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ON APPLICATIONS OF THE REPRODUCING KERNEL METHOD FOR CONSTRUCTION OF CUBATURE FORMULAS

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Abstract. In this paper we use the method of Reproducing Kernel and *Gegenbauer* polynomials for constructing cubature formulas on the unit ball B^d, and on the standard simplex. Also we study the relation between interpolation polynomials based on the zeros of quasi-orthogonal *Chebyshev* polynomials and the nodes of near minimal degree cubature formulas.

1. **Introduction**

1) The Reproducing Kernel of a Hilbert space of functions

One calls reproducing Kernel of the Hilbert space H of functions defined on D, real valued ($D \subset \mathbb{R}^d$), a function $K = K(x, y) : D \times D \to \mathbb{R}$, which verifies the following conditions

i) $K(\cdot, y) \in H$, for any fixed $y \in D$,

ii) $< f, K(\cdot, y) > = f(y), \ \forall f \in H$.

It is known that in the Hilbert space H are stated the following results.

Theorem 1.1. If the Hilbert space H has a Reproducing Kernel, then this kernel is unique and symmetric with respect to its arguments.

Theorem 1.2. If L is a linear and bounded functional defined on the Hilbert space H, which has a Reproducing Kernel, then the representation function corresponding to L is $g(x) = L_y[K(x, y)]$.

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We consider now, $H = \mathbb{P}_n^d$ the space of all polynomials of degree at most n, and $D \subset \mathbb{R}^d$.

It is known that $dim\mathbb{P}_n^d(D) = \binom{n+d}{d}$, if and only if $int(D) \neq \emptyset$.

Let $f \in \mathbb{P}_n^d$ be a polynomial of degree exact n, and we denote

$$\mu = \mu(d,n) = \binom{n+d}{d} = \frac{(n+d)!}{n!d!}.$$

It was shown that the number of terms in the representation of the polynomial f is equal to $\mu(d,n)$ and this number represents the number of the monomials in the expression of $f = f(x)$.

Let $W = W(x) : D \to \mathbb{R}_0^+$, $(D \subset \mathbb{R}^d)$, be a weight function.

Theorem 1.3. For a given region (domain) D, $D \subset \mathbb{R}^d$ and a given weight function $W = W(x) : D \to \mathbb{R}_0^+$, exists and are unique $r(d,n) = \mu(d,n-1) = \frac{(n-1+d)!}{(n-1)!d!}$ orthogonal polynomials of degree n, which are linearly independent.

Let now, $\{e_i(x)\}_{i=0}^{\infty}$, be the monomials which are ordered increasing, and for the same degree for certain terms, we use the lexicographic order.

So, the set $\{e_i(x)\}$, $i = \overline{1,\mu(d,n)}$ represents all the monomials of degree at most n.

By applying the Gram-Schmidt orthonormalization process, we can obtain an *orthonormalized* set with respect to the scalar product

$$(f,g) = I(f \cdot g) = \int_D f(x)g(x)W(x)dx. \tag{1.1}$$

2) The *Gegenbauer* (ultraspherical) orthogonal polynomials

We present now, some of the properties of *Gegenbauer* polynomials, which play an important role in the applications of the cubature formulas theory by using the Reproducing Kernel method.

The *Gegenbauer* polynomials are usually defined by the following generating function:

$$(1 - 2tz + z^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^{(\lambda)}(t)z^n, \tag{1.2}$$

where $|z| < 1$, $|t| \leq 1$, $\lambda > 0$.

The coefficients $C_n^{(\lambda)}(t)$ are algebraic polynomials of degree n which are called the *Gegenbauer* polynomials associated with λ. One can prove that the family of polynomials $\{C_n^{(\lambda)}\}_{n=0}^{\infty}$ is a complete orthogonal system for the weighted space $L_2(I, W)$, $I = [-1, 1]$, $W(t) = W_\lambda(t) := (1-t^2)^{\lambda-\frac{1}{2}}$, and we have

$$\int_{[-1,1]} C_n^{(\lambda)}(t) C_m^{(\lambda)}(t) W(t) dt = \begin{cases} 0, & m \neq n \\ \gamma_{n,\lambda} = \frac{\pi^{1/2}(2\lambda)_n \Gamma(\lambda+1/2)}{(n+\lambda)n!\Gamma(\lambda)}, & m = n \end{cases}$$

where we use $(a)_\lambda$, the *Pockhammer* symbol,

$$(a)_0 := 0, \quad (a)_n := a(a+1)\ldots(a+n-1) = \Gamma(a+n)/\Gamma(a).$$

Also we have,

$$C_n^{(\lambda)}(-t) = (-1)^n C_n^{(\lambda)}(t), \quad C_n^{(\lambda)}(1) = \frac{(2\lambda)n}{n!} \text{ and } C_0^{(\lambda)} = 1. \tag{1.3}$$

The *Gegenbauer* polynomials can also be defined by the well known *Rodrigues's* formula (see [7] *Szegö*)

$$C_n^{(\lambda)}(t) = (-1)^n \alpha_{n,\lambda} (1-t^2)^{-\lambda+\frac{1}{2}} \frac{d^n}{dt^n}[(1-t^2)^{n+\lambda-\frac{1}{2}}]$$

where,

$$\alpha_{n,\lambda} = \frac{(2\lambda)_n}{n! 2^n (\lambda + \frac{1}{2})_n}.$$

It is known that there exists the following identity which relates *Gegenbauer* polynomials with different weights

$$\frac{d^k}{dt^k} C_n^{(\lambda)}(t) = 2^k (\lambda)_k C_{n-k}^{(\lambda+k)}, \; k = 1, 2, \ldots n. \tag{1.4}$$

For $\lambda = 1/2$, we can obtain the *Legendre* polynomial

$$P_n(t) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dt^n}[(1-t^2)^n] = C_n^{(1/2)}(t)$$

and for $\lambda = 1$ we obtain the *Chebyshev* polynomial of second kind U_n,

$$U_n = \frac{sin[(n+1)arccost]}{\sqrt{1-t^2}} = C_n^{(1)}(t).$$

Also, we can obtain the *Chebyshev* polynomial of the first kind

$$T_n(t) := cos(narccost) = C_n^{(0)},$$

by considering $C_n^{(0)}$ associated with the weight function $W_0(t) = (1-t^2)^{-1/2}$.

We can also consider the *Gegenbauer* polynomials $C_n^{(\lambda)}$, for $\lambda < 0$, $\lambda \in \mathbb{Z}^-$ namely,

$$C_n^{(\lambda)}(t) := \alpha(1-t^2)^{-\lambda+\frac{1}{2}} \frac{d^n}{dt^n}\left[(1-t^2)^{n+\lambda-\frac{1}{2}}\right], \ \lambda < 0$$

where α is an constant independent of t and we can write the identity

$$\frac{d^k}{dt^k} C_n^{(\lambda)}(t) = c C_{n-k}^{(\lambda+k)}(t), \quad k = 1, 2 \ldots, n,$$

where c is independent of t.

3) The relation between Cubature Formulas and the Reproducing Kernels

The Reproducing Kernel method was first used by $I.PMysovskikh$ ([3]) and later studied by $M\ddot{o}ller$ ([2]).

Let a given weight function $W = W(x)$ be defined on a subset $D \subset \mathbb{R}^d$. Then, a cubature formula is a linear combination of function values on some points, that approximates $\int_D f(x)W(x)dx$.

Let $I^d[f] = \int_D f(x)W(x)dx$, $f \in C(D)$, $D \subset \mathbb{R}^d$ for which the moments $I^d[x^\alpha]$, $\alpha \in \mathbb{N}^d$ exists and W=W(x) is nonnegative.

We say that the cubature formula has the degree of exactness m, if it yields the exact value of the integrals for any function $f \in \mathbb{P}_m^d$, which is a polynomials of degree at most m.

We denote the space of polynomials of degree at most n by $\mathbb{P}_n^d$.

Let

$$\left\{P_k^n : 1 \leq k \leq r(d,n)\right\}, \ 0 \leq n < \infty,$$

(where $r(d,n) = \mu(d, n-1) = \binom{d+n-1}{d}$), denote a sequence of orthonormal polynomials of degree n with respect to the inner product (1.1), which are linearly independent, where the superscript n means that $P_k^n \in \mathbb{P}_n^d$ and let denote by $\mathbf{P}_n = (P_1^n, \ldots, P_{r(d,n)}^n)$, the vector of all these polynomials.

The $n-th$ Reproducing Kernel $K_n(x,y)$ of the Hilbert space $H=\mathbb{P}_n^d$ is defined by:

$$K_n(x,y)=\sum_{k=0}^{n}\mathbf{P}_k^T(x)\mathbf{P}_k(y)=\sum_{k=0}^{n}\sum_{j=1}^{r(d,k)}P_j^k(x)P_j^k(y),\ \forall\ x,y\in R^d. \tag{1.5}$$

The method of Reproducing Kernel requires to choose d points: $a^{(1)},\ldots,$ $a^{(d)}\in\mathbb{R}^d$, such that the hypersurfaces $H_1,\ldots H_d$, where H_i is the surface defined by $H_i=\{x\in\mathbb{R}^d : K_n(x,a^{(i)})=0\}$, intersect at n^d points. The points $a^{(1)},\ldots,a^{(d)}$ are chosen as follows.

For $a^{(1)}$ we choose any point that is not a common zero of the polynomial set $\mathbf{P}_n$. If the points $a^{(1)},\ldots,a^{(r-1)}$ have been chosen, then we choose $a^{(r)}\in\bigcap_{k=1}^{r-1}H_k$, and $a^{(r)}$ may be any point of this set, which is not a common zero of $\mathbf{P}_n$.

We assume that the infinity is not a common point of $H_1,\ldots,H_d$.

We present now the following results.

a) The Method of Reproducing Kernel

If $H_1,\ldots,H_d$ defined by $a^{(1)},\ldots,a^{(d)}$, intersect at n^d distinct points: $\{x^{(i)},\ i=\overline{1,n^d}\}$, then there is a cubature formula of degree $2n$,

$$Q_n(f)=\sum_{i=1}^{d}\lambda_i f(a^{(i)})+\sum_{j=1}^{n^d}\mu_j f(x^{(j)}),\ \forall f\in\mathbb{P}_{2n}^d, \tag{1.6}$$

where $\lambda_i=1/K_n(a^{(i)},a^{(i)})$.

If the weight function $W=W(x)$ is centrally symmetric, that is, $W=W(x)$ and its support set D satisfy $\forall x\in D\Rightarrow -x\in D,\ W(-x)=W(x)$, then there is a modified method of Reproducing Kernel due to *Möller* ([2]).

Let $\widetilde{K_n}$ denote:

$$\widetilde{K_n}(x,y)=\sum_{k=0}^{n}{}'\sum_{j=0}^{r(d,k)}{}'P_j^k(x)P_j^k(y),\ \forall x,y\in R^d, \tag{1.7}$$

where $\sum'$ means that the summation is taken over those j so that the corresponding P_j^k has the same parity as n. We choose the points $a^{(i)}$ as before except that we replace H_i by the *hypersurface* $\widetilde{H_i}$ defined by $\widetilde{H_i}=\{x\in\mathbb{R}^d:\widetilde{K_n}(x,a^{(i)})=0\}$ and we suppose that the infinity is not a common point of $\widetilde{H_1}\dots,\widetilde{H_d}$. Then we have, if $W = W(x)$ is centrally symmetric on $D \subset \mathbb{R}^d$.

b) The Modified method of Reproducing Kernel

If $\widetilde{H_1},\dots,\widetilde{H_d}$ defined by $a^{(1)},\dots,a^{(d)}$ intersect at n^d distinct points: $\{x^{(i)}, i=\overline{1,n^d}\}$, then there is a cubature formula of degree $2n+1$,

$$Q_n(f) = \sum_{i=1}^{d} \lambda_i [f(a^{(i)}) + f(-a^{(i)})]/2 + \sum_{j=1}^{n^d} \mu_j f(x^{(j)}),\ \forall f \in \mathbb{P}^d_{2n+1}, \tag{1.8}$$

where $\lambda_i = 1/\widetilde{K_n}(a^{(i)},a^{(i)})$.

If $d=2$, then the method requires to choose two points $a^{(1)}$ and $a^{(2)}$ so that the polynomial surface $\widetilde{K_n}(x,a^{(1)})$ and $\widetilde{K_n}(x,a^{(2)})$ have n^2 common zeros.

In the paper [12] Y. Xu was presented a compact formula of the Reproducing Kernel for the *Jacobi* type weight functions on the unit ball and on the standard simplex.

The method of Reproducing Kernel yields cubature formulas of degree $2n+1$ or $2n$ with $n^d + dn$ or $n^d + dn - 1$ nodes, which is greater than the theoretic lower bound for the number of nodes.

2. Cubature formulas on the unit ball using the reproducing kernel method

Let $x, y \in \mathbb{R}^d$ and we use the following notations:

$< x, y >= x_1y_1 + \cdots + x_dy_d$, the usual Euclidian inner product,

$|x|^2 = \parallel x \parallel^2 \ =< x, x >$, the Euclidian norm.

We consider cubature formulas on the unit ball $B^d = \{x \in \mathbb{R}^d :\ \parallel x \parallel \le 1\}$, with respect to the normalized weight function

$$W_\mu(x) = w_\mu(1- \parallel x \parallel^2)^{\mu-\frac{1}{2}},\ \mu \ge 0,\ x \in B^d, \tag{2.1}$$

where w_μ is a constant chosen so that the integral $\int_{B^d} W_\mu(x)dx = 1$, and we have

$$w_\mu = \frac{2}{\omega_{d-1}} \frac{\Gamma(\mu+\frac{d+1}{2})}{\Gamma(\mu+\frac{1}{2})\Gamma(\frac{d}{2})} = \frac{\Gamma(\mu+\frac{d+1}{2})}{\pi^{d/2}\Gamma(\mu+\frac{1}{2})},$$

where $\omega_{d-1} = 2\pi^{d/2}/\Gamma(d/2)$ is the surface area of the unit sphere in $\mathbb{R}^d$.

Let $K_n(.,.)$ be the $n-th$ Reproducing Kernel with respect to weight function W_μ. In [12] is presented the following compact formula for the representation of this kernel.

$$K_n(W_\mu;x,y) = c_\mu \int_{-1}^{1} \Big[C_n^{(\mu+\frac{d+1}{2})}(< x,y > + \sqrt{1- \| x \|^2}\sqrt{1- \| y \|^2}\ t) + \tag{2.2}$$

$$+ C_{n-1}^{(\mu+\frac{d+1}{2})}(< x,y > + \sqrt{1- \| x \|^2}\sqrt{1- \| y \|^2}\ t)\Big](1-t^2)^{\mu-1}dt,$$

where $c_\mu = 1/\int_{-1}^{1}(1-t^2)^{\mu-1}dt$ and $C_n^{(\lambda)}$ is the *Gegenbauer* polynomial of degree n defined by the generating function (1.2), which have the property

$$C_n^{(\lambda)}(-t) = (-1)^n C_n^{(\lambda)}(t).$$

If we take in consideration the expressions: $K_n(W_\mu;x,y) \pm K_n(W_\mu;x,-y)$ for n being even and odd, respectively then it follows from the formula (1.5) and (1.7) that the modified Reproducing Kernel function $\widetilde{K_n}(W_\mu;...)$ is given by the formula

$$\widetilde{K_n}(W_\mu;x,y) = c_\mu \int_{-1}^{1} C_n^{(\mu+\frac{d+1}{2})}(< x,y > + \sqrt{1- \| x \|^2}\sqrt{1- \| y \|^2}\ t)(1-t^2)^{\mu-1}dt. \tag{2.3}$$

For $\mu \to 0$, in (2.2) and (2.3), one can use the limit

$$\lim_{\mu\to 0} c_\mu \int_{-1}^{1} f(t)(1-t^2)^{\mu-1}dt = \frac{f(1)+f(-1)}{2}. \tag{2.4}$$

In the case $\mu = \frac{1}{2}$, we have: $W_{1/2}(x) = d/\omega_{d-1}$.

If $\mu = 0$ we have: $W_0(x) = w_0(1- \| x \|)^{-1/2}$ and we obtain:

$$\widetilde{K_n}(W_0;x,y) = \frac{1}{2}\Big[C_n^{(3/2)}(< x,y > + \sqrt{1- \| x \|^2}\sqrt{1- \| y \|^2}) + \tag{2.5}$$

$$+ C_n^{(3/2)}(< x,y > - \sqrt{1- \| x \|^2}\sqrt{1- \| y \|^2})\Big].$$

If we consider $\| a \| = 1$, we have

$$\widetilde{K_n}(W_\mu;x,a) = C_n^{(\mu+(d+1)/2)}(< x,a >). \tag{2.6}$$

In this case, if $\| a \| = 1$ then a is not a common zero of the polynomial set $\mathbf{P}_n$, because that $\mathbf{P}_n$ has no common zeros if n is even, and it has only origin as common zero if n is odd.

2.1 The construction of a family of cubature formulas on B^d by using the *Gegenbauer* polynomials

One can use the properties of the *Gegenbauer* polynomial $C_n^{(\lambda)}(t),\ \lambda = \mu + (d+1)/2$, that all its zeros are inside $(-1, 1)$, and we denote these zeros by:

$$-1 < t_{1,n} < t_{2,n} < \cdots < t_{n,n} < 1, \text{ where } \lambda = \mu + (d+1)/2.$$

It is known that these zeros are symmetric with respect to the origin, that is, they satisfy the relation $t_{i,n} = -t_{n-(i-1),n}$. So, in [12] was given the following strategy to choose the points $a^{(1)}, \ldots, a^{(d)}$ as follows.

Let n be fixed and let $t_{*,n}$ be a fixed zero of $C_n^{(\mu+(d+1)/2)}(t)$, and let define:

$$a^{(1)} = (1, 0, \ldots, 0), \quad a^{(k)} = (b_1,\ \ldots\ , b_{k-1}, \sqrt{1 - b_1^2 - \cdots - b_{k-1}^2}, 0, \ldots, 0),$$

$0 \le k \le d$, where the components $b_1, \ldots, b_{d-1}$ are determined inductively by the conditions: $< a^{(k)}, a^{(k+1)} > = t_{*,n}$, which is equivalent with

$$b_1^2 + \cdots + b_{k-1}^2 + \sqrt{1 - b_1^2 - \cdots - b_{k-1}^2}\ b_k = t_{*,n},\ k = \overline{1, d-1},$$

from which are obtained:

$$b_1 = t_{*,n},\ b_2 = (t_{*,n} - b_1^2)/\sqrt{1 - b_1^2},\ \ldots, \text{ and we have } b_k \le \sqrt{1 - b_1^2 - \cdots - b_{k-1}^2},$$

because $t_{*,n} < 1$, hence $a^{(k+1)}$ is well defined. It follows that

$$\bigcap_{i=1}^{k} H_k = \{x \in R^d :< x, a^{(1)} >= t_{i_1,n}, \ldots, < x, a^{(k)} >= t_{i_k,n},\ 1 \le i_1, \ldots, i_k \le n\}$$

for $k = \overline{2, d}$. If we assume that $a^{(k)} \in H_1 \bigcap \cdots \bigcap H_{k-1}$, and we require that $a^{(k+1)} \in \bigcap_{i=1}^{k} H_i$, one observe that $a^{(2)} = (t_{*,n}, \sqrt{1 - t_{*,n}^2}, 0, \ldots, 0) \in H_1$.

Inductively, if we assume that $a^{(k)} \in \bigcap_{i=1}^{k-1} H_i$, that is

$$< a^{(i)}, a^{(k)} >= t_{*,n},\ 1 \le i \le k-1.$$

Since $a^{(k+1)}$ satisfies $< a^{(k)}, a^{(k+1)} >= t_{*,n}$ it follows that:

$$< a^{(i)}, a^{(k+1)} >= t_{*,n}, \quad i = \overline{1, k},$$

that is $a^{(k+1)} \in H_1 \bigcap \cdots \bigcap H_k$.

One can observe that $H_1 \bigcap \cdots \bigcap H_d$ contains n^d distinct points, which are given by the relations:

$$< x, a^{(1)} > = \ t_{i_1,n}, \ \ldots, \ < x, a^{(d)} > = \ t_{i_d,n}, \ 1 \leq i_1, \ldots, i_d \leq n. \qquad (2.7)$$

Theorem 2.1. Let $a^{(1)}, \ldots, a^{(d)}$ be defined as above and let H_k be the surface defined by $H_k = \{x \in \mathbb{R}^d : \widetilde{K_n}(W_\mu; x, a^{(k)}) = 0\}$. Then the modified method of the Reproducing Kernel yields, a cubature formula of degree $2n+1$, based on $a^{(1)}, \ldots, a^{(d)}$ and the n^d distinct points determined by (2.7) and have the form

$$Q_n(f) = \sum_{i=1}^{d} \lambda_i [f(a^{(i)}) + f(-a^{(i)})]/2 + \sum_{j=1}^{n^d} \mu_j f(x^{(j)}), \ \forall f \in \mathbb{P}^d_{2n+1}, \qquad (2.8)$$

where $\lambda_i = 1/\widetilde{K_n}(a^{(i)}, a^{(i)})$.

We obtain by using (2.6) that

$$\lambda_i = 1/\widetilde{K_n}(W_\mu; a^{(i)}, a^{(i)}) = 1/C_n^{(\mu+(d+1)/2)}(1) = 1/\binom{n+2\mu+d}{n}. \qquad (2.9)$$

For fixed d and n, the others weights μ_j in (2.8) can be determined by solving a linear system of equations.

From the fact that in definition of $a^{(k)}$, if we use the condition

$$< a^{(k-1)}, a^{(k)} >= t_{*,n},$$

we remark that one can choose $t_{*,n}$ to be any zero of the polynomial $C_n^{(\mu+(d+1)/2)}(t)$ and we can get many different formulas from this method.

Remark 2.1. When n is an odd integer, then $C_n^{(\mu+(d+1)/2)}$ is an odd polynomial, and it follows that $t = 0$ is a zero of this polynomial.

If we take $t_{*,n} = 0$ in the definition of $a^{(k)}$ in the above construction, then we obtain: $a^{(1)} = e_1, \ldots, a^{(d)} = e_d$, where $\{e_i, \ i = \overline{1, d}\}$ is the standard basis of $\mathbb{R}^d$, that is, $e_1 = (1, 0, \ldots, 0)$, $\quad e_2 = (0, 1, 0, \ldots, 0), \ldots, e_d = (0, \ldots, 0, 1)$.

But from (2.6) it follows that

$$\widetilde{K_n}(W_\mu; x, e_k) = C_n^{(\mu+(d+1)/2)}(x_k), \ k = \overline{1, d}$$

and we observe that the n^d intersection points of $H_1 \bigcap \cdots \bigcap H_d$, namely $\{x^{(i)},\ i = \overline{1, n^d}\}$, are the tensor product of the zeros obtained from (2.7).

Let n be an odd integer and let $t_{1,n}, \ldots, t_{n,n}$ be the zeros of $C_n^{(\mu+(d+1)/2)}(t)$. Then there is a cubature of degree $2n+1$ on B^d of the form:

$$\int_{B^d} f(x)W_\mu(x)dx = \frac{1}{\begin{pmatrix} n+2\mu+d \\ n \end{pmatrix}} \sum_{k=1}^{n} \Big[f(e_k) + f(-e_k) \Big] /2+ \tag{2.10}$$

$$+ \sum_{k_1=1}^{n} \cdots \sum_{k_d=1}^{n} \mu_{k_1,\ldots,k_d} f(t_{k_1,n}, \ldots, t_{k_d,n}),\ \forall f \in \mathbb{P}_{2n+1}^d.$$

The weights μ_j in the formula (2.10) can be computed by solving a linear system equations for a given n and d.

In the case $d = 2$, we can consider the polynomials $l_{k,n}$ defined by:

$$l_{k,n} = \prod_{i=1, i\neq k}^{n} \frac{x - t_{i,n}}{t_{k,n} - t_{i,n}} = \frac{C_n^{(\mu+(d+1)/2)}(t)}{(2\mu+d+1)C_{n-1}^{\mu+(d+3)/2)}(t_{k,n})(x - t_{k,n})},$$

which are the fundamental interpolation polynomials based on the zeros of $C_n^{(\mu+(d+1)/2)}(t)$ which satisfies the interpolation conditions: $l_{k,n}(t_{j,n}) = \delta_{k,j}$, by using (1.4).

One observe that the polynomial $l_{k_1,n}(x_1)l_{k_2,n}(x_2)(1 - x_1^2 - x_2^2)$ is of degree $2(n-1) + 2 = 2n$, then it will be integrated exactly by the cubature formula (2.10), and from the interpolation property of $l_{k,n}$ we will obtain the values of the weights are

$$\mu_{k_1,k_2} = \int_{B^2} l_{k_1,n}(x_1)l_{k_2,n}(x_2)(1 - x_1^2 - x_2^2)W_\mu(x_1, x_2)dx_1dx_2.$$

The formula (2.10) uses the tensor product of nodes of an one variable quadrature rule. The points $\{t_{1,n}, \ldots, t_{n,n}\}$ are nodes of a Gaussian quadrature formula of degree $2n-1$ on $[-1, 1]$ for the measure: $W(x) = (1-x^2)^{\mu+d/2}dx$ on $[-1, 1]$. Moreover, $\{-1, t_{1,n}, \ldots, t_{n,n}, 1\}$ form the nodes of a *Gauss − Lobatto* type quadrature formula of degree $2n+1$,

$$\int_{-1}^{1} f(x)(1-x^2)^{\mu+d/2}dx = Af(-1) + \sum_{k=1}^{n} \lambda_k f(t_{k,n}) + Af(1),\ \forall f \in \mathbb{P}_{2n+1}^1. \tag{2.11}$$

The tensor product of $\{-1, t_{1,n}, ..., t_{n,n}, 1\}$, can be used as nodes in the following product formula of degree $2n+1$ for the product weight function:

$$W(x) = \prod_{k=1}^{d} (1 - x_k)^{\mu+d/2} \text{ on } [-1,1]^d,$$

$$\int_{[-1,1]^d} f(x) \prod_{k=1}^{d} (1 - x_k)^{\mu+d/2} dx = \sum_{k_1=0}^{n+1} \cdots \sum_{k_d=0}^{n+1} \lambda_{k_1} \ldots \lambda_{k_d} f(t_{k_1,n}, \ldots, t_{k_d,n}), \quad (2.12)$$

for $\forall f \in \mathbb{P}^d_{2n+1}$, where $t_{0,n} = -1, \quad t_{n+1,n} = 1$ and $\lambda_0 = \lambda_{n+1} = A$.

It was showed that some nodes of the cubature formulas constructed above can lie outside of the unit ball B^d. But we can choose different values $a^{(k)}$ in order to construct formulas with all nodes inside of B^d.

2.2 Samples of cubature formulas of lower degree with nodes inside B^d

We use the modified method of the Reproducing Kernel to construct cubature formulas of lower degree with nodes inside B^d.

a. Formulas of degree 5

We choose $a^{(1)} = (0, 0, \ldots, 0)$ the origin of $\mathbb{R}^d$ and we define $a^{(k+1)}$, $1 \le k \le d-1$ by

$$a^{(k+1)} = (\sqrt{\frac{1}{2\mu+d+3}}, \ldots, \sqrt{\frac{1}{2\mu+d+3}}, \sqrt{\frac{d+3-k}{2\mu+d+3}}, 0 \ldots, 0) \quad (2.13)$$

which has $d-k$ zero components.

From the properties of the *Gegenbauer* polynomials [7], we have:

$$C_2^{(\lambda)}(t) = \lambda[2(\lambda+1)t^2 - 1], \text{ for } n = 2,$$

where $\lambda = \mu + (d+1)/2$, and follows that

$$\tilde{K}_2(W_\mu; x, y) = \lambda\Big[(2\mu+d+3) < x, y >^2 + (2\mu+d+3)(1-|x|^2)(1-|y|^2)/(2\mu+1) - 1\Big]. \quad (2.14)$$

If we take, $a^{(1)} = (0, \ldots, 0)$, it follows from the formula of $\tilde{K}_2(W_\mu; x, y)$ that $H_1 = \{x : K_2(x, a^{(1)}) = 0\} = \{x : |x|^2 = (d+2)/(2\mu+d+3)\}$ and we require that the chosen point $a^{(k+1)}$ from (2.13), belongs to H_k and we obtain:

$$\tilde{K}_2(W_\mu; x, a^{k+1}) = (\mu + \frac{d+1}{2})\Big[x_1^2 + \cdots + x_{k-1}^2 + (d+3-k)x_k^2 - \|x\|^2\Big],$$

from which we obtain $a^{(k+1)} \in \bigcap_{i=1}^{k} H_i$ and

$$\bigcap_{i=1}^{d} H_i = \{(\pm\sqrt{\frac{1}{2\mu+d+3}}, \dots, \pm\sqrt{\frac{1}{2\mu+d+3}}, \pm\sqrt{\frac{3}{2\mu+d+3}})\} \tag{2.15}$$

Thus, $\bigcap_{i=1}^{d} H_i$ has the 2^d intersection points obtained from (2.15).

Now we can apply the modified Reproducing Kernel method, from which one results that the nodes of the cubature formula are $\{a^{(i)},\ i = \overline{1,d}\}$ from (2.13) and $(x^{(j)},\ j = 1, 2^d)$ from (2.15) and these nodes generates a cubature formula of degree 5 on B^d of the form (2.8). Using the formula of $\tilde{K}_2(W_\mu; x, y)$ one get the coefficients of the formula for this choice of the nodes $a^{(k+1)}$, if we consider $n = 2$

$$\lambda_1 = 1/\tilde{K}_2(W_\mu; 0, 0) = \frac{2(2\mu+1)}{(2\mu+d+1)(d+2)}, \tag{2.16}$$

$$\lambda_{k+1} = 1/\tilde{K}_2(W_\mu; a_{k+1}, a_{k+1}) = \frac{2(2\mu+d+3)}{(2\mu+d+1)(d+2-k)(d+3-k)},\ k = \overline{2,d}.$$

Then there exists the weights μ_ξ such that the following cubature formula is of degree 5 for W_μ on B^d [12].

$$\int_{B^d} f(x)W_\mu(x)dx = \frac{2(2\mu+1)}{(2\mu+d+1)(d+2)} f(0)$$

$$+\frac{2\mu+d+3}{2\mu+d+1} \sum_{k=1}^{d-1} \frac{f(a^{(k+1)}) + f(-a^{(k+1)})}{(d+2-k)(d+3-k)}$$

$$+ \sum_{\xi \in \{-1,1\}^d} \mu_\xi f\left(\xi_1\sqrt{\frac{1}{2\mu+d+3}}, \dots, \xi_{d-1}\sqrt{\frac{1}{2\mu+d+3}}, \xi_d\sqrt{\frac{3}{2\mu+d+3}}\right). \tag{2.17}$$

In this formula the weights $\mu_\xi,\ \xi = (\xi_1, \dots, \xi_d) \in \{-1,1\}^d$ can be determined by the condition that the formula must be exact for polynomials of degree 5.

In the case of $d = 2$, we have the explicit formula

$$\int_{B^2} f(x)W_\mu(x)dx = \frac{2(2\mu+1)}{4(2\mu+3)} f(0) + \frac{2\mu+5}{12(2\mu+3)} \big[f(2/\sqrt{2\mu+5}, 0) \tag{2.18}$$

$$+f(-2/\sqrt{2\mu+5}, 0)\big] + \frac{2\mu+5}{12(2\mu+3)} \sum f(\pm 1/\sqrt{2\mu+5}, \pm\sqrt{3}/\sqrt{2\mu+5}).$$

The formula on B^d uses $N = 2^d + 2d - 1$ nodes. According with *Möller's* lower bound [2], the cubature formula of degree 5 must have at least $N^* \geq d(d+1)+1$ nodes, then the formula (2.18) which have $N = 2^2 + 2 \cdot 2 - 1 = 7$ is minimal.

For $d = 3$, the cubature formula on B^3, which was constructed by using (2.10) in [12], have $N = 13$ nodes and is minimal; for $d = 5$, $N = 2^5 + 2 \cdot 5 - 1 = 41$ nodes which is more that the lower bound of $N^* = 5(5+1) + 1 = 31$.

Finally we obtain the formula (2.17). To determine the other coefficients, one can require that the formula be exact for the polynomials of degree at most 5.

For $d = 3$, we can choose $f(x)$ to be the test functions $x_1,\ x_1x_2,\ x_1^2,\ x_1x_2x_3$.

For the case of $d > 3$, it is useful the following formula for the nonzero moments of the weight function $W_\mu = W_\mu(x)$ ([12])

$$\int_{B^d} x_1^{2k_1} \dots x_d^{2k_d} W_\mu(x)dx = \frac{\Gamma(\mu + (d+1)/2)\ \Gamma(k_1 + 1/2) \dots\ \Gamma(k_d + 1/2)}{\pi^{d/2}\Gamma(\mu + (d+1)/2 + k_1 + \cdots + k_d)}.$$

3. **Cubature formulas on the triangle using the reproducing kernel method**

We consider now, cubature formulas on the triangle using the compact formula in [12], for a family of weight functions on a d-dimensional simplex.We use the following notations: $x \in \mathbb{R}^d$, $|x|_1 = |x_1| + \cdots + |x_d|$, the l^1 norm of x, $|\alpha|_1 = \alpha_1 + \cdots + \alpha_d$, the length of multiindex $\alpha \in \mathbb{N}^d$ and the standard simplex:

$$T^d = \{x \in \mathbb{R}^d :\ x_1 \geq 0, \dots, x_d \geq 0,\ 1 - |x|_1 \geq 0\}.$$

We remark that, for $d = 2$ we have T^2 which is the triangle with vertices $(0,0), (1,0)$ and $(0,1)$.

In [12] was found the compact formula for the Reproducing Kernel with respect to the weight function:

$$W_\alpha(x) = w_\alpha x_1^{\alpha_1 - 1/2} \dots x_d^{\alpha_d - 1/2}(1 - |x|_1)^{\alpha_{d+1} - 1/2},\ \alpha_i \geq 0, \tag{3.1}$$

where w_α is the normalization constant such that $\int_{T^d} W_\alpha(x)dx = 1$, namely,

$$w_\alpha = \frac{\Gamma(|\alpha|_1 + (d+1)/2)}{\Gamma(\alpha_1 + 1/2) \dots \Gamma(\alpha_{d+1} + 1/2)}.$$

Then the reproducing Kernel $K_n(W_\alpha)$ given in terms of *Gegenbauer* polynomials, has the expression [12]:

$$K_n(W_\alpha; x, y) = \int_{[-1,1]^{d+1}} C_{2n}^{(|\alpha|_1+(d+1)/2)}(\sqrt{x_1 y_1}\ t_1 + \cdots + \sqrt{x_{d+1} y_{d+1}}\ t_{d+1}). \quad (3.2)$$

$$\cdot \prod_{i=1}^{d+1} c_{\alpha_i}(1-t_i^2)^{\alpha_i - 1} dt,$$

where

$$x, y \in T^d,\ x_{d+1} = 1 - |x|_1,\ y_{d+1} = 1 - |y|_1,$$

and we use limit (2.4) in the case when have one $\alpha_i = 0$.

If we take $y = e_i = (0, \ldots 0, 1, 0, \ldots 0)$, the i-th element of the standard basis, with the i-th component =1, of $\mathbb{R}^d$, $1 \le i \le d$, then we have the following explicit formula:

$$K_n(W_\alpha; x, e_i) = A_{\alpha,i} P_n^{(|\alpha|_1 + d/2 - \alpha_i, \alpha_i - 1/2)}(2x_i - 1)$$

where

$$A_{\alpha,i} = C_{2n}^{(|\alpha|_1+(d+1)/2)}(0) / P_n^{(|\alpha|_1 + d/2 - \alpha_i, \alpha_i - 1/2)}(-1)$$

(see [12]).

This formula was derived in [14] from (3.2) using a product formula for *Jacobi* polynomials.

We observe that, e_i is not a common zero of $\mathbf{P}_n$. This follows from the expression of $\mathbf{P}_n^T(x)\mathbf{P}_n(y) = \sum_k P_k^n(x) P_k^n(y)$.

Let $d = 2$ and $\alpha_1 = \alpha_2 = \alpha_3 = 1/2$. Then the weight function W_α becomes a multiple of unit weight function, denoted by $W_{1/2}$, and we have: $W_{1/2}(x) = 2$.

In this case, the Reproducing Kernel takes the form:

$$K_n(W_{1/2}; x, y) = \frac{1}{\pi^3} \int_{[-1,1]^3} C_{2n}^{(3)}(\sqrt{x_1 y_1} t_1 + \sqrt{x_2 y_2} t_2 + \sqrt{x_3 y_3} t_3) \prod_{i=1}^{3} (1 - t_i^2)^{-1/2} dt$$

For $\alpha = 0,$ we have $W_0(x) = (x_1 x_2 x_3)^{-1/2}/2\pi$.

In [11] was shown that any cubature formula for W_0 with all nodes inside T^2 corresponds to a cubature formula on a sphere S^2. In this case, the Reproducing

Kernel can be represented in the following simple form:

$$K_n(W_0; x, y) = \frac{1}{4}\sum C_{2n}^{(3/2)}(\sqrt{x_1y_1} \pm \sqrt{x_2y_2} \pm \sqrt{x_3y_3}),$$

where the sum is over all possible sign changes, and this formula follows from (3.2) by taking limits (2.4).

Samples of cubature formulas on the triangle

For $n = 2$, we have the following explicit formula for $K_n(W_{1/2}; x, y)$

$$K_2(W_{1/2}; x, y) = 6(1 - 10(x_1y_1 + x_2y_2 + x_3y_3) + 60(x_1x_2y_1y_2 + x_1x_3y_1y_3+$$

$$+x_2x_3y_2y_3) + 15(x_1^2y_1^2 + x_2^2y_2^2 + x_3^2y_3^2)).$$

If we take $a^{(1)} = (1, 0)$, one obtain that $K_2(W_{1/2}, x, (1, 0))$ has two zeros,

$$z_1 = (5 - \sqrt{10})/15,\ z_2 = (5 + \sqrt{10})/15.$$

From this fact, it follows that $K_2(W_{1/2}, x, (1, 0))$ and $K_2(W_{1/2}, x, (z_1, 0))$ have 4 distinct common zeros:

$$\left((5 - \sqrt{10})/15, (70 - 7\sqrt{10} \pm \sqrt{10(233 - 62\sqrt{10})}/90\right)$$

$$\left((5 + \sqrt{10})/15, (30 - 3\sqrt{10} \pm \sqrt{3(110 - 20\sqrt{10})}/90\right).$$

4. The construction of cubature formulas by using the *Chebyshev* orthogonal polynomials and the reproducing kernel method

Let us consider, the *Chebyshev* polynomial of degree n,

$$T_n^*(x) = cosn\theta,\ x = cos\theta,$$

that is

$$T_n^*(x) = cos(narcosx).$$

The zeros of T_n^* are $x_k = \frac{(2k-1)\pi}{2n}$, $k = \overline{1, n}$, and T_n^* are orthogonal with respect to the *Chebyshev* weight function $w_1(x) = (1 - x^2)^{-1/2}$ on $[-1, 1]$.

The zeros of T_n^* can be selected as the nodes of the Gaussian quadrature formula with respect to $w(x)$ and these zeros can be used to construct a compact interpolation formula. Let we denote the classical *Chebyshev* weight of the first kind

$$w_1(x) = \frac{1}{\pi}\frac{1}{\sqrt{1-x^2}},\ x \in (-1,1)$$

Then the orthonormal polynomials with respect to w_1 are

$$T_0(x) = 1,\ T_k(x) = \sqrt{2}cosk\theta,\ k \geq 1,\ x = cos\theta \quad \text{and} \quad \int_{-1}^{1} w_1(x)dx = 1.$$

Next, we can consider the product *Chebyshev* weight function on $[-1,1]^2$ defined by

$$W^{(2)}(x,y) = w_1(x)w_1(y) = \frac{1}{\pi^2}\frac{1}{\sqrt{1-x^2}}\frac{1}{\sqrt{1-y^2}},\ (x,y) \in [-1,1]^2. \tag{4.1}$$

One can verify that the polynomials defined by

$$P_k^n(x,y) = T_{n-k}(x)T_k(y),\ k = \overline{0,n},\ n \in \mathbb{N}_0, \tag{4.2}$$

where P_k^n is of degree exactly n are orthogonal with respect to $W^{(2)}(x,y)$.

In [10] was established the following relations. If we denote $\mathbf{P}_n = (P_0^n, ..., P_n^n)^T, n \in \mathbb{N}_0$, the vector of the polynomials of degree exactly n in (4.2) and the matrices,

$$A_{n,1} = \frac{1}{2}\begin{bmatrix} 1 & 0 & & 0 & \dots & 0 \\ 0 & 1 & & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & & \vdots & \\ 0 & 0 & & \dots & \sqrt{2} & 0 \end{bmatrix}, \qquad A_{n,2} = \frac{1}{2}\begin{bmatrix} 0 & \sqrt{2} & & 0 & \dots & 0 \\ 0 & 0 & & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & & \vdots & \\ 0 & 0 & & \dots & \dots & 1 \end{bmatrix},$$

it can be verified that product *Chebyshev* polynomials satisfy the three-term relation

$$x_i\mathbf{P}_n(x) = A_{n,i}\mathbf{P}_{n+1}(x) + A_{n-1,i}^T\mathbf{P}_{n-1}(x),\ i = 1,2, x = (x_1,x_2) \text{ or } x = (x,y) \tag{4.3}$$

For $x, y \in \mathbb{R}^2$, the Reproducing Kernel of the product *Chebyshev* polynomials is defined by

$$K_n(x,y) = \sum_{k=0}^{n-1}\sum_{j=0}^{k} P_j^k(x)P_j^k(y) = \sum_{k=0}^{n-1}\mathbf{P}_k^T(x)\mathbf{P}_k(y)$$

and $\mathbf{P}_n^T(x)\mathbf{P}_n(y) = K_n(x,y) - K_{n-1}(x,y)$.

If one consider $x = (cos\theta_1, cos\theta_2),\ y = (cos\varphi_1, cos\varphi_2)$, then we have the compact formula [10].

$$K_n(x,y) = D_n(\theta_1+\varphi_1, \theta_2+\varphi_2) + D_n(\theta_1+\varphi_1, \theta_2-\varphi_2) + D_n(\theta_1-\varphi_1, \theta_2+\varphi_2)$$

$$+D_n(\theta_1-\varphi_1, \theta_2-\varphi_2),$$

where the function D_n has the form

$$D_n(\theta_1,\theta_2) = \frac{1}{2}\frac{cos(n-\frac{1}{2})\theta_1 cos\frac{\theta_1}{2} - cos(n-\frac{1}{2})\theta_2 cos\frac{\theta_2}{2}}{cos\theta_1 - cos\theta_2}.$$

One can use these formulas in order to obtain a compact formula for the Lagrange interpolation, which will be used to construct a cubature formula of degree $2n-1$ with respect to $W^{(2)}(x,y)$ of the form

$$I_n(f) = \int_{[-1,1]^2} f(x,y)W^{(2)}(x,y)dxdy \simeq Q_n(f), \tag{4.4}$$

where $Q_n(f) = \sum_{k=0}^{N} \lambda_k f(x_k),\ \lambda_k > 0,\ x_k \in \mathbb{R}^2$, so that we have

$$I_n(P) = Q_n(P),\ \forall P \in \mathbb{P}^2_{2n-1}.$$

According to a general result of *Möller* for centrally symmetric weight functions, for example one can consider $W^{(2)}(x,y) = w_1(x)w_1(y)$, the number of nodes in the cubature formula satisfies

$$N \geq dim\mathbb{P}^2_{n-1} + [n/2] = \binom{n+1}{2} + [n/2].$$

Let consider z_k be the points $z_k = z_{k,n} = cos\frac{k\pi}{n},\ k = \overline{0,n}.$

In [10] was stated, based on the three-term recurrence relation (4.3), that a cubature formula exists when the following matrix equations in the variable V are solvable

$$A_{n-1,1}(VV^T - I)A^T_{n-1,2} = A_{n-1,2}(VV^T - I)A^T_{n-1,1} \tag{4.5}$$

$$\text{and} \quad V^T A^T_{n-1,1}A_{n-1,2}V = V^T A^T_{n-1,2}A_{n-1,1}V,$$

where V is a matrix of size $(n+1)\times\sigma,\ \sigma = [n/2]$ or $\sigma = [n/2]+1$.

If $n = 2m$ in [10] was showed that a solution of (4.5) is

$$T_{n-(k-1)}(x)T_{k-1}(y) - T_{k-1}(x)T_{n-(k-1)}(y), 1 \leq k \leq n/2 + 1, \tag{4.6}$$

which corresponds to (A), and if $n = 2m - 1$, a solution of (4.5) is

$$T_{n-(k-1)}(x)T_{k-1}(y) - T_{k-1}(x)T_{n-(k-1)}(y), \quad 1 \leq k \leq (n+1)/2, \tag{4.7}$$

corresponds to (B).

If a cubature formula exists, we can consider the Lagrange interpolation problem based on the nodes of the cubature formula which consists in construction of a unique polynomial which is the solution of the problem to determining $P = P(x)$ so that $P(x_k) = f(x_k),\ k = \overline{1, N}$.

In [8], was proved that one can consider the subspace

$$\mathcal{V}_n^2 = \mathbb{P}_{n-1}^2 \bigcup span\{V^+\mathbf{P}_n\},$$

where V^+ is the unique *Moore-Penrose* generalized inverse of V, and in our case we have V with full rank and we have $V^+ = (V^TV)^{-1}V^T$.

For $(x, y) \in \mathbb{R}^2$, was used the following expression of the Reproducing Kernel

$$K_n^*(x, y) = K_n(x, y) + [V^+\mathbf{P}_n(x)]^T V^+\mathbf{P}_n(y). \tag{4.8}$$

Using a modified *Christoffel-Darboux* formula, was showed in [10] that $K_n^*(x_k, x_j) = 0$ for $k \neq j$ and $K_n^*(x_k, x_k) \neq 0$.

Finally, it follows that

$$(L_nf)(x) = \sum_{k=1}^{N} \frac{K_n^*(x, x_k)}{K_n^*(x_k, x_k)} f(x_k) \tag{4.9}$$

and we have

$$\int_{[-1,1]^2} (L_nf)(x)W_0(x)dx = \sum_{k=1}^{N} \lambda_k f(x_k) = I_n(f).$$

From the condition on P_j^k and the definition of $K_n^*(\cdot, \cdot)$ it follows that the coefficients in the cubature formula are given by the expression $\lambda_k = 1/K_n^*(x_k, x_k)$

If $n = 2m$, the interpolation nodes are

$$x_{2i,2j+1} = (z_{2i}, z_{2j+1}), \quad i = \overline{0, m},\ j = \overline{0, m-1} \tag{4.10}$$

$$x_{2i+1,2j} = (z_{2i+1}, z_{2j}), \quad i = \overline{0, m-1},\ j = \overline{0, m}.$$

From (4.7) and the expression of $K_n^*(x,y)$ one can obtain

$$K_n^*(x, x_{k,l}) = \frac{1}{2}[K_n(x, x_{k,l}) + K_{n-1}(x, x_{k,l})] - \frac{1}{2}(-1)^k[T_n(x) - T_n(y)].$$

Finally, one can obtain

$$K_n^*(x_{0,2j+1}, x_{0,2j+1}) = n^2,\ K_n^*(x_{2i,2j+1}, x_{2i,2j+1}) = n^2/2,$$

$$K_n^*(x_{2i+1,0}, x_{2i+1,0}) = n^2,\ K_n^*(x_{2i+1,2j}, x_{2i+1,2j}) = n^2/2,\ i > 0, j > 0.$$

If $n = 2m-1$, the interpolation nodes are

$$x_{2i,2j} = (z_{2i}, z_{2j}), \quad i, j = \overline{0, m-1}$$

$$x_{2i+1,2j+1} = (z_{2i+1}, z_{2j+1}), \quad i, j = \overline{0, m-1},$$

from which, was derived

$$K_n^*(x, x_{k,l}) = \frac{1}{2}[K_n(x, x_{k,l}) + K_{n-1}(x, x_{k,l})] - \frac{1}{2}(-1)^k[T_n(x) + T_n(y)],$$

from which was obtained

$$K_n^*(x_{2i,2j}, x_{2i,2j}) = \begin{cases} n^2/2, & \text{if } 0 < i, j \le m-1 \\ n^2, & \text{if } i = 0 \text{ or } j = 0,\ i + j > 0 \\ 2n^2, & \text{if } i = j = 0, \end{cases}$$

$$K_n^*(x_{2i+1,2j+1}, x_{2i+1,2j+1}) = \begin{cases} n^2/2, & \text{if } 0 \le i, j < m-1 \\ n^2, & \text{if } i = m-1 \text{ or } j = m-1,\ i + j < 2m-2 \\ 2n^2, & \text{if } i = j = m-1. \end{cases}$$

In [14] was proved the mean convergence of Lagrange interpolation formula corresponding to the weight function $W^{(2)}(x,y)$ and by integrating this formula one can arrive to the following cubature formulas

Based on the nodes (x_i, x_j), we obtain the cubature formulas:

A) For $n = 2m$,

$$(A)\ \frac{1}{\pi^2}\int_{-1}^{1}\int_{-1}^{1} f(x,y)\frac{dxdy}{\sqrt{1-x^2}\sqrt{1-y^2}} = \frac{2}{n^2}{\sum_{i=0}^{\frac{n}{2}}}'' \sum_{j=0}^{\frac{n}{2}-1} f(z_{2i}, z_{2j+1}) +$$

$$+\frac{2}{n^2}\sum_{i=0}^{\frac{n}{2}-1} {\sum_{j=0}^{\frac{n}{2}}}'' f(z_{2i+1}, z_{2j}), \forall f \in \mathbb{P}^2_{2n-1}$$

B) For $n = 2m - 1$,

$$(B)\ \frac{1}{\pi^2}\int_{-1}^{1}\int_{-1}^{1} f(x,y)\frac{dxdy}{\sqrt{1-x^2}\sqrt{1-y^2}} = \frac{2}{n^2}\sum_{i=0}^{\frac{n-1}{2}}\sum_{j=0}^{\frac{n-1}{2}} f(z_{2i}, z_{2j}) +$$

$$+\frac{2}{n^2}\sum_{i=0}^{\frac{n-1}{2}}\sum_{j=0}^{\frac{n-1}{2}} f(z_{n-2i}, z_{n-2j}), \forall f \in \mathbb{P}^2_{2n-1},$$

where Σ' means that the first term in summation is halved.

References

[1] Cools, R., Mysovskikh, I.P., Schmid, H.J., *Cubature Formulae and Orthogonal Polynomials*, J.Comput.& Appl. Math, **127**(2001), 121-152.

[2] Möller, H.M., *Polynomideale und Kubaturformeln*, Thesis, Univ. Dortmund, 1973.

[3] Mysovskikh, I.P., *Interpolatory Cubature formulas*, Nauka, Moscow, (1981), (Russian).

[4] Mysovskikh, I.P., *A representation of the reproducing Kernel of a sphere*, Comp. Math. Math. Phys., **36**(1996), 303-308.

[5] Stancu, D.D., *Generalizarea unor formule de interpolare pentru funcţiile de mai multe variabile şi unele consideraţii asupra formulelor de integrare numerică a lui Gauss*, Ed. Acad. R.P.R., Bul.Şt. Sec. St. Mat. şi Fiz., **2**(1957).

[6] Stroud, A.H., *Approximate calculation of multiple integrals*, Prentice Hall, Englewood Cliffs, NJ, (1971).

[7] Szegö, H., *Orthogonal polynomials*, 4th ed., Amer. Math. Soc. Collaq. Publ. vol. 23, Providence, RI, (1975).

[8] Xu, Y., *Common zeros of polynomials in several variables and Higher dimensional quadrature*, Pitman Research Notes in Mathematics series, Longman, Essex, (1994).

[9] Xu, Y., *A Class of Bivariate orthogonal polynomials and cubature formulas*, Numer.Math., **69**(1994), 233-241.

[10] Xu, Y., *Lagrange interpolation on Chebyshev points of two variables*, J. Approx. Theory, **87**(1996), 220-238.

[11] Xu, Y., *Orthogonal polynomials and cubature formulae on spheres and on balls*, SIAM J. Math. Anal., **29**(1998), 779-793.

[12] Xu, Y., *Constructing cubature formulae by the method of reproducing kernel*, Numer. Math., **85**(2000), 155-173.

[13] Xu, Y., *Li - Sumability of the product Jacobi series*, J. Approx Theory, **104**(2000), 287-301.

[14] Xu, Y., *Summability of Fourier orthogonal series for Jacobi weight on a ball in* $\mathbb{R}^d$, Trans. Amer. Math. Soc.

[15] Xu, Y., *Representation of reproducing Kernels and the Lebesque constants on the Ball*, J. Approx. Theory, **112**(2001), 295-310.

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