STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 4, December 2009

SUBORDINATION CHAINS AND QUASICONFORMAL EXTENSIONS OF HOLOMORPHIC MAPPINGS IN $\mathbb{C}^n$

PAULA CURT

Abstract. Let B be the unit ball in $\mathbb{C}^n$ with respect to the Euclidean norm. In this paper, by using the method of subordination chains, we obtain a sufficient condition for a normalized quasiregular mapping f to be extended to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

1. **Introduction and preliminaries**

J.A. Pfaltzgraff [12] proved that if $0 \leq q < 1$ and $f \in \mathcal{H}(B)$ is a quasiregular mapping, which satisfies the condition

$$(1 - \|z\|^2)\|[Df(z)]^{-1}D^2f(z)(z,\cdot)\| \leq q, \quad z \in B,$$

then f is biholomorphic on B and extends to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

The problem of quasiconformal extensions for quasiregular holomorphic mappings on the unit ball in $\mathbb{C}^n$ has been studied by H. Hamada and G. Kohr [11], P. Curt [5], P. Curt and G. Kohr [7], [8], [9].

In this paper we shall generalize the results due to J.A. Pfaltzgraff [12], P. Curt [5].

Let $\mathbb{C}^n$ denote the space of n-complex variables $z = (z_1, \ldots, z_n)$ with the usual inner product $\langle z, w\rangle = \sum_{i=1}^{n} z_i\overline{w}_i$ and Euclidean norm $\|z\| = \langle z, z\rangle^{1/2}$. Let B denote the open unit ball in $\mathbb{C}^n$ and let U be the unit disc in $\mathbb{C}$.

Received by the editors: 11.10.2008.

2000 *Mathematics Subject Classification.* Primary 32H, Secondary 30C45.

Key words and phrases. Loewner chains, quasiregular mapping, quasiconformal mapping.

Let $\mathcal{H}(\Omega)$ be the set of holomorphic mappings from a domain Ω in $\mathbb{C}^n$ into $\mathbb{C}^n$. If $f \in \mathcal{H}(B)$, let $J_f(z) = \det Df(z)$ be the complex jacobian determinant of f at z. Also let $\mathcal{L}(\mathbb{C}^n)$ be the space of continuous linear mappings from $\mathbb{C}^n$ into $\mathbb{C}^n$ with the standard operator norm

$$\|A\| = \sup\{\|Az\| : \|z\| = 1\},$$

and let I be the identity in $\mathcal{L}(\mathbb{C}^n)$. A mapping $f \in \mathcal{H}(B)$ is said to be normalized if $f(0) = 0$ and $Df(0) = I$.

We say that a mapping $f \in \mathcal{H}(B)$ is K-quasiregular, $K \geq 1$, if

$$\|Df(z)\|^n \leq K|\det Df(z)|, \quad z \in B.$$

A mapping $f \in \mathcal{H}(B)$ is called quasiregular if f is K-quasiregular for some $K \geq 1$. It is well known that quasiregular holomorphic mappings are locally biholomorphic.

Definition 1.1. Let G and G' be domains in $\mathbb{R}^m$. A homeomorphism $f : G \to G'$ is said to be K-quasiconformal if it is differentiable a.e., ACL (absolutely continuous on lines) and

$$\|Df(z)\|^m \leq K|\det Df(x)| \text{ a.e. } x \in G,$$

where $Df(x)$ denotes the real Jacobian matrix of f and K is a constant.

Note that a K-quasiregular biholomorphic mapping is K^2-quasiconformal.

If $f, g \in \mathcal{H}(B)$, we say that f is subordinate to g (and write $f \prec g$) if there exists a Schwarz mapping v (i.e. $v \in \mathcal{H}(B)$ and $\|v(z)\| \leq \|z\|$, $z \in B$) such that $f(z) = g(v(z))$, $z \in B$.

Definition 1.2. A mapping $L : B \times [0, \infty) \to \mathbb{C}^n$ is called a subordination chain if the following conditions hold:

(i) $L(0, t) = 0$ and $L(\cdot, t) \in \mathcal{H}(B)$ for $t \geq 0$;

(ii) $L(\cdot, s) \prec L(\cdot, t)$ for $0 \leq s \leq t < \infty$.

If $L(z, t)$ is a subordination chain such that $L(\cdot, t)$ is biholomorphic on B for $t \in [0, \infty)$, then we say that $L(z, t)$ is a univalent subordination chain (or a Loewner chain).

If $L(z,t)$ is a univalent subordination chain such that $DL(0,t) = e^t I$, we say that $L(z,t)$ is a normalized Loewner chain.

An important role in our discussion is played by the n-dimensional version of the class of holomorphic functions on the unit disc with positive real part

$$\mathcal{N} = \{h \in \mathcal{H}(B): \ h(0) = 0, \ \mathrm{Re}\ \langle h(z), z\rangle > 0, \ z \in B \setminus \{0\}\}$$

$$\mathcal{M} = \{h \in \mathcal{N}, \ Dh(0) = I\}.$$

The authors ([10, Theorem 1.10]) and [6, Theorem 2.3]) proved that normalized univalent subordination chains satisfy the generalized Loewner differential equation. Using an elementary change of variable, it is not difficult to reformulate the mentioned result in the case of normalized subordination chains $L(z,t) = a(t)z + \dots$, where $a : [0,\infty) \to \mathbb{C}$, $a \in C^1([0,\infty))$, $a(0) = 1$, and $a(t) \to \infty$ as $t \to \infty$.

Theorem 1.1. *Let $L(z,t) : B \times [0,\infty) \to \mathbb{C}^n$ be a Loewner chain such that $L(z,t) = a(t)z + \dots$, where $a \in C^1([0,\infty))$, $a(0) = 1$, and $\lim_{n\to\infty} |a(t)| = \infty$. Then there exists a mapping $h = h(z,t) : B \times [0,\infty) \to \mathbb{C}^n$ such that $h(\cdot,t) \in \mathcal{N}$ for $t \geq 0$, $h(z,\cdot)$ is measurable on $[0,\infty)$ for $z \in B$, and*

$$\frac{\partial L}{\partial z}(z,t) = DL(z,t)h(z,t), \ \text{a.e. } t \geq 0, \ z \in B.$$

Recently P. Curt and G. Kohr [9] proved the following result.

Theorem 1.2. *Let $L(z,t) : B \times [0,\infty) \to \mathbb{C}^n$, $L(z,t) = a(t)z + \dots$, be a Loewner chain such that $a(\cdot) \in C^1[0,\infty)$, $a(0) = 1$ and $\lim_{t\to\infty} |a(t)| = \infty$. Assume that the following conditions hold:*

(i) There exists $K > 0$ such that $L(\cdot,t)$ is K-quasiregular for each $t \geq 0$.

(ii) There exist some constants $M > 0$ and $\alpha \in [0,1)$ such that

$$\|DL(z,t)\| \leq \frac{M|a(t)|}{(1-\|z\|)^{\alpha}}, \quad z \in B, \ t \in [0,\infty).$$

(iii) There exists a sequence $\{t_m\}_{m\in\mathbb{N}}$, $t_m > 0$, $\lim_{m\to\infty} t_m = \infty$, and a mapping $F \in \mathcal{H}(B)$ such that

$$\lim_{m\to\infty} \frac{L(z,t_m)}{a(t_m)} = F(z)$$

locally uniformly on B.

Further, assume that the mapping $h(z,t)$ defined by Theorem 2 satisfies the following conditions:

(iv) There exists a constant $C>0$ such that

$$C\|z\|^2 \le R\langle h(z,t),z\rangle, \quad z\in B,\ t\in[0,\infty).$$

(v) There exists a constant $C_1>0$ such that

$$\|h(z,t)\|\le C_1, \quad z\in B,\ t\in[0,\infty).$$

Then $f=L(\cdot,0)$ extends to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

In this paper we obtain a sufficient condition for a normalized quasiregular holomorphic mapping on B, which can be embedded as the first element of a nonnormalized univalent subordination chain, to be extended to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

2. Main results

Theorem 2.1. *Let $f,g\in\mathcal{H}(B)$ be such that $f(0)=g(0)=0$, $Df(0)=Dg(0)=I$ and g is quasiregular in B. Also let $a\ge 2$. If there is $q\in[0,1)$ such that $1-\frac{2}{\alpha}\le q<\frac{2}{\alpha}$,*

$$\frac{2}{\alpha}\left\|[Dg(z)]^{-1}Df(z)-\frac{\alpha}{2}I\right\|\le q<1 \tag{2.1}$$

and

$$\frac{2}{\alpha}\Big\|\|z\|^{\alpha}\{[Dg(z)]^{-1}Df(z)-I\} \tag{2.2}$$

$$+(1-\|z\|^{\alpha})[Dg(z)]^{-1}D^2g(z)(z,\cdot)+\left(1-\frac{\alpha}{2}\right)I\Big\|\le q<1,\ z\in B,$$

then f extends to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

Proof. We shall show that the conditions (2.1) and (2.2) enable us to embed f as the initial element $f(z)=L(z,0)$ of a suitable subordination chain.

We define

$$L(z,t)=f(e^{-t}z)+(e^{\alpha t}-1)e^{-t}Df(ze^{-t})(z),\ t\in[0,\infty),\ z\in B. \tag{2.3}$$

In [4] the authors proved that the mapping L defined by (2.3) is a subordination chain. In the same paper the authors showed that the subordination chain defined by (2.3) satisfies the generalized Loewner equation where the mapping h is defined by:

$$h(z,t) = [I - E(z,t)]^{-1}[I + E(z,t)](z),\ q \in B,\ t \in [0,\infty) \tag{2.4}$$

and the mapping $E : B \times [0,\infty) \to \mathcal{L}(\mathbb{C}^n)$ is defined by

$$E(z,t) = -\frac{2}{\alpha}e^{-\alpha t}\{[Dg(ze^{-t})]^{-1}Df(ze^{-t}) - I\} \tag{2.5}$$

$$-\frac{2}{\alpha}(1-e^{-\alpha t})[Dg(ze^{-t})]^{-1}D^2g(ze^{-t})(ze^{-t},\cdot) - I\left(\frac{2}{\alpha}-1\right).$$

Further, we shall show that $\|E(z,t)\| \le q$ for all $(z,t) \in B \times [0,\infty)$.

We have

$$\|E(z,0)\| = \frac{2}{\alpha}\left\|[Dg(z)]^{-1}Df(z) - \frac{\alpha}{2}I\right\| \le q < 1, \quad z \in B,$$

by the condition (2.1). Next, fix $t \in (0,\infty)$. In view of the maximum principle for holomorphic mappings into complex Banach spaces, we obtain that

$$\|E(z,t)\| \le \max_{\|w\|=1} \|E(w,t)\|$$

$$= \frac{2}{\alpha}\max_{\|w\|=1}\Big\| \|we^{-t}\|^{\alpha}[Dg(we^{-t})]^{-1}[Df(we^{-t}) - I_n]$$

$$+(1-\|we^{-t}\|^{\alpha})[Dg(we^{-t})]^{-1}D^2g(we^{-t})(we^{-t},\cdot) + I\left(1-\frac{\alpha}{2}\right)\Big\|,\ z \in B.$$

Hence, we deduce from the condition (2.2) that

$$\|E(z,t)\| \le q < 1, \quad z \in B^n.$$

Therefore

$$\|E(z,t)\| \le q < 1, \quad z \in B,\ t \in [0,\infty)$$

and hence $I - E(z,t)$ is an invertible linear operator.

Further calculations show that

$$\frac{\partial L(z,t)}{\partial t} = \frac{\alpha}{2}e^{(\alpha-1)t}Dg(ze^{-t})[I + E(z,t)](z) \tag{2.6}$$

$$= DL(z,t)[I - E(z,t)]^{-1}[I + E(z,t)](z)$$

$$= DL(z,t)h(z,t), \quad t \in [0,\infty),\ z \in B.$$

On the other hand, taking into account the conditions (i) and (ii) in the hypothesis, we deduce that

$$(1-\|z\|^\alpha)\|[Dg(z)]^{-1}D^2g(z)(z,\cdot)\| \tag{2.7}$$

$$\leq q\cdot\frac{\alpha}{2}+\|z\|^\alpha\cdot\frac{\alpha}{2}\cdot q+(1-\|z\|^\alpha)\left(\frac{\alpha}{2}-1\right)$$

$$= \|z\|^\alpha\left(q\frac{\alpha}{2}-\frac{\alpha}{2}+1\right)+q\frac{\alpha}{2}+\frac{\alpha}{2}-1$$

$$\leq \max_{x\in[0,1]}\left\{x\left(q\frac{\alpha}{2}-\frac{\alpha}{2}+1\right)+q\frac{\alpha}{2}+\frac{\alpha}{2}-1\right\}$$

$$= \max\left\{q\frac{\alpha}{2}+\frac{\alpha}{2}-1, q\alpha\right\} = q\alpha = 2\beta, \quad z \in B,$$

where $\beta = \dfrac{q\alpha}{2} < 1$. Since $\alpha \geq 2$, we deduce from the above relation that

$$(1-\|z\|^2)\|[Dg(z)]^{-1}D^2g(z)(z,\cdot)\| \leq 2\beta, \quad \beta \leq \|z\| < 1.$$

From (2.6), by using a similar argument with that used in the proof of Theorem 2.1 [9] we obtain that there exists $M > 0$ such that

$$|\det Dg(z)| \leq \frac{M}{(1-\|z\|)^{n\beta}}, \quad z \in B, \tag{2.8}$$

and hence

$$\|Dg(z)\| \leq \frac{L}{(1-\|z\|)^{\beta}}. \tag{2.9}$$

It remains to prove that the mappings $L(\cdot,t)$, $t \geq 0$ are quasiregular. For the subordination chain defined by (2.3) we have

$$DL(z,t) = e^{(\alpha-1)t}\frac{\alpha}{2}Dg(ze^{-t})[I-E(z,t)], \quad z \in B,\ t \geq 0$$

where $L = \sqrt[n]{ML}$.

Since g is a quasiregular holomorphic mapping and the following inequality holds

$$1-q \leq \|I-E(z,t)\| \leq 1+q, \quad z \in B,\ t \geq 0$$

we easily obtain

$$\|DL(z,t)\| \leq \frac{\alpha}{2}e^{(\alpha-1)t}(1+q)\frac{L}{(1-\|z\|)^{\beta}} \tag{2.10}$$

$$= \frac{L^* a(t)}{(1 - \|z\|)^\beta}, \quad z \in B,\ t \in [0, \infty).$$

On the other hand, we have

$$\|DL(z,t)\|^n \leq \left(\frac{\alpha}{2}\right)^n e^{n(\alpha-1)t} \|Dg(ze^{-t})\|^n (1+q)^n \tag{2.11}$$

$$\leq \left(\frac{\alpha}{2}\right)^n e^{n(\alpha-1)t} K |\det Dg(ze^{-t})| (1+q)^n$$

$$\leq \left(\frac{1+q}{1-q}\right)^n K |\det DL(z,t)|, \quad z \in B,\ t \geq 0.$$

Since the conditions of Theorem 1.2 are satisfied we obtain that the function $f(z) = L(z, \cdot)$ admits a quasiconformal extension defined on $\mathbb{R}^{2n}$.

Observe that:

a) if $f = g$ and $\alpha = 2$ we obtain Theorem 3.1 of [12],

b) if $\alpha = 2$ we obtain Theorem 2.1 of [5].

References

[1] Becker, J., *Löewnersche Differentialgleichung und quasikonform fortsetzbare schlichte Functionen*, J. Reine Angew. Math., **255** (1972), 23-43.

[2] Bojarski, B., and Iwaniec, T., *Analytical foundations of the theory of quasiconformal mappings in* $\mathbb{R}^n$, Ann. Acad. Sci. Fenn. Ser. A Math., **8**(1983), 257-324.

[3] Brodksii, A.A., *Quasiconformal extension of biholomorphic mappings*, In: Theory of Mappings and Approximation of Functions, 3-34, Naukiva Durka, Kiew, 1983.

[4] Curt, P., *A univalence criterion for holomorphic mappings in* $\mathbb{C}^n$, Mathematica, **37(60)**, 1-2, 1995, 65-69.

[5] Curt, P., *Quasiconformal extensions of holomorphic maps in* $\mathbb{C}^n$, Mathematica, **46(69)**, 1, 2004, 55-60.

[6] Curt, P. and Kohr, G., *Subordination chains and Loewner differential equations in several complex variables*, Ann. Univ. M. Curie-Sklodowska, Sect. A, **57**(2003), 35-43.

[7] Curt, P. and Kohr, G., *Quasiconformal extensions and q-subordination chains in* $\mathbb{C}^n$, Mathematica (Cluj), **49(72)**(2007), 149-159.

[8] Curt, P. and Kohr, G., *The asymptotical case of certain quasiconformal extensions results for holomorphic mappings in* $\mathbb{C}^n$, Bull. Belgian Math. Soc. Simon Stevin, **14**(2007), 653-667.

[9] Curt, P. and Kohr, G., *Some remarks concerning quasiconformal extensions in several complex variables*, J. Inequalities Appl., 2008, 16 pages, Article ID690932.

[10] Graham, I., Hamada, H., Kohr, G., *Parametric representation of univalent mappings in several complex variables*, Canadian Journal of Mathematics, vol. 54, no. 2, 2002, pp. 324-351.

[11] Hamada, H. and Kohr, G., *Quasiconformal extension of biholomorphic mappings in several complex variables*, J. Anal. Math., **96**(2005), 269-282.

[12] Pfaltzgraff, J.A., *Subordination chains and quasiconformal extension of holomorphic maps in* $\mathbb{C}^n$, Ann. Acad. Sci. Fenn. Ser. A, Math., **1**(1975), 13-25.

Faculty of Economics and Business Administration
Babeş-Bolyai University, 58-60 Teodor Mihali Str.
400591 Cluj-Napoca, Romania
E-mail address: paula.curt@econ.ubbcluj.ro