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ON A GENERAL CLASS OF BETA APPROXIMATING OPERATORS OF SECOND KIND

VASILE MIHEŞAN

Abstract. We shall define a general linear transform, from which we obtain as special case the beta second kind transform. We obtain several positive linear operators as a special case of this beta second kind transform. We apply the beta second kind transform to Baskakov's operator B_n and we obtain different generalization of it.

1. **Introduction**

In this paper we continue our earlier investigations [5], [6], [7], [8], [9], [10] concerning to use Euler's beta function for constructing linear positive operators.

Euler's beta function is defined for $p, q > 0$ by the following formula

$$B(p,q) = \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}} du. \tag{1.1}$$

The beta second kind transform of the function f is defined by the following formula

$$T_{p,q} f = \frac{1}{B(p,q)} \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}} f(u) du. \tag{1.2}$$

We shall define a more general linear transform from which we obtain as special case the beta second kind transform.

Let us denote by $M[0,\infty)$ the linear space of functions defined for $t \geq 0$, bounded and Lebesgue measurable in each interval $[c,d]$, where $0 < c < d < \infty$.

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For $a, b \in \mathbb{R}$ we define the (a, b)-beta transform of a function f (see [5])

$$\mathcal{T}_{p,q}^{(a,b)} f = \frac{1}{B(p,q)} \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}} f\left(\frac{u^a}{(1+u)^{a+b}}\right) du, \tag{1.3}$$

where $B(\cdot,\cdot)$ is the beta function (1.1) and $f \in M[0,\infty)$ such that $\mathcal{T}_{p,q}^{(a,b)}|f| < \infty$.

If we consider in (1.3) $a + b = 0$ we obtain the second kind transform of function $f \in M[0,\infty)$

$$T_{p,q}^{(a)} = \mathcal{T}_{p,q}^{(a,-a)} f = \frac{1}{B(p,q)} \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}} f(u^a) du \tag{1.4}$$

such that $T_{p,q}^{(a)}|f| < \infty$. Clearly $T_{p,q}^{(a)}$ is a positive linear functional.

We shall consider here only the special case $a = 1$.

2. **The beta second kind transform. Case** $a = 1$

If we put in (1.4) $a = 1$ we obtain the beta second kind transform

$$T_{p,q} f = T_{p,q}^{(1)} f = \frac{1}{B(p,q)} \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}} f(u) du \tag{2.1}$$

for $f \in M[0,\infty)$ such that $T_{p,q}|f| < \infty$ considered by D.D. Stancu [13] (see also [7]).

Remark. If $a = -1$ we obtain $T_{p,q}^{(-1)} f = T_{p,q}^{(1)} f = T_{p,q} f$ (see [7]).

Theorem 2.1. [13] *The moment of order k $(1 \le k < q)$ of the functional $T_{p,q}$ has the following value*

$$T_{p,q} e_k = \frac{p(p+1)\dots(p+k-1)}{(q-1)\dots(q-k)}, \quad 1 \le k < q. \tag{2.2}$$

Consequently we obtain

$$T_{p,q} e_1 = \frac{p}{q-1}, \quad T_{p,q} e_2 = \frac{p(p+1)}{(q-1)(q-2)}, \quad q > 2. \tag{2.3}$$

We impose that $T_{p,q} e_1 = e_1$, that is $\dfrac{p}{q-1} = x$, or $p = \dfrac{\beta}{\alpha} x$, $q = 1 + \dfrac{\beta}{\alpha}$, $x > 0$, $\alpha, \beta > 0$ and we obtain the following linear positive operators

$$(\mathcal{T}^{(\alpha,\beta)} f)(x) = \frac{1}{B\left(\frac{\beta}{\alpha}x, 1 + \frac{\beta}{\alpha}\right)} \int_0^\infty \frac{u^{\frac{\beta}{\alpha}-1}}{(1+u)^{1+\frac{\beta}{\alpha}(x+1)}} f(u) du. \tag{2.4}$$

Corollary 2.2. *One has*

$$\mathcal{T}^{(\alpha,\beta)}((t-x)^2;x)=\frac{\alpha}{\beta-\alpha}x(1+x),\quad \beta>\alpha>0. \tag{2.5}$$

Proof. It is obtained from (2.3) for $p=\frac{\beta}{\alpha}x$, $q=1+\frac{\beta}{\alpha}$, $p+q=1+\frac{\beta}{\alpha}(1+x)$.

$$(\mathcal{T}^{(\alpha,\beta)}e_2)(x)=\frac{\beta x(\beta x+\alpha)}{\beta(\beta-\alpha)}=x^2+\left(\frac{\beta x^2+\alpha x}{\beta-\alpha}-x^2\right)=x^2+\frac{\alpha(x+x^2)}{\beta-\alpha}$$

and

$$\mathcal{T}^{(\alpha,\beta)}((t-x)^2;x)=\frac{\alpha}{\beta-\alpha}x(1+x).\quad \beta>\alpha>0.$$

Special cases

1. Let $\mathcal{T}_1^{(\alpha)}$ be the beta second kind operator defined by

$$(\mathcal{T}_1^{(\alpha)}f)(x)=\frac{1}{B\left(\frac{x}{\alpha},1+\frac{1}{\alpha}\right)}\int_0^\infty \frac{u^{\frac{x}{\alpha}-1}}{(1+u)^{\frac{1+x}{\alpha}+1}}f(u)du. \tag{2.6}$$

The operator (2.6) has been considered by Stancu [13] (see also [1], [2], [7], [11]) and it is obtained by (2.4) if we choose in (2.4) $\beta=1$ and $\alpha\in(0,1)$.

Corollary 2.3. [7] *One has*

$$\mathcal{T}_1^{(\alpha)}((t-x)^2;x)=\frac{\alpha}{1-\alpha}x(1+x),\quad \alpha\in(0,1). \tag{2.7}$$

For $\alpha=\frac{1}{n}$ *we obtain*

$$\mathcal{T}_1^{(1/n)}((t-x)^2;x)=\frac{x(1+x)}{n-1}.$$

2. Another beta second kind operator it is obtained by (2.4) for $\beta=\frac{1}{1+x}$, $\beta>\alpha$, $x\in\left(0,\frac{1}{\alpha}-1\right)$, $\alpha\in(0,1)$

$$(\mathcal{T}_2^{(\alpha)}f)(x)=\frac{1}{B\left(\frac{x}{\alpha(1+x)},1+\frac{1}{\alpha(1+x)}\right)}\int_0^\infty \frac{u^{\frac{x}{\alpha(1+x)}-1}}{(1+u)^{\frac{1}{\alpha}+1}}f(t)dt \tag{2.8}$$

where $f\in M[0,\infty)$ such that $\mathcal{T}_2^{(\alpha)}|f|<\infty$, considered by J. Adell [2] (see also [7]).

Corollary 2.4. [7] *One has*

$$\mathcal{T}_2^{(\alpha)}((t-x)^2;x)=\frac{\alpha x(1+x)^2}{1-\alpha(1+x)},\quad x<\frac{1}{\alpha}-1.$$

For $\alpha=1/n$, $n\in\mathbb{N}$ we obtain

$$\mathcal{T}_2^{(1/n)}((t-x)^2;x)=\frac{x(1+x)^2}{n-1-x},\quad x<n-1.$$

3. Let $\mathcal{T}_3^{(\alpha)}$ be the operator defined by

$$(\mathcal{T}_3^{(\alpha)}f)(x)=\frac{1}{B\left(\frac{1}{\alpha};1+\frac{1}{\alpha x}\right)}\int_0^\infty \frac{u^{\frac{1}{\alpha}-1}}{(1+u)^{\frac{1+x}{\alpha x}+1}}f(t)dt, \tag{2.9}$$

$x\in\left(0,\frac{1}{\alpha}\right)$, $\alpha\in(0,1)$.

The operator (2.9) is obtained by (2.4) if we choose in (2.4) $\beta=\frac{1}{x}$.

Corollary 2.5. *One has*

$$\mathcal{T}_3^{(\alpha)}((t-x)^2;x)=\frac{\alpha x^2(1+x)}{1-\alpha x},\quad x<\frac{1}{\alpha}.$$

For $\alpha=1/n$, $n\in\mathbb{N}$ we obtain

$$\mathcal{T}_3^{(1/n)}((t-x)^2;x)=\frac{x^2(1+x)}{n-x},\quad x<n.$$

4. For $\beta=\frac{x}{1+x}>\alpha$, $x\in\left(\frac{\alpha}{1-\alpha},\infty\right)$, $\alpha\in(0,1)$ we obtain by (2.4) the following operator

$$(\mathcal{T}_4^{(\alpha)}f)(x)=\frac{1}{B\left(\frac{x^2}{\alpha(1+x)},1+\frac{x}{\alpha(1+x)}\right)}\int_0^\infty \frac{u^{\frac{x^2}{\alpha(1+x)}-1}}{(1+u)^{\frac{x}{\alpha}-1}}f(u)du. \tag{2.10}$$

Corollary 2.6. *One has*

$$\mathcal{T}_4^{(\alpha)}((t-x)^2;x)=\frac{\alpha x(1+x)^2}{x-\alpha(1+x)},\quad x>\frac{\alpha}{1-\alpha}.$$

For $\alpha=1/n$, $n\in\mathbb{N}$ we obtain

$$\mathcal{T}_4^{(1/n)}((t-x)^2;x)=\frac{x(1+x)^2}{(n-1)x-1},\quad x>\frac{1}{n-1}.$$

5. Let $\mathcal{T}_5^{(\alpha)}$ be the operator

$$(\mathcal{T}_5^{(\alpha)} f)(x) = \frac{1}{B\left(\dfrac{1+x}{\alpha}, \dfrac{1+x}{\alpha x} + 1\right)} \int_0^\infty \frac{u^{\frac{1+x}{\alpha}-1}}{(1+u)^{\frac{(1+x)^2}{\alpha x}+1}} f(u) du \tag{2.11}$$

$\alpha \in (0,1)$, $\alpha > 0$. The operator (2.11) is obtained by (2.4) if we put in (2.4) $\beta = \dfrac{1+x}{x}$.

Corollary 2.7. *One has*

$$\mathcal{T}_5^{(\alpha)}((t-x)^2; x) = \frac{\alpha x^2 (1+x)}{1+(1-\alpha)x}, \quad x > 0.$$

For $\alpha = 1/n$, $n \in \mathbb{N}$,

$$\mathcal{T}_5^{(1/n)}((t-x)^2; x) = \frac{1}{n(1+x) - x}.$$

6. For $\beta = x$, $x \in (\alpha, \infty)$, $\alpha \in (0,1)$ we obtain by (2.4) the following operator

$$(\mathcal{T}_6^{(\alpha)} f)(x) = \frac{1}{B\left(\dfrac{x^2}{\alpha}, 1 + \dfrac{x}{\alpha}\right)} \int_0^\infty \frac{u^{\frac{x^2}{\alpha}-1}}{(1+u)^{\frac{x(1+x)}{\alpha}+1}} f(u) du \tag{2.12}$$

Corollary 2.8. *One has*

$$\mathcal{T}_6^{(\alpha)}((t-x)^2; x) = \frac{\alpha x(1+x)}{x-\alpha}, \quad x > \alpha.$$

For $\alpha = 1/n$, $n \in \mathbb{N}$ *we obtain*

$$\mathcal{T}_6^{(1/n)}((t-x)^2; x) = \frac{x(1+x)}{nx-1}, \quad x > \frac{1}{n}.$$

7. Let $\mathcal{T}_7^{(\alpha)}$ be the beta operator defined by

$$(\mathcal{T}_7^{(\alpha)} f)(x) = \frac{1}{B\left(\dfrac{x(1+x)}{\alpha}, 1 + \dfrac{1+x}{\alpha}\right)} \int_0^\infty \frac{u^{\frac{x(1+x)}{\alpha}-1}}{(1+u)^{\frac{(1+x)^2}{\alpha}+1}} f(u) du \tag{2.13}$$

$\alpha \in (0,1)$, $x > 0$. The operator (2.13) is obtained by (2.4) if we put in (2.4) $\beta = 1+x$.

Corollary 2.9. *One has*

$$\mathcal{T}_7^{(\alpha)}((t-x)^2; x) = \frac{\alpha x(1+x)}{1-\alpha+x}, \quad x > 0.$$

For $\alpha = 1/n$ we obtain

$$\mathcal{T}_7^{(1/n)}((t-x)^2;x) = \frac{x(1+x)}{nx+n-1}.$$

8. Another beta second kind operator is obtained for $\beta = \dfrac{1}{x(1+x)} > \alpha$, $x \in \left(0, \dfrac{\sqrt{1+4/\alpha}-1}{2}\right)$, $\alpha \in (0,1)$

$$(\mathcal{T}_8^{(\alpha)} f)(x) = \frac{1}{B\left(\dfrac{1}{\alpha(1+x)}, \dfrac{1}{\alpha x(1+x)}+1\right)} \int_0^\infty \frac{u^{\frac{1}{\alpha(1+x)}-1}}{(1+u)^{\frac{1}{\alpha x}+1}} f(u)du. \tag{2.14}$$

Corollary 2.10. *One has*

$$\mathcal{T}_8^{(\alpha)}((t-x)^2;x) = \frac{\alpha x^2(1+x)^2}{1-\alpha x(1+x)}, \quad x < \frac{\sqrt{1+4/\alpha}-1}{2}.$$

For $\alpha = 1/n$, $n \in \mathbb{N}$ we obtain

$$\mathcal{T}_8^{(1/n)}((t-x)^2;x) = \frac{x^2(1+x)^2}{n-x(1+x)}, \quad x(1+x) < n.$$

9. For $\beta = x(1+x) > \alpha$, $x \in \left(\dfrac{\sqrt{1+4\alpha}-1}{2}, \infty\right)$, $\alpha \in (0,1)$ we obtain by (2.4) the following operator

$$(\mathcal{T}_9^{(\alpha)} f)(x) = \frac{1}{B\left(\dfrac{x^2(1+x)}{\alpha}, \dfrac{x(1+x)}{\alpha}+1\right)} \int_0^\infty \frac{u^{\frac{x^2(1+x)}{\alpha}-1}}{(1+u)^{\frac{x(1+x)^2}{\alpha}+1}} f(u)du. \tag{2.15}$$

Corollary 2.11. $\mathcal{T}_9^{(\alpha)}((t-x)^2;x) = \dfrac{\alpha x(1+x)}{x(1+x)-\alpha}$, $x(1+x) > \alpha$.

For $\alpha = 1/n$, $n \in \mathbb{N}$ we obtain

$$\mathcal{T}_9^{(1/n)}((t-x)^2;x) = \frac{x(1+x)}{nx(1+x)-1}, \quad nx(1+x) > 1.$$

3. The functional $B_n^{(p,q)} f = \mathcal{T}_{p,q}(B_n f)$

Now let us apply the transform (2.1) to the Baskakov operator B_n, defined by [3]

$$(B_n f)(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right). \tag{3.1}$$

Theorem 3.1. [7] *The $\mathcal{T}_{p,q}$ transform of $B_n f$ can be expressed by the following formula*

$$B_n^{(p,q)} f = \mathcal{T}_{p,q}(B_n f) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{(p)_k (q)_n}{(p+q)_{n+k}} f\left(\frac{k}{n}\right) \tag{3.2}$$

where $(a)_m = a(a+1)\dots(a+m-1)$.

Theorem 3.2. [7] *One has*

$$B_n^{(p,q)} e_1 = \frac{p}{q-1}, \quad B_n^{(p,q)} e_2 = \frac{p(p+1)}{(q-1)(q-2)} + \frac{1}{n} \cdot \frac{p(p+q-1)}{(q-1)(q-2)}, \quad q > 2. \tag{3.3}$$

We impose that $B_n^{(p,q)} e_1 = e_1$, that is $\frac{p}{q-1} = x$, or $p = \frac{\beta}{\alpha} x$, $q = 1 + \frac{\beta}{\alpha}$, $x > 0$; $\alpha, \beta > 0$, $\alpha < \beta$ and we obtain from Theorem 3.1 and Theorem 3.2 the following results.

Corollary 3.3. *One has*

$$(B_n^{(\alpha,\beta)} f)(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} b_{n,k}^{(\alpha,\beta)}(x) f\left(\frac{k}{n}\right) \tag{3.4}$$

where

$$b_{n,k}^{(\alpha,\beta)}(x) = \frac{\beta x(\beta x + \alpha)\dots(\beta x + (k-1)\alpha)(\beta+\alpha)(\beta+2\alpha)\dots(\beta+n\alpha)}{(\beta(1+x)+\alpha)(\beta(1+x)+2\alpha)\dots(\beta(1+x)+(n+k)\alpha)}.$$

Corollary 3.4. *One has*

$$(B_n^{(\alpha,\beta)} e_1)(x) = x, \quad (B_n^{(\alpha,\beta)} e_2)(x) = x^2 + \frac{\alpha n + \beta}{\beta - \alpha} \cdot \frac{x(1+x)}{n}$$

$$B_n^{(\alpha,\beta)}((t-x)^2; x) = \frac{\alpha n + \beta}{\beta - \alpha} \cdot \frac{x(1+x)}{n}, \quad \beta > \alpha. \tag{3.5}$$

Special cases

1. If we put in (3.4) $\beta = 1$, $\alpha \in (0,1)$, we obtain the operator considered by D.D. Stancu [13] (see also [1], [7])

$$(C_n^{(\alpha)} f)(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} c_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right) \tag{3.6}$$

$$c_{n,k}^{(\alpha)} = \frac{x(x+\alpha)\dots(x+(k-1)\alpha)(1+\alpha)\dots(1+n\alpha)}{(1+x+\alpha)(1+x+2\alpha)\dots(1+x+(n+k)\alpha)}$$

Corollary 3.5. *One has*

$$C_n^{(\alpha)}((t-x)^2; x) = \frac{1+\alpha n}{1-\alpha} \cdot \frac{x(1+x)}{n}. \tag{3.7}$$

For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$C_n^{(1/n)}((t-x)^2; x) = \frac{2x(1+x)}{n-1}.$$

2. Another operator it is obtained by (3.4) for $\beta = \dfrac{1}{1+x}$, $\alpha \in (0,1)$, $x \in \left(0, \dfrac{1}{\alpha} - 1\right)$

$$(D_n^{(\alpha)} f)(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} d_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right) \tag{3.8}$$

$$d_{n,k}^{(\alpha)}(x) = \frac{x(x+\alpha(1+x))\dots(x+(k-1)\alpha(1+x))(1+\alpha(1+x)\dots(1+n\alpha(1+x))}{(1+\alpha)(1+2\alpha)\dots(1+(n+k)\alpha)(1+x)^{n+k}}$$

Corollary 3.6. *One has*

$$D_n^{(\alpha)}((t-x)^2; x) = \frac{1+n\alpha(1+x)}{1-\alpha(1+x)} \cdot \frac{x(1+x)}{n}, \quad x \in \left(0, \frac{1}{\alpha} - 1\right).$$

For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$D_n^{(1/n)}((t-x)^2; x) = \frac{x(1+x)(2+x)}{n-1-x}, \quad x \in (0, n-1).$$

3. Let $E_n^{(\alpha)}$ be the operator defined by

$$(E_n^{(\alpha)} f)(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} e_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right) \tag{3.9}$$

$$e_{n,k}^{(\alpha)}(x) = \frac{(1+\alpha)(1+2\alpha)\dots(1+(k-1)\alpha)(1+\alpha x)\dots(1+n\alpha x)}{(1+x+\alpha x)\dots(1+x+(n+k)\alpha x)} \cdot x^k$$

$\alpha \in (0,1)$, $x \in (0, 1/\alpha)$. This operator is obtained by (3.4) for $\beta = 1/x$.

Corollary 3.7. *One has*

$$E_n^{(\alpha)}((t-x)^2; x) = \frac{\alpha n x + 1}{1 - \alpha x} \cdot \frac{x(1+x)}{n}, \quad x < \frac{1}{\alpha}.$$

For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$E_n^{(1/n)}((t-x)^2; x) = \frac{x(1+x)^2}{n-x}, \quad x < n.$$

4. For $\beta = \dfrac{x}{1+x}$, $\alpha \in (0,1)$, $x > \dfrac{\alpha}{1-\alpha}$ we obtain by (3.4) the following operator

$$(F_n^{(\alpha)} f)(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} f_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right) \tag{3.10}$$

$$f_{n,k}^{(\alpha)}(x) = \frac{x^2(x^2+\alpha(1+x))\dots(x^2+(k-1)\alpha(1+x))(x+\alpha(1+x))\dots(x+n\alpha(1+x))}{(x+\alpha)(x+2\alpha)\dots(x+(n+k)\alpha)(1+x)^{n+k}}$$

Corollary 3.8. *One has*

$$F_n^{(\alpha)}((t-x)^2; x) = \frac{\alpha n(1+x)+x}{x-\alpha(1+x)}, \quad x > \frac{\alpha}{1-\alpha}.$$

For $\alpha = \dfrac{1}{n}$, $n \in \mathbb{N}$, we obtain

$$F_n^{(1/n)}((t-x)^2; x) = \frac{x(1+x)(1+2x)}{(n-1)x-1}, \quad x > \frac{1}{n-1}.$$

5. Let $G_n^{(\alpha)}$ be the operator

$$(G_n^{(\alpha)} f)(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} g_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right) \tag{3.11}$$

$$g_{n,k}^{(\alpha)}(x) = \frac{(1+x)(1+x+\alpha)\dots(1+x+(k-1)\alpha)(1+x+\alpha x)\dots(1+x+n\alpha x)}{((1+x)^2+\alpha x)((1+x)^2+2\alpha x)\dots((1+x)^2+(n+k)\alpha x)} \cdot x^k.$$

The operator (3.11) is obtained by (3.4) if we put in (3.4) $\beta = \dfrac{1+x}{x}$, $\alpha \in (0,1)$, $x > 0$.

Corollary 3.9. *One has*

$$G_n^{(\alpha)}((t-x)^2;x) = \frac{\alpha nx+1+x}{1+x-\alpha x}\cdot\frac{x(1+x)}{n}.$$

For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$G_n^{(1/n)}((t-x)^2;x) = \frac{x(1+x)(1+2x)}{n+(n-1)x}.$$

6. For $\beta = x$, $\alpha \in (0,1)$, $x \in (\alpha,\infty)$ we obtain by (3.4) the following operator

$$(H_n^{(\alpha)}f)(x) = \sum_{k=0}^{\infty}\binom{n+k-1}{k}h_{n,k}^{(\alpha)}(x)f\left(\frac{k}{n}\right) \tag{3.12}$$

$$h_{n,k}^{(\alpha)}(x) = \frac{x^2(x^2+\alpha)\dots(x^2+(k-1)\alpha)(x+\alpha)(x+2\alpha)\dots(x+n\alpha)}{(x(1+x)+\alpha)\dots(x(1+x)+(n+k)\alpha)}.$$

Corollary 3.10. *One has*

$$H_n^{(\alpha)}((t-x)^2;x) = \frac{\alpha n+x}{x-\alpha}\cdot\frac{x(1+x)}{n},\quad x>\alpha.$$

For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$H_n^{(\alpha)}((t-x)^2;x) = \frac{x(1+x)^2}{nx-1},\quad x>\frac{1}{n}.$$

7. Let $K_n^{(\alpha)}$ be the operator

$$(K_n^{(\alpha)}f)(x) = \sum_{k=0}^{\infty}\binom{n+k-1}{k}k_{n,k}^{(\alpha)}(x)f\left(\frac{k}{n}\right) \tag{3.13}$$

$$k_{n,k}^{(\alpha)}(x) = \frac{x(1+x)(x(1+x)+\alpha)\dots(x(1+x)+(k-1)\alpha)(1+x+\alpha)\dots(1+x+n\alpha)}{((1+x)^2+\alpha)((1+x)^2+2\alpha)\dots((1+x)^2+(n+k)\alpha)}$$

The operator $K_n^{(\alpha)}$ is obtained by (3.4) if we put in (3.4) $\beta = 1+x$, $\alpha \in (0,1)$, $x \in (0,\infty)$.

Corollary 3.11. *One has*

$$K_n^{(\alpha)}((t-x)^2;x) = \frac{\alpha n+x+1}{1+x-\alpha}\cdot\frac{x(1+x)}{n}.$$

For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$K_n^{(1/n)}((t-x)^2;x) = \frac{x(1+x)(2+x)}{n(1+x)-1}.$$

8. For $\beta = \dfrac{1}{x(1+x)}$, $\alpha \in (0,1)$, $\alpha x(1+x) < 1$ we obtain by (3.4) the following operator

$$(L_n^{(\alpha)} f)(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} l_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right) \tag{3.14}$$

$$l_{n,k}^{(\alpha)}(x) = \sum_{k=0}^{\infty} \frac{(1+\alpha(1+x))\dots(1+(k-1)\alpha(1+x))(1+\alpha x(1+x))\dots(1+n\alpha x(1+x))x^k}{(1+\alpha x)(1+2\alpha x)\dots(1+(n+k)\alpha x)(1+x)^{n+k}}$$

Corollary 3.12. *One has*

$$L_n^{(\alpha)}((t-x)^2;x) = \frac{\alpha n x(1+x)+1}{1-\alpha x(1+x)} \cdot \frac{x(1+x)}{n}, \quad \alpha x(1+x) < 1.$$

For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$L_n^{(1/n)}((t-x)^2;x) = \frac{x(1+x)(1+x(1+x))}{n-x(1+x)}, \quad x(1+x) < n.$$

9. Another operator it is obtained for $\beta = x(1+x)$, $\alpha \in (0,1)$, $x(1+x) > \alpha$.

$$(M_n^{(\alpha)} f)(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} m_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right) \tag{3.15}$$

$$m_{n,k}^{(\alpha)}(x) = \frac{x^2(1+x)(x^2(1+x)+\alpha)\dots(x^2(1+x)+(k-1)\alpha)(x(1+x)+\alpha)\dots(x(1+x)+n\alpha)}{(x(1+x)^2+\alpha)\dots(x(1+x)^2+(n+k)\alpha)}.$$

Corollary 3.13. *One has*

$$M_n^{(\alpha)}((t-x)^2;x) = \frac{\alpha n + x(1+x)}{x(1+x)-\alpha} \cdot \frac{x(1+x)}{n}, \quad x(1+x) > \alpha.$$

For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$M_n^{(1/n)}((t-x)^2;x) = \frac{x(1+x)(1+x(1+x))}{nx(1+x)-1}, \quad nx(1+x) > 1.$$

References

[1] Adell, J.A., De la Cal, J., *On a Bernstein-type operator associated with the inverse Polya-Eggenberger distribution*, Rend. Circolo Mat. Palermo (2), Nr. **33**(1993), 143-154.

[2] Adell, J. A., Badia, F. G., De la Cal, J., Plo, L., *On the property of monotonic convergence for Beta operators*, J. of Approx. Theory, **84**(1996), 61-73.

[3] Baskakov, V.A., *An example of a sequence of linear positive operators in the space of the continuous functions*, Dokl. Akad. Nauk SSSR, **113**(1957), 249-251.

[4] Bleimann, G., Butzer, P.L., Hahn, L., *Bernstein-type operator approximating continuous functions on the semi-axis*, Indag. Math., **42**(1980), 255-262.

[5] Miheşan, V., *Approximation of continuous functions by means of linear positive operators*, Ph. D. Thesis, "Babeş-Bolyai" University, Faculty of Mathematics and Computer Science, Cluj-Napoca, 1997.

[6] Miheşan, V., *The beta approximating operators of first kind*, Studia Univ. Babeş-Bolyai, Mathematica, vol. XLIX, **2**(2004), 65-77.

[7] Miheşan, V., *The beta approximating operators of second kind*, Studia Univ. Babeş-Bolyai, Mathematica, XLIX, **2**(2004), 79-88.

[8] Miheşan, V., *On a general class of Beta approximating operators of first kind*, Studia Univ. Babeş-Bolyai, Mathematica, 2008 (in press).

[9] Miheşan, V., *On the modified Beta approximating operators of first kind*, Rev. Anal. Numer. Theor. Approx., 33, no. **1**(2004), 67-71.

[10] Miheşan, V., *On the modified Beta approximating operators of second kind*, Rev. Anal. Numer. Theor. Approx., 34, no. **2**(2005), 135-138.

[11] Rathore, R. K. S., *Linear combinations of linear positive operators and generating relations on special functions*, Ph. D. Thesis, Delhi, 1973.

[12] Stancu, D. D., *Two classes of positive linear operators*, Anal. Univ. Timişoara, Ser. Sti. Mat., **8**(1970), 213-220.

[13] Stancu, D. D., *On the Beta approximating operators of second kind*, Rev. Anal. Numer., Théor. Approx., 24, **1-2**(1995), 231-239.

Technical University of Cluj-Napoca
Department of Mathematics
400020 Cluj-Napoca, Romania
E-mail address: Vasile.Mihesan@math.utcluj.ro