STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 4, December 2009

COMPLETE SUBMANIFOLDS IN A HYPERBOLIC SPACE

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Abstract. In this paper, we study n-dimensional ($n \geq 3$) complete submanifolds M^n in a hyperbolic space $H^{n+p}(-1)$ with the scalar curvature $n(n-1)R$ and the mean curvature H being linearly related. Suppose that the normalized mean curvature vector field is parallel and the mean curvature is positive and obtains its maximum on M^n. We prove that if the squared norm $\|h\|^2$ of the second fundamental form of M^n satisfies $\|h\|^2 \leq nH^2 + (B_H)^2, (p \leq 2)$, and $\|h\|^2 \leq nH^2 + (\widetilde{B}_H)^2, (p \geq 3)$, then M^n is totally umbilical, or M^n is isometric to $S^{n-1}(r) \times H^1(-1/(r^2+1))$ for some $r > 0$, where B_H and $\widetilde{B}_H$ are denoted by (1.1) and (1.2), respectively.

1. Introduction

Let $M_p^{n+p}(c)$ be a $(n+p)$-dimensional space form of constant curvature c, M^n be an n-dimensional submanifold in $M^{n+p}(c)$ with parallel mean curvature vector. If $c = 0$, Cheng and Nonaka [3] obtained some intrinsic rigidity theorems of complete submanifolds with parallel mean vector in Euclidean space R^{n+p}. If $c > 0$, Xu [16] obtained the intrinsic rigidity theorems of these kind of submanifolds in a sphere $S^{n+p}(c)(c = 1)$. If $c < 0$, Yu [18] and Hu [10] proved some intrinsic rigidity theorems of complete hypersurfaces with constant mean curvature in a hyperbolic space $H^{n+1}(c)$

Let M^n be an n-dimensional complete submanifold with constant normalized scalar curvature in $M^{n+p}(c)$. If $c = 0$, for hypersurfaces ($p = 1$), Cheng and

Received by the editors: 13.05.2008.
2000 *Mathematics Subject Classification.* 53C40, 53C42.
Key words and phrases. Submanifold, scalar curvature, mean curvature, hyperbolic space.
Project supported by NSF of Shaanxi Province (SJ08A31) and NSF of Shaanxi Educational Committee (2008JK484).

Yau [6] obtained an intrinsic rigidity theorem of these kind of hypersurfaces in Euclidean space R^{n+1}, and for submanifolds ($p > 1$), Cheng [4] studied the problem and obtained a rigidity and classification theorem. If $c > 0$, Li [10] proved a rigidity and classification theorem of compact hypersurfaces with constant normalized scalar curvature in a sphere $S^{n+1}(c)(c = 1)$. As a generalization, Cheng [4] obtained a rigidity and classification theorem of higher codimension compact submanifolds in $S^{n+p}(c)(c = 1)$. If $c < 0$, the authors [15] studied the submanifolds with constant normalized scalar curvature in hyperbolic space $H^{n+p}(c)(c = -1)$ and obtained some rigidity and classification theorems.

It is well-know that the investigation on hypersurfaces with the scalar curvature $n(n-1)R$ and the mean curvature H being linearly related is also important and interesting. Fox example, Cheng [5] and Li [11] obtained some characteristic theorems of such space-like hypersurfaces in a de Sitter space and such compact hypersurfaces in a unit sphere in terms of sectional curvature, respectively. It is natural and very important to study n-dimensional submanifolds with the scalar curvature $n(n-1)R$ and the mean curvature H being linearly related and with higher codimension in a space form $M^{n+p}(c)$. But there are few results about it. In this paper, we shall investigate n-dimensional complete submanifolds in a hyperbolic space $H^{n+p}(-1)$ with the scalar curvature and the mean curvature being linearly related. We shall prove the following:

Main Theorem. *Let M^n be a n-dimensional $(n \geq 3)$ complete submanifold with $n(n-1)R = k'H$, $(H^2 \geq 1)$ in a hyperbolic space $H^{n+p}(-1)$, where k' is a positive constant. Suppose that the normalized mean curvature vector field is parallel and the mean curvature H is positive and obtains its maximum on M^n. If the norm square $\|h\|^2$ of the second fundamental form of M^n satisfies*

$$\|h\|^2 \leq nH^2 + (B_H^+)^2, (p \leq 2),$$

and

$$\|h\|^2 \leq nH^2 + (\widetilde{B}_H^+)^2, (p \geq 3),$$

then M^n is totally umbilical, or M^n is isometric to $S^{n-1}(r)\times H^1(-1/(r^2+1))$ for some $r>0$, where B_H^+ and $\widetilde{B}_H^+$ are denoted by

$$B_H^+ = -\frac{1}{2}(n-2)\sqrt{\frac{n}{n-1}}H + \sqrt{\frac{n^3H^2}{4(n-1)} - n}, \tag{1.1}$$

$$\widetilde{B}_H^+ = -\frac{1}{3}(n-2)\sqrt{\frac{n}{n-1}}H + \frac{1}{3}\sqrt{\frac{n}{n-1}(n^2+2n-2)H^2 - 6n}. \tag{1.2}$$

2. **Preliminaries**

Let M^n be a n-dimensional complete submanifold in a hyperbolic space $H^{n+p}(-1)$, we choose a local field of orthonormal frames $e_1,\cdots,e_{n+p}$ in $H^{n+p}(-1)$ such that at each point of $M^n, e_1,\cdots,e_n$ span the tangent space of M^n. Let $\omega_1,\cdots,\omega_{n+p}$ be the dual frame field, then the structure equations of $H^{n+p}(-1)$ are given by

$$d\omega_A = -\sum_{B=1}^{n+p}\omega_{AB}\wedge\omega_B, \quad \omega_{AB}+\omega_{BA}=0, \tag{2.1}$$

$$d\omega_{AB} = -\sum_{C=1}^{n+p}\omega_{AC}\wedge\omega_{CB} + \frac{1}{2}\sum_{C,D=1}^{n+p}K_{ABCD}\omega_C\wedge\omega_D, \tag{2.2}$$

$$K_{ABCD} = -(\delta_{AC}\delta_{BD}-\delta_{AD}\delta_{BC}). \tag{2.3}$$

Restricting these form to M^n, we have

$$\omega_\alpha = 0, \quad \alpha = n+1,\cdots,n+p. \tag{2.4}$$

$$\omega_{\alpha_i} = \sum_{j=1}^{n}h_{ij}^\alpha\omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha, \tag{2.5}$$

$$d\omega_i = -\sum_{j=1}^{n}\omega_{ij}\wedge\omega_j, \quad \omega_{ij}+\omega_{ji}=0, \tag{2.6}$$

$$d\omega_{ij} = -\sum_{k=1}^{n}\omega_{ik}\wedge\omega_{kj} + \frac{1}{2}\sum_{k,l=1}^{n}R_{ijkl}\omega_k\wedge\omega_l, \tag{2.7}$$

$$R_{ijkl} = -(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}) + \sum_{\alpha=n+1}^{n+p}(h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha). \tag{2.8}$$

The normal curvature tensor $R_{\alpha\beta ij}$ and Ricci curvature are

$$R_{\alpha\beta ij} = \sum_{l=1}^{n}(h_{il}^{\alpha}h_{lj}^{\beta} - h_{jl}^{\alpha}h_{li}^{\beta}), \tag{2.9}$$

$$R_{jk} = -(n-1)\delta_{jk} + \sum_{\alpha=n+1}^{n+p}(\sum_{i=1}^{n}h_{ii}^{\alpha}h_{jk}^{\alpha} - \sum_{i=1}^{n}h_{ik}^{\alpha}h_{ji}^{\alpha}), \tag{2.10}$$

$$n(n-1)(R+1) = n^2H^2 - \|h\|^2, \tag{2.11}$$

where R is the normalized scalar curvature, H is the mean curvature of M^n, $\|h\|^2$ is the squared norm of the second fundamental form of M^n. Define the first and second covariant derivatives of h_{ij}^{α} by

$$\sum_{k=1}^{n}h_{ijk}^{\alpha}\omega_k = dh_{ij}^{\alpha} - \sum_{k=1}^{n}h_{ik}^{\alpha}\omega_{kj} - \sum_{k=1}^{n}h_{jk}^{\alpha}\omega_{ki} - \sum_{\beta=n+1}^{n+p}h_{ij}^{\beta}\omega_{\beta\alpha}, \tag{2.12}$$

$$\sum_{l=1}^{n}h_{ijkl}^{\alpha}\omega_l = dh_{ijk}^{\alpha} - \sum_{l=1}^{n}h_{ljk}^{\alpha}\omega_{li} - \sum_{l=1}^{n}h_{ilk}^{\alpha}\omega_{lj} - \sum_{l=1}^{n}h_{ijl}^{\alpha}\omega_{lk} - \sum_{\beta=n+1}^{n+p}h_{ijk}^{\beta}\omega_{\beta\alpha}. \tag{2.13}$$

The Codazzi equation and Ricci identities are

$$h_{ijk}^{\alpha} = h_{ikj}^{\alpha} = h_{jik}^{\alpha}, \tag{2.14}$$

$$h_{ijkl}^{\alpha} - h_{ijlk}^{\alpha} = \sum_{m=1}^{n}h_{mj}^{\alpha}R_{mikl} + \sum_{m=1}^{n}h_{im}^{\alpha}R_{mjkl} + \sum_{\beta=n+1}^{n+p}h_{ij}^{\beta}R_{\beta\alpha kl}. \tag{2.15}$$

The Laplacian of h_{ij}^{α} is defined by $\Delta h_{ij}^{\alpha} = \sum_{k=1}^{n}h_{ijkk}^{\alpha}$. From (2.14) and (2.15), we get

$$\Delta h_{ij}^{\alpha} = \sum_{k=1}^{n}h_{kkij}^{\alpha} + \sum_{k,m=1}^{n}h_{km}^{\alpha}R_{mijk} + \sum_{k,m=1}^{n}h_{mi}^{\alpha}R_{mkjk} + \sum_{k=1}^{n}\sum_{\beta=n+1}^{n+p}h_{ki}^{\beta}R_{\beta\alpha jk}. \tag{2.16}$$

Denote by ξ the mean curvature vector field. When $\xi \neq 0$, since we suppose $H > 0$, $e_{n+1} = \frac{\xi}{H}$ is the normal vector field on M^n. We define S_1 and S_2 by

$$S_1 = \sum_{i,j=1}^{n}(h_{ij}^{n+1} - H\delta_{ij})^2, \quad S_2 = \sum_{\alpha=n+2}^{n+p}\sum_{i,j=1}^{n}(h_{ij}^{\alpha})^2. \tag{2.17}$$

Obviously, we have

$$\|h\|^2 = nH^2 + S_1 + S_2. \tag{2.18}$$

By the definition of the mean curvature vector ξ, we have

$$nH = \sum_{i=1}^{n} h_{ii}^{n+1}, \quad \sum_{i=1}^{n} h_{ii}^{\alpha} = 0, n+2 \le \alpha \le n+p. \tag{2.19}$$

From (2.11), (2.17) and (2.18), we get

$$\Delta(n^2H^2) = \Delta\|h\|^2 + n(n-1)\Delta R = \Delta(\mathrm{tr}H_{n+1}^2) + \Delta S_2 + n(n-1)\Delta R. \tag{2.20}$$

Hence, from (2.8), (2.9) and (2.16), by a direct and simple calculation we conclude

$$\begin{aligned}
\frac{1}{2}\Delta(\mathrm{tr}H_{n+1}^2) &= \sum_{i,j,k=1}^{n} (h_{ijk}^{n+1})^2 + \sum_{i,j=1}^{n} h_{ij}^{n+1}\Delta h_{ij}^{n+1} \qquad (2.21)\\
&= \sum_{i,j,k=1}^{n} (h_{ijk}^{n+1})^2 + \sum_{i,j=1}^{n} h_{ij}^{n+1}(nH)_{ij} - n\sum_{i,j=1}^{n}(h_{ij}^{n+1})^2 - (\sum_{i,j=1}^{n}(h_{ij}^{n+1})^2)^2\\
&\quad + nH\sum_{i,j,k=1}^{n} h_{ij}^{n+1}h_{jk}^{n+1}h_{ki}^{n+1} + n^2H^2 - \sum_{\beta=n+2}^{n+p}\{\sum_{i,j=1}^{n}(h_{ij}^{n+1} - H\delta_{ij})h_{ij}^{\beta}\}^2\\
&\quad + \sum_{\beta=n+2}^{n+p}\{\sum_{i,j,k=1}^{n}[h_{ij}^{n+1}h_{kj}^{n+1} - (h_{ij}^{n+1})^2](h_{ik}^{\beta})^2\},
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2}\Delta S_2 &= \sum_{\alpha=n+2}^{n+p}\sum_{i,j,k=1}^{n}(h_{ijk}^{\alpha})^2 + \sum_{\alpha=n+2}^{n+p}\sum_{i,j=1}^{n} h_{ij}^{\alpha}\Delta h_{ij}^{\alpha} \qquad (2.22)\\
&= \sum_{\alpha=n+2}^{n+p}\sum_{i,j,k=1}^{n}(h_{ijk}^{\alpha})^2 - n\sum_{\alpha=n+2}^{n+p}\sum_{i,j=1}^{n}(h_{ij}^{\alpha})^2 + nH\sum_{\alpha=n+2}^{n+p}\mathrm{tr}(H_{n+1}H_{\alpha}^2)\\
&\quad - \sum_{\alpha=n+2}^{n+p}[\mathrm{tr}(H_{n+1}H_{\alpha})]^2 - \sum_{\alpha,\beta=n+2}^{n+p} N(H_{\alpha}H_{\beta} - H_{\beta}H_{\alpha})\\
&\quad - \sum_{\alpha,\beta=n+2}^{n+p}[\mathrm{tr}(H_{\alpha}H_{\beta})]^2 + \sum_{\alpha=n+2}^{n+p}\mathrm{tr}(H_{n+1}H_{\alpha})^2 - \sum_{\alpha=n+2}^{n+p}\mathrm{tr}(H_{n+1}^2H_{\alpha}^2).
\end{aligned}$$

We need the following lemmas:

Lemma 2.1 ([12], [1]). *Let μ_i, $i = 1, \cdots, n$ be real numbers, with $\sum_i \mu_i = 0$ and $\sum_i \mu_i^2 = \beta^2 \ge 0$. Then*

$$-\frac{n-2}{\sqrt{n(n-1)}}\beta^3 \le \sum_i \mu_i^3 \le \frac{n-2}{\sqrt{n(n-1)}}\beta^3, \tag{2.23}$$

and equality holds if and only if either $(n-1)$ *of the numbers* μ_i *are equal to* $\beta/\sqrt{n(n-1)}$ *or* $(n-1)$ *of the numbers* μ_i *are equal to* $-\beta/\sqrt{n(n-1)}$.

Lemma 2.2 ([14]). *Let* A, B *be symmetric* $n \times n$ *matrices satisfying* $AB = BA$, *and* $\mathrm{tr}A = \mathrm{tr}B = 0$. *Then*

$$|\mathrm{tr}A^2B| \leq \frac{n-2}{\sqrt{n(n-1)}}(\mathrm{tr}A^2)(\mathrm{tr}B^2)^{\frac{1}{2}}. \tag{2.24}$$

Lemma 2.3 ([4]). *Let* $a_1, \cdots, a_n, b_{ij} (i, j = 1, 2, \cdots, n)$ *be real numbers satisfying* $\sum_{i=1}^n a_i = 0$, $\sum_{i=1}^n b_{ii} = 0$, $\sum_{i,j=1}^n b_{ij}^2 = b$ *and* $b_{ij} = b_{ji} (i, j = 1, 2, \cdots, n)$. *Then*

$$-(\sum_{i=1}^n b_{ii}a_i)^2 + \sum_{i,j=1}^n b_{ij}^2 a_i a_j - \sum_{i,j=1}^n b_{ij}^2 a_i^2 \geq -\sum_{i=1}^n a_i^2 b. \tag{2.25}$$

Lemma 2.4 ([9]). *Let* $A_1, A_2, \cdots, A_p$ *be* $(n \times n)$ *symmetric matrices* $(p \geq 2)$. *Denote* $S_{\alpha\beta} = trA_\alpha A'_\beta, S_\alpha = S_{\alpha\alpha} = N(A_\alpha), S = S_1 + \cdots + S_p$. *Then*

$$\sum_{\alpha,\beta=1}^n N(A_\alpha A_\beta - A_\beta A_\alpha) + \sum_{\alpha,\beta=1}^p S_{\alpha\beta}^2 \leq \frac{3}{2}S^2, \tag{2.26}$$

and the equality holds if and only if one of the following conditions hold: (1) $A_1 = A_2 = \cdots = A_p = 0$*; (2) Only two of* $A_1, \cdots, A_p$ *are different from zero. Assuming* $A_1 \neq 0, A_2 \neq 0, A_3 = \cdots = A_p = 0$. *Then* $S_{11} = S_{22}$, *and there exists* $(n \times n)$ *orthogonal matrix* T *such that*

$$TA_1T' = \sqrt{\frac{S_{11}}{2}}\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \end{pmatrix}, \quad TA_2T' = \sqrt{\frac{S_{22}}{2}}\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

In order to represent our theorems, we need some notations, for details see Lawson [8] and Ryan [13]. First we give a description of the real hyperbolic space $H^{n+1}(c)$ of constant curvature $c(< 0)$.

For any two vectors x and y in R^{n+2}, we set

$$g(x, y) = x_1y_1 + \cdots + x_{n+1}y_{n+1} - x_{n+2}y_{n+2},$$

(R^{n+2}, g) is the so-called Minkowski space-time. Denote $\rho = \sqrt{-1/c}$. We define

$$H^{n+1}(c) = \{x \in R^{n+2} \mid g(x,x) = -\rho^2, x_{n+2} > 0\}.$$

Then $H^{n+1}(c)$ is a simply-connected hypersurface of R^{n+2}. Hence, we obtain a model of a real hyperbolic space.

We define

$$\begin{aligned} M_1 =& \{x \in H^{n+1}(c) \mid x_1 = 0\}, \\ M_2 =& \{x \in H^{n+1}(c) \mid x_1 = r > 0\}, \\ M_3 =& \{x \in H^{n+1}(c) \mid x_{n+2} = x_{n+1} + \rho\}, \\ M_4 =& \{x \in H^{n+1}(c) \mid x_1^2 + \cdots + x_{n+1}^2 = r^2 > 0\}, \\ M_5 =& \{x \in H^{n+1}(c) \mid x_1^2 + \cdots + x_{k+1}^2 = r^2 > 0, \\ & x_{k+2}^2 + \cdots + x_{n+1}^2 - x_{n+2}^2 = -\rho^2 - r^2\}. \end{aligned}$$

$M_1, \cdots, M_5$ are often called the standard examples of complete hypersurfaces in $H^{n+1}(c)$ with at most two distinct constant principal curvatures. It is obvious that $M_1, \cdots, M_4$ are totally umbilical. In the sense of Chen [2], they are called the hyperspheres of $H^{n+1}(c)$. M_3 is called the horosphere and M_4 the geodesic distance sphere of $H^{n+1}(c)$. Ryan [13] obtained the following:

Lemma 2.5 ([13]). *Let M^n be a complete hypersurface in $H^{n+1}(c)$. Suppose that, under a suitable choice of a local orthonormal tangent frame field of TM^n, the shape operator over TM^n is expressed as a matrix A. If M^n has at most two distinct constant principal curvatures, then it is congruent to one of the following:*

(1) M_1. In this case, $A = 0$, and M_1 is totally geodesic. Hence M_1 is isometric to $H^n(c)$;

(2) M_2. In this case, $A = \frac{1/\rho^2}{\sqrt{1/\rho^2+1/r^2}} I_n$, where I_n denotes the identity matrix of degree n, and M_2 is isometric to $H^n(-1/(r^2+\rho^2))$;

(3) M_3. In this case, $A = \frac{1}{\rho} I_n$, and M_3 is isometric to a Euclidean space R^n;

(4) M_4. In this case, $A = \sqrt{1/r^2 + 1/\rho^2} I_n, M_4$ is isometric to a round sphere $S^n(r)$ of radius r;

(5) M_5. *In this case,* $A = \lambda I_k \oplus \mu I_{n-k}$, *where* $\lambda = \sqrt{1/\rho^2 + 1/r^2}$, *and* $\mu = \frac{1/\rho^2}{\sqrt{1/r^2+1/\rho^2}}$, M_5 *is isometric to* $S^k(r) \times H^{n-k}(-1/(r^2+\rho^2))$.

3. **Proof of main theorem**

For a C^2-function f defined on M^n, we defined its gradient and Hessian (f_{ij}) by

$$df = \sum_{i=1}^{n} f_i\omega_i, \quad \sum_{j=1}^{n} f_{ij}\omega_j = df_i + \sum_{j=1}^{n} f_j\omega_{ji}. \tag{3.1}$$

Let $T = \sum T_{ij}\omega_i \otimes \omega_j$ be a symmetric tensor on M^n defined by

$$T_{ij} = nH\delta_{ij} - h_{ij}^{n+1}. \tag{3.2}$$

Follow Cheng-Yau [6], we introduce operator $\Box$ associated to T acting on f by

$$\Box f = \sum_{i,j=1}^{n} T_{ij} f_{ij} = \sum_{i,j=1}^{n} (nH\delta_{ij} - h_{ij}^{n+1}) f_{ij}. \tag{3.3}$$

By a simple calculation and from (2.20), we obtained

$$\begin{aligned}\Box(nH) =& \sum_{i,j=1}^{n} (nH\delta_{ij} - h_{ij}^{n+1})(nH)_{ij} \\ =& \frac{1}{2}\Delta(n^2H^2) - n^2\|\nabla H\|^2 - \sum_{i,j=1}^{n} h_{ij}^{n+1}(nH)_{ij} \\ =& \frac{1}{2}n(n-1)\Delta R + \frac{1}{2}\Delta(\mathrm{tr}H_{n+1}^2) + \frac{1}{2}\Delta S_2 - n^2\|\nabla H\|^2 - \sum_{i,j=1}^{n} h_{ij}^{n+1}(nH)_{ij}.\end{aligned} \tag{3.4}$$

By making use of the similar method in [5], we prove the following:

Proposition 3.1. *Let* M^n *be an* n*-dimensional submanifold in a hyperbolic space* $H^{n+p}(-1)$ *with* $n(n-1)R = k'H(k' = const. > 0)$. *If the mean curvature* $H > 0$, *then the operator*

$$L = \Box - (k'/2n)\Delta$$

is elliptic.

Proof. For a fixed α, we choose a orthonormal frame field $\{e_j\}$ at each point in M^n so that $h_{ij}^\alpha = \lambda_i^\alpha \delta_{ij}$. From (2.19), we have, for any i,

$$\begin{aligned}
(nH - \lambda_i^{n+1} - k'/2n) =& \sum_j \lambda_j^{n+1} - \lambda_i^{n+1} \\
& - (1/2)[-\sum_{j,\alpha}(\lambda_j^\alpha)^2 + n^2H^2 - n(n-1)]/(nH) \\
\geq & \sum_j \lambda_j^{n+1} - \lambda_i^{n+1} \\
& - (1/2)[-\sum_j(\lambda_j^{n+1})^2 + (\sum_j \lambda_j^{n+1})^2 - n(n-1)]/(nH) \\
=&[(\sum_j \lambda_j^{n+1})^2 - \lambda_i^{n+1}(\sum_j \lambda_j^{n+1}) \\
& - (1/2)\sum_{l\neq j} \lambda_l^{n+1}\lambda_j^{n+1} + (1/2)n(n-1)](nH)^{-1} \\
=&[\sum_j (\lambda_j^{n+1})^2 + (1/2)\sum_{l\neq j}\lambda_l^{n+1}\lambda_j^{n+1} \\
& - \lambda_i^{n+1}(\sum_j \lambda_j^{n+1}) + (1/2)n(n-1)](nH)^{-1} \\
=&[\sum_{i\neq j}(\lambda_j^{n+1})^2 + (1/2)\sum_{\substack{l\neq j\\ l,j\neq i}} \lambda_l^{n+1}\lambda_j^{n+1} + (1/2)n(n-1)](nH)^{-1} \\
=&(1/2)[\sum_{j\neq i}(\lambda_j^{n+1})^2 + (\sum_{j\neq i}\lambda_j^{n+1})^2 + n(n-1)](nH)^{-1} > 0.
\end{aligned}$$

Thus, L is an elliptic operator. This completes the proof of Proposition 3.1.

Proposition 3.2. *Let M^n be a n-dimensional submanifold in a hyperbolic space $H^{n+p}(-1)$ with $n(n-1)R = k'H, (k' = const. > 0)$. If the mean curvature $H > 0$, then*

$$\|\nabla h\|^2 \geq n^2 \|\nabla H\|^2.$$

Proof. Since $H > 0$, we have $\|h\|^2 \neq 0$. In fact, if $\|h\|^2 = \sum_{i,\alpha}(\lambda_i^\alpha)^2 = 0$ at a point of M^n, then $\lambda_i^\alpha = 0$ for all i and α at this point. This implies that $H = 0$ at this point. This is impossible.

From (2.11) and $n(n-1)R = k'H$, we have

$$k'\nabla_i H = 2n^2 H\nabla_i H - 2\sum_{j,k,\alpha} h^\alpha_{kj} h^\alpha_{kji},$$

$$(\frac{1}{2}k' - n^2 H)\nabla_i H = -\sum_{j,k,\alpha} h^\alpha_{kj} h^\alpha_{kji},$$

$$(\frac{1}{2}k' - n^2 H)^2 \|\nabla H\|^2 = \sum_i (\sum_{j,k,\alpha} h^\alpha_{kj} h^\alpha_{kji})^2 \le \sum_{i,j,\alpha} (h^\alpha_{ij})^2 \sum_{i,j,k,\alpha} (h^\alpha_{ijk})^2 \quad = \|h\|^2 \|\nabla h\|^2.$$

Therefore, we have

$$\begin{aligned} \|\nabla h\|^2 - n^2\|\nabla H\|^2 \ge & [(\frac{k'}{2} - n^2 H)^2 - n^2\|h\|^2]\|\nabla H\|^2 \frac{1}{\|h\|^2} \\ = & [\frac{(k')^2}{4} + n^3(n-1)]\|\nabla H\|^2 \frac{1}{\|h\|^2} \ge 0. \end{aligned}$$

This completes the proof of Proposition 3.2.

Proof of Main Theorem. By making use of the similar method in [4], we choose a local orthonornmal frame field $\{e_1, \cdots, e_n\}$ such that $h^{n+1}_{ij} = \lambda_i \delta_{ij}$. Let $\mu_i = \lambda_i - H$. Then $\sum\limits_{n=1}^{n} \mu_i = 0, \sum\limits_{i=1}^{n} \mu_i^2 = \sum\limits_{i=1}^{n} \lambda_i^2 - nH^2 = \mathrm{tr}H^2_{n+1} - nH^2 = S_1$. By Lemma 2.1, we get

$$\begin{aligned} nH \sum_{i,j,k=1}^{n} h^{n+1}_{ii} h^{n+1}_{jk} h^{n+1}_{ki} = & nH\sum_{i=1}^{n} \lambda_i^3 = 3nH^2 S_1 + n^2 H^4 + nH\sum_{i=1}^{n} \mu_i^3 \\ \ge & 3nH^2 S_1 + n^2 H^4 - \frac{n(n-2)}{\sqrt{n(n-1)}} H (S_1)^{\frac{3}{2}}. \end{aligned} \tag{3.5}$$

From Lemma 2.3, we obtain

$$\begin{aligned} & -\sum_{\beta=n+2}^{n+p} \{\sum_{i,j=1}^{n} (h^{n+1}_{ij} - H\delta_{ij}) h^\beta_{ij}\}^2 + \sum_{\beta=n+2}^{n+p} \{\sum_{i,j,k=1}^{n} [h^{n+1}_{ij} h^{n+1}_{kj} - (h^{n+1}_{ij})^2](h^\beta_{ik})^2\} \\ & = -\sum_{\beta=n+2}^{n+p} \{\sum_{i=1}^{n} (\lambda_i - H) h^\beta_{ii}\}^2 + \sum_{\beta=n+2}^{n+p} \{\sum_{i,k=1}^{n} (\lambda_i\lambda_k - \lambda_i^2)(h^\beta_{ik})^2\} \\ & = \sum_{\beta=n+2}^{n+p} \{-(\sum_{i=1}^{n} \mu_i h^\beta_{ii})^2 + \sum_{i,k=1}^{n} (\mu_i\mu_k - \mu_i^2)(h^\beta_{ik})^2\} \\ & \ge \sum_{\beta=n+2}^{n+p} \{-\sum_{i=1}^{n} \mu_i^2 \sum_{i,j=1}^{n} (h^\beta_{ij})^2\} = -S_1 S_2. \end{aligned} \tag{3.6}$$

Hence from (2.21), (3.5), (3.6) we have

$$\frac{1}{2}\Delta(trH_{n+1}^2) \geq \sum_{i,j,k=1}^{n}(h_{ijk}^{n+1})^2 + \sum_{i,j=1}^{n} h_{ij}^{n+1}(nH)_{ij} - n\sum_{i=1}^{n}\lambda_i^2 - (\sum_{i=1}^{n}\lambda_i^2)^2 \quad (3.7)$$

$$+ n^2H^2 + 3nH^2S_1 + n^2H^4 - \frac{n(n-2)}{\sqrt{n(n-1)}}H(S_1)^{\frac{3}{2}} - S_1S_2$$

$$= \sum_{i,j,k=1}^{n}(h_{ijk}^{n+1})^2 + \sum_{i,j=1}^{n} h_{ij}^{n+1}(nH)_{ij}$$

$$+ S_1\{-n + nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H\sqrt{S_1} - S_1 - S_2\}.$$

Let M^n be complete connect submanifold in $H^{n+p}(-1)$ with positive mean curvature. Suppose that the normalized mean curvature vector $\frac{\xi}{H}$ is parallel in $T^{\perp}M^n$. If we choose $e_{n+1} = \frac{\xi}{H}$, then $\omega_{\alpha n+1} = 0$, for all α. Consequently $R_{\alpha n+1 jk} = 0$. From (2.9) we have

$$\sum_{i=1}^{n} h_{ij}^{\alpha}h_{ik}^{n+1} = \sum_{i=1}^{n} h_{ik}^{\alpha}h_{ij}^{n+1}. \quad (3.8)$$

Hence, we obtain

$$H_\alpha H_{n+1} = H_{n+1}H_\alpha. \quad (3.9)$$

We set $B = H_{n+1} - HI$, (I is the unit matrix) then $\mathrm{tr}B = 0$, since $\mathrm{tr}H_\alpha = 0(\alpha > n+1)$. By (3.9) we get for $\alpha > n+1, H_\alpha B = BH_\alpha$. By virtue of Lemma 2.2, we see that

$$|\mathrm{tr}(H_\alpha^2 B)| \leq \frac{n-2}{\sqrt{n(n-1)}}\mathrm{tr}H_\alpha^2\sqrt{\mathrm{tr}B^2}, \quad \alpha > n+1. \quad (3.10)$$

Since

$$\mathrm{tr}(H_\alpha^2 B) = \mathrm{tr}(H_\alpha^2 H_{n+1}) - H\mathrm{tr}H_\alpha^2, \quad \alpha > n+1, \quad (3.11)$$

$$\mathrm{tr}B^2 = \mathrm{tr}H_{n+1}^2 - nH^2 = S_1. \quad (3.12)$$

By (3.10), (3.11) and (3.12), we have

$$\mathrm{tr}(H_\alpha^2 H_{n+1}) \leq (H + \frac{n-2}{\sqrt{n(n-1)}}\sqrt{S_1})\mathrm{tr}H_\alpha^2, \quad (\alpha > n+1). \quad (3.13)$$

From Lemma 2.4 and definition of S_2

$$-\sum_{\alpha,\beta=n+2}^{n+p} N(H_\alpha H_\beta - H_\beta H_\alpha) - \sum_{\alpha,\beta=n+2}^{n+p}[\mathrm{tr}(H_\alpha H_\beta)]^2 \geq -\frac{3}{2}S_2^2. \quad (3.14)$$

When $p=2$, we have

$$-\sum_{\alpha,\beta=n+2}^{n+p} N(H_\alpha H_\beta - H_\beta H_\alpha) - \sum_{\alpha,\beta=n+2}^{n+p} [\mathrm{tr}(H_\alpha H_\beta)]^2 = -S_2^2. \tag{3.15}$$

For a fixed $\alpha, n+2 \le \alpha \le n+p$, we choose a local orthonormal frame field $\{e_1, \cdots, e_n\}$ such that $h_{ij}^\alpha = \lambda_i^\alpha \delta_{ij}$. Thus, we have $\sum_{i=1}^n \lambda_i^\alpha = 0$ and $\mathrm{tr}H_\alpha^2 = \sum_{i=1}^n (\lambda_i^\alpha)^2$. Let $B = H_{n+1} - HI = (b_{ij})$. We have $b_{ij} = b_{ji}(i,j=1,2,\cdots,n)$, $\sum_{i=1}^n b_{ii} = 0$ and $\sum_{i,j=1}^n b_{ij}^2 = S_1$. Since $\lambda_i^\alpha, b_{ij}(i,j=1,2,\cdots,n)$ satisfy Lemma 2.3, from Lemma 2.3, we get

$$\begin{aligned}
&-\sum_{\alpha=n+2}^{n+p} [\mathrm{tr}(H_{n+1}H_\alpha)]^2 + \sum_{\alpha=n+2}^{n+p} \mathrm{tr}(H_{n+1}H_\alpha)^2 - \sum_{\alpha=n+2}^{n+p} \mathrm{tr}(H_{n+1}^2 H_\alpha^2) \\
&= \sum_{\alpha=n+2}^{n+p} \{-[\mathrm{tr}((H_{n+1}-HI)H_\alpha)]^2 + \mathrm{tr}[(H_{n+1}-HI)H_\alpha]^2 - \mathrm{tr}[(H_{n+1}-HI)^2 H_\alpha^2]\} \\
&= \sum_{\alpha=n+2}^{n+p} \{-[\mathrm{tr}(BH_\alpha)]^2 + \mathrm{tr}(BH_\alpha)^2 - \mathrm{tr}(B^2H_\alpha^2)\} \\
&= \sum_{\alpha=n+2}^{n+p} \{-(\sum_{i=1}^n b_{ii}\lambda_i^\alpha)^2 + \sum_{i=1}^n b_{ij}^2 (\lambda_i^\alpha)^2(\lambda_j^\alpha)^2 - \sum_{i=1}^n b_{ij}^2(\lambda_i^\alpha)^2\} \\
&\ge \sum_{\alpha=n+2}^{n+p} [-\sum_{i=1}^n (\lambda_i^\alpha)^2 \sum_{i,j=1}^n b_{ij}^2] = -S_1 \sum_{\alpha=n+2}^{n+p} \mathrm{tr}H_\alpha^2 = -S_1S_2.
\end{aligned} \tag{3.16}$$

Therefore, by (2.22), (3.13), (3.14) and (3.16), when $p \ge 3$, we get

$$\frac{1}{2}\Delta S_2 \ge \sum_{\alpha=n+2}^{n+p} \sum_{i,j,k=1}^n (h_{ijk}^\alpha)^2 + S_2\{-n + nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} H\sqrt{S_1} - S_1 - \frac{3}{2}S_2\}. \tag{3.17}$$

When $p=2$, from (2.22), (3.13), (3.15), (3.16), we have

$$\frac{1}{2}\Delta S_2 \ge \sum_{\alpha=n+2}^{n+p} \sum_{i,j,k=1}^n (h_{ijk}^\alpha)^2 + S_2\{-n + nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} H\sqrt{S_1} - S_1 - S_2\}. \tag{3.18}$$

Case 1. If $p=1$, we have $S_2=0, S_1=\|h\|^2-nH^2$. Therefore, by (3.4), (3.7) and Proposition 3.2, we have

$$\begin{aligned}\square(nH)=&\frac{1}{2}n(n-1)\Delta R+\|\nabla h\|^2-n^2\|\nabla H\|^2 \\ &+S_1\{-n+nH-\frac{n(n-2)}{\sqrt{n(n-1)}}H\sqrt{S_1}-S_1\} \\ \geq&\frac{1}{2}n(n-1)\Delta R+\|g\|^2\{-n+nH^2-\frac{n(n-2)}{\sqrt{n(n-1)}}H\|g\|-\|g\|^2\},\end{aligned} \tag{3.19}$$

where $\|g\|^2$ is a non-negative C^2-function on M^n defined by $\|g\|^2=\|h\|^2-nH^2$.

Therefore, from (3.19), we have

$$\begin{aligned}nLH=&n[\square H-(k'/2n)\Delta H] \\ =&\square(nH)-(1/2)n(n-1)\Delta R \\ \geq&\|g\|^2\{-n+nH^2-\frac{n(n-2)}{\sqrt{n(n-1)}}H\|g\|-\|g\|^2\} \\ =&\|g\|^2P_H(\|g\|),\end{aligned} \tag{3.20}$$

where

$$P_H(\|g\|)=-n+nH^2-\frac{n(n-2)}{\sqrt{n(n-1)}}H\|g\|-\|g\|^2. \tag{3.21}$$

Since $H^2\geq 1$, we know that $P_H(\|g\|)$ has two real roots B_H^+ and B_H^- given by

$$B_H^{\pm}=-\frac{1}{2}(n-2)\sqrt{\frac{n}{n-1}}H\pm\sqrt{\frac{n^3H^2}{4(n-1)}-n}. \tag{3.22}$$

Therefore, we know that

$$P_H(\|g\|)=(\|g\|-B_H^-)(-\|g\|+B_H^+).$$

Clearly, we know that $\|g\|-B_H^->0$. From the assumption of Main Theorem, we infer that $P_H(\|g\|)\geq 0$ on M^n. This implies that the right-hand side of (3.20) is non-negative. From Proposition 3.1, we know that L is elliptic. Since H obtains its maximum on M^n, from (3.20), we have $H=const.$ on M^n. From (3.20) again, we get $\|g\|^2P_H(\|g\|)=0$. Therefore, we have $\|g\|^2=0$ and M^n is totally umbilical, or $P_H(\|g\|)=0$. In the latter case, we infer that the equalities hold in (3.20), (3.19) and (2.23) of Lemma 2.1. Therefore, we know that $(n-1)$ of the numbers $\lambda_i - H$ are equal to $\|g\|/\sqrt{n(n-1)}$. This implies that M^n has $(n-1)$ principal curvatures equal and constant. As H is constant, the other principal curvature is constant as well. Therefore we know that M^n is isoparametric. From the result of Lemma 2.5, M^n is isometric to $S^{n-1}(r) \times H^1(-1/(r^2+1))$ for some $r > 0$.

Case 2. If $p = 2$, from (2.18), we have

$$S_1 \leq \|h\|^2 - nH^2. \tag{3.23}$$

From (3.4), (3.7), (3.18), (3.23), Proposition 3.2 and (2.18) we have

$$\begin{aligned}\Box(nH) \geq & \frac{1}{2}n(n-1)\Delta R + (S_1+S_2)\{-n+nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H\sqrt{S_1} - (S_1+S_2)\} \\ & \hspace{28em} (3.24) \\ \geq & \frac{1}{2}n(n-1)\Delta R + \|g\|^2\{-n+nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H\|g\| - \|g\|^2\},\end{aligned}$$

where $\|g\|^2 = \|h\|^2 - nH^2$.

Therefore, from (3.22), we have

$$\begin{aligned}nLH = & \Box(nH) - (1/2)n(n-1)\Delta R \\ \geq & \|g\|^2\{-n+nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H\|g\| - \|g\|^2\} \\ = & \|g\|^2 P_H(\|g\|),\end{aligned} \tag{3.25}$$

where $P_H(\|g\|)$ is denoted by (3.21). $P_H(\|g\|)$ has two real roots B_H^+ and B_H^- denoted by (3.22). Therefore, we know that

$$P_H(\|g\|) = (\|g\| - B_H^-)(-\|g\| + B_H^+).$$

Since $\|g\| - B_H^- > 0$, from the assumption of Main Theorem, we infer that $P_H(\|g\|) \geq 0$ on M^n. This implies that the right-hand side of (3.25) is non-negative. By making use of the same method in Case 1, we can obtain $\|g\|^2 P_H(\|g\|) = 0$. Therefore, we have $\|g\|^2 = 0$ and M^n is totally umbilical, or $P_H(\|g\|) = 0$. If $P_H(\|g\|) = 0$, we infer that the equalities hold in (3.25), (3.24), (3.23) and (2.23) of Lemma 2.1. If the equality holds in (3.23), we have $S_1 = \|h\|^2 - nH^2$. This implies $S_2 = 0$. Since e_{n+1} is parallel on the normal bundle $T^{\perp}(M^n)$ of M^n, using the method of Yau [17], we know that M^n lies in a totally geodesic submanifold $H^{n+1}(-1)$ of $H^{n+p}(-1)$. If the equality holds in Lemma 2.1, by making use of the same assertion as in the proof of Case 1, we infer that M^n has two distinct principal curvatures and is isoparametric. Therefore, from Lemma 2.5, we know that M^n is isometric to $S^{n-1}(r) \times H^1(-1/(r^2+1))$ for some $r > 0$.

Case 3. If $p \geq 3$, from (3.4),(3.7),(3.17),(3.23) and Proposition 3.2, we have

$$\begin{aligned}\Box(nH) \geq & \frac{1}{2}n(n-1)\Delta R + (S_1+S_2)\{-n+nH^2 \\ & -\frac{n(n-2)}{\sqrt{n(n-1)}}H\sqrt{S_1} - (S_1+S_2)\} - \frac{1}{2}S_2^2 \\ \geq & \frac{1}{2}n(n-1)\Delta R + (S_1+S_2)\{-n+nH^2 \\ & -\frac{n(n-2)}{\sqrt{n(n-1)}}H\sqrt{S_1} - (S_1+S_2)\} - \frac{1}{2}(S_1+S_2)^2 \\ \geq & \frac{1}{2}n(n-1)\Delta R + (S_1+S_2)\{-n+nH^2 \\ & -\frac{n(n-2)}{\sqrt{n(n-1)}}H\sqrt{\|h\|^2 - nH^2} - \frac{3}{2}(S_1+S_2)\} \\ = & \frac{1}{2}n(n-1)\Delta R + \|g\|^2\{-n+nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H\|g\| - \frac{3}{2}\|g\|^2\},\end{aligned} \tag{3.26}$$

where $\|g\|^2 = \|h\|^2 - nH^2$.

Therefore, we have

$$\begin{aligned} nLH = & \Box(nH) - (1/2)n(n-1)\Delta R \\ \geq & \|g\|^2\{-n+nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H\|g\| - \frac{3}{2}\|g\|^2\} \\ = & \frac{3}{2}\|g\|^2\{\frac{2}{3}(nH^2-n) - \frac{2}{3}\frac{n(n-2)}{\sqrt{n(n-1)}}H\|g\| - \|g\|^2\} \\ = & \frac{3}{2}\|g\|^2 Q_H(\|g\|),\end{aligned} \tag{3.27}$$

where

$$Q_H(\|g\|) = \frac{2}{3}(nH^2-n) - \frac{2}{3}\frac{n(n-2)}{\sqrt{n(n-1)}}H\|g\| - \|g\|^2.$$

Since $H^2 \geq 1$, we know that $Q_H(\|g\|)$ has two real roots $\widetilde{B}_H^+$ and $\widetilde{B}_H^-$ given by

$$\widetilde{B}_H^{\pm} = -\frac{1}{3}(n-2)\sqrt{\frac{n}{n-1}}H \pm \frac{1}{3}\sqrt{\frac{n}{n-1}(n^2+2n-2)H^2 - 6n},$$

Therefore, we know that

$$Q_H(\|g\|) = (\|g\| - \widetilde{B}_H^-)(-\|g\| + \widetilde{B}_H^+).$$

Clearly, we know that $\|g\| - \widetilde{B}_H^- > 0$. From the assumption of Main Theorem, we infer that $Q_H(\|g\|) \geq 0$ on M^n. This implies that the right-hand side of (3.27) is non-negative. From Proposition 3.1, we know that L is elliptic. Since H obtains its maximum on M^n, from (3.27), we have $H = const.$ on M^n. From (3.27) again, we get $\|g\|^2 Q_H(\|g\|) = 0$. Therefore, we have $\|g\|^2 = 0$ and M^n is totally umbilical, or $Q_H(\|g\|) = 0$. If $Q_H(\|g\|) = 0$, we infer that the equalities hold in (3.27), (3.26) and (3.23). Therefore, we know that

$$S_1 = \|h\|^2 - nH^2, \quad S_2 = S_1 + S_2.$$

From (2.18), this implies that $S_2 = 0$ and $S_1 = 0$. Therefore, we have $\|g\|^2 = \|h\|^2 - nH^2 = 0$ on M^n and M^n is totally umbilical. This completes the proof of Main Theorem.

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