zh'(z)}{\gamma+(1-\gamma)h(z)} , \end{aligned}$$

or

$$\frac{z(I^n f(z))'}{I^n f(z)} - \gamma = (1-\gamma)h(z) + \frac{(1-\gamma)zh'(z)}{\gamma+(1-\gamma)h(z)} . \tag{2.3}$$

Taking $h(z) = u = u_1 + iu_2$ and $zh'(z) = v = v_1 + iv_2$, we define the function $\varphi(u, v)$ by:

$$\varphi(u, v) = (1-\gamma)u + \frac{(1-\gamma)v}{\gamma+(1-\gamma)u} . \tag{2.4}$$

Then it follows from (2.4) that

(i) $\varphi(u, v)$ is continuous in $D = (C - \left\{\frac{\gamma}{\gamma-1}\right\}) \times C$;

(ii) $(1, 0) \in D$ and $\operatorname{Re}\{\varphi(1, 0)\} = 1-\gamma > 0$;

(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \le -\frac{1}{2}(1+u_2^2)$,

$$\begin{aligned} \operatorname{Re}\{\varphi(iu_2, v_1)\} &= \operatorname{Re}\left\{\frac{(1-\gamma)v_1}{\gamma+(1-\gamma)iu_2}\right\} \\ &= \frac{\gamma(1-\gamma)v_1}{\gamma^2+(1-\gamma)^2u_2^2} \\ &\le -\frac{\gamma(1-\gamma)(1+u_2^2)}{2[\gamma^2+(1-\gamma)^2u_2^2]} < 0 , \end{aligned}$$

for $0 \le \gamma < 1$. Therefore, the function $\varphi(u, v)$ satisfies the conditions in Lemma. It follows from the fact that if $\operatorname{Re}\{\varphi(h(z), zh'(z))\} > 0, z \in U$, then $\operatorname{Re}\{h(z)\} > 0$ for $z \in U$, that is, if $f(z) \in S_n^*(\gamma)$ then $f(z) \in S_{n+1}^*(\gamma)$. This completes the proof of Theorem 2.1. □

We next prove:

Theorem 2.2. $C_n(\gamma) \subset C_{n+1}(\gamma) (0 \le \gamma < 1, n \in N_0)$.

Proof. $f \in C_n(\gamma) \Leftrightarrow I^n f \in C(\gamma) \Leftrightarrow z(I^n f)' \in S^*(\gamma) \Leftrightarrow I^n(zf') \in S^*(\gamma) \Leftrightarrow zf' \in S_n^*(\gamma) \Rightarrow zf' \in S_{n+1}^*(\gamma) \Leftrightarrow I^{n+1}(zf') \in S^*(\gamma) \Leftrightarrow z(I^{n+1}f)' \in S^*(\gamma) \Leftrightarrow I^{n+1}f \in C(\gamma) \Leftrightarrow f \in C_{n+1}(\gamma)$.

This completes the proof of Theorem 2.2. □

Theorem 2.3. $K_n(\beta,\gamma) \subset K_{n+1}(\beta,\gamma)(0 \le \gamma < 1, 0 \le \beta < 1, n \in N_0)$.

Proof. Let $f(z) \in K_n(\beta,\gamma)$. Then there exists a function $k(z) \in S^*(\gamma)$ such that

$$\operatorname{Re}\left\{\frac{z(I^n f(z))'}{k(z)}\right\} > \beta \ \ (z \in U) \ .$$

Taking the function $g(z)$ which satisfies $I^n g(z) = k(z)$, we have $g(z) \in S_n^*(\gamma)$ and

$$\operatorname{Re}\left\{\frac{z(I^n f(z))'}{I^n g(z)}\right\} > \beta \ \ (z \in U) \ . \tag{2.5}$$

Now put

$$\frac{z(I^{n+1} f(z))'}{I^{n+1} g(z)} - \beta = (1-\beta)h(z) \ , \tag{2.6}$$

where $h(z) = 1 + c_1 z + c_2 z^2 + ...$ Using (1.12) we have

$$\frac{z(I^n f(z))'}{I^n g(z)} = \frac{I^n(zf'(z))}{I^n g(z)} = \frac{z(I^{n+1}(zf'(z)))'}{z(I^{n+1}g(z))'}$$

$$= \frac{\dfrac{z(I^{n+1}(zf'(z))'}{I^{n+1}g(z)}}{\dfrac{z(I^{n+1}g(z))'}{I^{n+1}g(z)}} \ . \tag{2.7}$$

Since $g(z) \in S_n^*(\gamma)$ and $S_n^*(\gamma) \subset S_{n+1}^*(\gamma)$, we let $\dfrac{z(I^{n+1}g(z))'}{I^{n+1}g(z)} = \gamma + (1-\gamma)H(z)$, where $\operatorname{Re} H(z) > 0 (z \in U)$. Thus (2.7) can be written as

$$\frac{z(I^n f(z))'}{I^n g(z)} = \frac{\dfrac{z(I^{n+1}(zf'(z))'}{I^{n+1}g(z)}}{\gamma + (1-\gamma)H(z)} \ . \tag{2.8}$$

Consider

$$z(I^{n+1} f(z))' = I^{n+1} g(z)[\beta + (1-\beta)h(z)] \ . \tag{2.9}$$

Differentiating both sides of (2.9), we have

$$\frac{z(I^{n+1}(zf'(z)))'}{I^{n+1}g(z)} = (1-\beta)zh'(z) + [\beta + (1-\beta)h(z)] \ . \ [\gamma + (1-\gamma)H(z)] \ . \tag{2.10}$$

Using (2.10) and (2.8), we have

$$\frac{z(I^n f(z))'}{I^n g(z)} - \beta = (1-\beta)h(z) + \frac{(1-\beta)zh'(z)}{\gamma + (1-\gamma)H(z)} \ . \tag{2.11}$$

Taking $u = h(z) = u_1 + iu_2, v = zh^{'}(z) = v_1 + iv_2$ in (2.11), we form the function $\Psi(u, v)$ as follows:

$$\Psi(u,v) = (1-\beta)u + \frac{(1-\beta)v}{\gamma + (1-\gamma)H(z)} \ . \tag{2.12}$$

It is clear that the function $\Psi(u, v)$ defined in $D = C \times C$ by (2.12) satisfies conditions (i) and (ii) of Lemma easily. To verify condition (iii), we proceed as follows:

$$\operatorname{Re}\Psi(iu_2, v_1) = \frac{(1-\beta)v_1[\gamma + (1-\gamma)h_1(x,y)]}{[\gamma + (1-\gamma)h_1(x,y)]^2 + [(1-\gamma)h_2(x,y)]^2} \ ,$$

where $H(z) = h_1(x, y) + ih_2(x, y), h_1(x, y)$ and $h_2(x, y)$ being the functions of x and y and $\operatorname{Re} H(z) = h_1(x, y) > 0$. By putting $v_1 \leq -\frac{1}{2}(1 + u_2^2)$, we obtain

$$\operatorname{Re}\Psi(iu_2, v_1) \leq -\frac{(1-\beta)(1+u_2^2)[\gamma + (1-\gamma)h_1(x,y)]}{2\left\{[\gamma + (1-\gamma)h_1(x,y)]^2 + [(1-\gamma)h_2(x,y)]^2\right\}} < 0 \ .$$

Hence $\operatorname{Re} h(z) > 0 (z \in U)$ and $f(z) \in K_{n+1}(\beta, \gamma)$. The proof of Theorem 2.3 is complete. □

Using the same method as in Theorem 2.3 with the fact that $f(z) \in K_n^*(\beta, \gamma) \Leftrightarrow zf^{'}(z) \in K_n(\beta, \gamma)$, we can deduce from Theorem 2.3 the following:

Theorem 2.4. $K_n^*(\beta, \gamma) \subset K_{n+1}^*(\beta, \gamma) (0 \leq \beta, \gamma < 1, n \in N_0)$.

3. Integral operator

For $c > -1$ and $f(z) \in A$, we recall here the generalized Bernardi-Libera-Livingston integral operator as:

$$L_c(f) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt \ . \tag{3.1}$$

The operator $L_c(f)$ when $c \in N$ was studied by Bernardi [2]. For $c = 1, L_1(f)$ was investigated ealier by Libera [5] and Livingston [7].

The following theorems deal with the generalized Bernardi-Libera-Livingston integral operator $L_c(f)$ defined by (3.1).

Theorem 3.1. *Let $c > -\gamma$. If $f(z) \in S_n^*(\gamma)$, then $L_c(f) \in S_n^*(\gamma)$.*

Proof. From (3.1), we have

$$z(I^n L_c(f))' = (c+1)I^n f(z) - cI^n L_c(f) \ . \tag{3.2}$$

Set

$$\frac{z(I^n L_c(f))'}{I^n L_c(f)} = \frac{1+(1-2\gamma)w(z)}{1-w(z)} \ , \tag{3.3}$$

where $w(z)$ is analytic or meromorphic in $U, w(0)=0$. Using (3.2) and (3.3) we get

$$\frac{I^n f(z)}{I^n L_c(f)} = \frac{c+1+(1-c-2\gamma)w(z)}{(c+1)(1-w(z))} \ . \tag{3.4}$$

Differentiating (3.4) with respect to z logarithmically, we obtain

$$\frac{z(I^n f(z))'}{I^n f(z)} = \frac{1+(1-2\gamma)w(z)}{1-w(z)} + \frac{zw'(z)}{1-w(z)} + \frac{(1-c-2\gamma)zw'(z)}{1+c+(1-c-2\gamma)w(z)} \ . \tag{3.5}$$

Now we claim that $|w(z)| < 1 (z \in U)$. Otherwise, there exists a point $z_0 \in U$ such that $\max\limits_{|z|\le|z_0|} |w(z)| = |w(z_0)| = 1$. Then by Jack's lemma [3], we have $z_0 w'(z_0) = kw(z_0)(k \ge 1)$.

Putting $z = z_0$ and $w(z_0) = e^{i\theta}$ in (3.5), we have

$$\text{Re}\left\{\frac{1+(1-2\gamma)w(z_0)}{1-w(z_0)}\right\} = \text{Re}\left\{(1-\gamma)\frac{1+w(z_0)}{1-w(z_0)} + \gamma\right\} = \gamma \ ,$$

and

$$\text{Re}\left\{\frac{z_0(I^n f(z_0))'}{I^n f(z_0)} - \gamma\right\} = \text{Re}\left\{\frac{2(1-\gamma)ke^{i\theta}}{(1-e^{i\theta})[1+c+(1-c-2\gamma)e^{i\theta}]}\right\}$$

$$= 2k(1-\gamma)\text{Re}\left\{\frac{(e^{i\theta}-1)\left[1+c+(1-c-2\gamma)e^{-i\theta}\right]}{2(1-\cos\theta)\left[(1+c)^2+2(1+c)(1-c-2\gamma)\cos\theta+(1-c-2\gamma)^2\right]}\right\}$$

$$= \frac{-2k(1-\gamma)(c+\gamma)}{(1+c)^2+2(1+c)(1-c-2\gamma)\cos\theta+(1-c-2\gamma)^2} \le 0 \ ,$$

which contradicts the hypothesis that $f(z) \in S_n^*(\gamma)$. Hence $|w(z)| < 1$ for $z \in U$, and it follows from (3.3) that $L_c(f) \in S_n^*(\gamma)$. The proof of Theorem 3.1 is complete. □

Theorem 3.2. *Let $c > -\gamma$. If $f(z) \in C_n(\gamma)$, then $L_c(f) \in C_n(\gamma)$.*

Proof. $f \in C_n(\gamma) \Leftrightarrow zf' \in S_n^*(\gamma) \Rightarrow L_c(zf') \in S_n^*(\gamma) \Leftrightarrow z(L_c f)' \in S_n^*(\gamma) \Leftrightarrow L_c(f) \in C_n(\gamma)$. □

Theorem 3.3. *Let $c > -\gamma$. If $f(z) \in K_n(\beta,\gamma)$, then $L_c(f) \in K_n(\beta,\gamma)$.*

Proof. Let $f(z) \in K_n(\beta,\gamma)$. Then, by definition, there exists a function $g(z) \in S_n^*(\gamma)$ such that

$$\operatorname{Re}\left\{\frac{z(I^n f(z))'}{I^n g(z)}\right\} > \beta \ (z \in U) \ .$$

Put

$$\frac{z(I^n L_c(f))'}{I^n L_c(g)} - \beta = (1-\beta)h(z) \ , \tag{3.6}$$

where $h(z) = 1 + c_1 z + c_2 z^2 + \ldots$. From (3.2), we have

$$\begin{aligned}\frac{z(I^n f(z))'}{I^n g(z)} &= \frac{I^n(zf'(z))}{I^n g(z)} \\ &= \frac{z(I^n L_c(zf'))' + cI^n L_c(zf')}{z(I^n L_c(g))' + cI^n L_c(g)} \\ &= \frac{\dfrac{z(I^n L_c(zf'))'}{I^n L_c(g)} + \dfrac{cI^n L_c(zf')}{I^n L_c(g)}}{\dfrac{z(I^n L_c(g))'}{I^n L_c(g)} + c} \ .\end{aligned} \tag{3.7}$$

Since $g(z) \in S_n^*(\gamma)$, then from Theorem 3.1, we have $L_c(g) \in S_n^*(\gamma)$. Let

$$\frac{z(I^n L_c(g))'}{I^n L_c(g)} = \gamma + (1-\gamma)H(z) \ ,$$

where $\operatorname{Re} H(z) > 0 (z \in U)$. Using (3.7), we have

$$\frac{z(I^n f(z))'}{I^n g(z)} = \frac{\dfrac{z(I^n L_c(zf'))'}{I^n L_c(g)} + c[(1-\beta)h(z)+\beta]}{\gamma + c + (1-\gamma)H(z)} \ . \tag{3.8}$$

Also, (3.6) can be written as

$$z(I^n L_c(f))' = I^n L_c(g)[\beta + (1-\beta)h(z)] \ . \tag{3.9}$$

Differentiating both sides of (3.9), we have

$$z\left\{z(I^n L_c(f))'\right\}' = z(I^n L_c(g))'[\beta + (1-\beta)h(z)] + (1-\beta)zh'(z)I^n L_c(g) \ ,$$

or

$$\frac{z\left\{z(I^n L_c(f))'\right\}'}{I^n L_c(g)} = \frac{z(I^n L_c(zf'))'}{I^n L_c(g)}$$

$$= (1-\beta)zh'(z) + [\beta + (1-\beta)h(z)]\,[\gamma + (1-\gamma)H(z)] \ .$$

From (3.8), we have

$$\frac{z(I^n f(z))'}{I^n g(z)} - \beta = (1-\beta)h(z) + \frac{(1-\beta)zh'(z)}{\gamma + c + (1-\gamma)H(z)} \ . \tag{3.10}$$

We form the function $\Psi(u, v)$ by taking $u = h(z)$ and $v = zh'(z)$ in (3.10) as:

$$\Psi(u, v) = (1-\beta)u + \frac{(1-\beta)v}{\gamma + c + (1-\gamma)H(z)} \ . \tag{3.11}$$

It is clear that the function $\Psi(u, v)$ defined by (3.11) satisfies the conditions (i), (ii) and (iii) of Lemma 2.1. Thus we have $I_n(f(z)) \in K_n(\beta, \gamma)$. The proof of Theorem 3.3 is complete. ☐

Similarly, we can prove:

Theorem 3.4. *Let $c > -\gamma$. If $f(z) \in K_n^*(\beta, \gamma)$, then $I_n(f(z)) \in K_n^*(\beta, \gamma)$.*

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