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GRONWALL LEMMAS AND COMPARISON THEOREMS FOR THE CAUCHY PROBLEM ASSOCIATED TO A SET DIFFERENTIAL EQUATION

IOANA CAMELIA TIŞE

Abstract. Let $P_{cp,cv}(\mathbb{R}^n)$ be the family of all nonempty compact, convex subset of $\mathbb{R}^n$. We consider the following Cauchy problem:

$$(1)\quad \begin{cases} D_H U = F(t,U),\ t \in J \\ \hfill U(t_0) = U^0 \end{cases}$$

where $U^0 \in P_{cp,cv}(\mathbb{R}^n),\ t_0 \geq 0,\ J = [t_0, t_0 + a],\ a > 0$, and

$$F : J \times P_{cp,cv}(\mathbb{R}^n) \to P_{cp,cv}(\mathbb{R}^n).$$

The purpose of the paper is to study the existence of a solution as well as some comparison theorems and Gronwall type lemmas for the above Cauchy problem.

1. Introduction

Let $\mathbb{R}^n$ be the real n-dimensional space and $P_{cp,cv}(\mathbb{R}^n)$ the family of all nonempty compact, convex subset of $\mathbb{R}^n$ endowed with the Pompeiu-Hausdorff metric H.

We consider the following Cauchy problem with respect to a set differential equation:

$$(1)\quad \begin{cases} D_H U = F(t,U),\ t \in J \\ \hfill U(t_0) = U^0 \end{cases}$$

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where $U^0 \in P_{cp,cv}(\mathbb{R}^n),\ t_0 \geq 0,\ J = [t_0, t_0 + a],\ a > 0,$
$F \in C(J \times P_{cp,cv}(\mathbb{R}^n), P_{cp,cv}(\mathbb{R}^n))$ and D_H is the Hukuhara derivative of U.

A solution of (1) is a continuous function $U : J \to P_{cp,cv}(\mathbb{R}^n)$ which satisfies (1) for each $t \in J$.

The aim of the article is to study the existence of a solution as well as some comparison theorems and Gronwall type lemmas for the above Cauchy problem.

The paper is organized as follows. The next section, Preliminaries, contains some basic notations and notions used throughout the paper. The third section presents some comparison theorems and Gronwall type lemmas for the above Cauchy problem (1).

2. **Preliminaries**

The aim of this section is to present some notions and symbols used in the paper.

Definition 1. *$U \in C^1(J, P_{cp,cv}(\mathbb{R}^n))$ is a solution of the problem (1)$\Longleftrightarrow$ U satisfies (1) for all $t \in J$.*

Let us consider the following equations:

$$(2)\ \ U(t) = U^0 + \int_{t_0}^{t} D_H(U(s))ds,\ t \in J,$$

$$(3)\ \ U(t) = U^0 + \int_{t_0}^{t} F(s, U(s))ds,\ t \in J.$$

Lemma 2. *If $U \in C^1(J, P_{cp,cv}(\mathbb{R}^n))$, then* (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3).

We consider on $C(J, P_{cp,cv}(\mathbb{R}^n))$ the metric H_*^B defined by:

$$H_*^B(U, V) := \max_{t \in [t_0, t_0+a]} [H(U(t), V(t))e^{-\tau(t-t_0)}],\ \tau > 0.$$

The pair $(C(J, P_{cp,cv}(\mathbb{R}^n)), H_*^B)$ forms a complete metric space.

We consider on $P_{cp,cv}(\mathbb{R}^n)$ the order relation $\leq_m$ defined by:

$$U, V \in P_{cp,cv}(\mathbb{R}^n) : U \leq_m V \Longleftrightarrow U \subseteq V.$$

Definition 3. *The operator $F(t, \cdot) : J \times P_{cp,cv}(\mathbb{R}^n) \to P_{cp,cv}(\mathbb{R}^n)$, is called increasing if*

$$A, B \in P_{cp,cv}(\mathbb{R}^n),\ A \leq_m B \Rightarrow F(t, A) \leq_m F(t, B),\ \textit{for all } t \in J.$$

Define on $C(J, P_{cp,cv}(\mathbb{R}^n))$ an order relation " $\leq$ " defined by:

$$X, Y \in C(J, P_{cp,cv}(\mathbb{R}^n)),\ X \leq Y \Leftrightarrow X(t) \leq_m Y(t),\ \text{for all } t \in J.$$

The space $(C(J, P_{cp,cv}(\mathbb{R}^n)), H_*^B, \leq)$ being an ordered and complete metric space is also an L-space (see[3]).

Let $(X, d, \leq)$ be a ordered metric space and $T : X \to X$ an operator.

We note: $F_T := \{x \in X | Tx = x\}$the fixed point set of T;

$$(UF)_T := \{x \in X | Tx \leq x\} \text{ the upper fixed point set for T;}$$

$$(LF)_T := \{x \in X | Tx \geq x\} \text{ the lower fixed point set for T.}$$

Definition 4. *([4]) Let X be an L-space. Then, the operator $T : X \to X$ is a Picard operator (PO) if*

(i) $F_T = \{x_T^*\}$;

(ii) $T^n x \to x_T^*$ as $n \to \infty$, for all $x \in X$.

Definition 5. *([4]) Let X be an L-space. Then, the operator $T : X \to X$ is a weakly Picard operator (WPO) if the sequence $(T^n x)_{n \in \mathbb{N}}$ converges for all $x \in X$ and the limit (witch may depend on x) is a fixed point of T.*

3. **Main results**

Theorem 6. *We consider the problem (1) and $F : J \times P_{cp,cv}(\mathbb{R}^n) \longrightarrow P_{cp,cv}(\mathbb{R}^n)$ be an operator.*

Suppose that:

i) F is continuous on $J \times P_{cp,cv}(\mathbb{R}^n)$ and $U^0 \in P_{cp,cv}(\mathbb{R}^n)$;

ii) $F(t, \cdot)$ is Lipschitz, i.e. there exists $L \geq 0$

such that $H(F(t, U), F(t, V)) \leq LH(U, V)$ for all $U, V \in P_{cp,cv}(\mathbb{R}^n)$ and $t \in J$.

Then the problem (1) *has a unique solution* U^* *and* $U^*(t) = \lim\limits_{n\to\infty} U_n(t)$, *where* $U_n \in C(J, P_{cp,cv}(\mathbb{R}^n))$ *is recurrently defined by the relation:*

$$\begin{cases} U_{n+1}(t) = U^0 + \int_{t_0}^{t} F(s, U_n(s))ds, \ n \in \mathbb{N} \\ \hfill U_0 \in P_{cp,cv}(\mathbb{R}^n). \end{cases}$$

Proof. Consider the operator: $\Gamma : C(J, P_{cp,cv}(\mathbb{R}^n)) \to C(J, P_{cp,cv}(\mathbb{R}^n))$ where

$$\Gamma U(t) = U^0 + \int_{t_0}^{t} F(s, U(s))ds, \ t \in J.$$

We will verify the contraction condition for Γ.

$$H(\Gamma(U)(t), \Gamma(V)(t)) = H(U^0 + \int_{t_0}^{t} F(s, U(s))ds, U^0$$

$$+ \int_{t_0}^{t} F(s, V(s))ds) \le H(U^0, U^0) + H(\int_{t_0}^{t} F(s, U(s))ds, \int_{t_0}^{t} F(s, V(s))ds)$$

$$\le \int_{t_0}^{t} H(F(s, U(s)), F(s, V(s)))ds \le \int_{t_0}^{t} LH(U(s), V(s))ds =$$

$$= L \int_{t_0}^{t} H(U(s), V(s))e^{-\tau(s-t_0)}e^{\tau(s-t_0)}ds \le LH_*^B(U, V) \int_{t_0}^{t} e^{\tau(s-t_0)}ds =$$

$$= \frac{L}{\tau} H_*^B(U, V)(e^{\tau(t-t_0)} - 1) \le \frac{L}{\tau} H_*^B(U, V)e^{\tau(t-t_0)},$$

then we have:

$$H(\Gamma(U)(t), \Gamma(V)(t))e^{-\tau(t-t_0)} \le \frac{L}{\tau} H_*^B(U, V), \text{ for all } t \in J.$$

Taking the maximum for $t \in J$, then we have:

$$H_*^B(\Gamma(U), \Gamma(V)) \le \frac{L}{\tau} H_*^B(U, V), \text{ for all } U, V \in C(J, P_{cp,cv}(\mathbb{R}^n)), \ \tau > 0.$$

Thus, the integral operator Γ is Lipschitz with constant $L_\Gamma = \frac{L}{\tau}$, $\tau > 0$.

Choosing τ such as $\frac{L}{\tau} < 1$, then Γ is an contraction and by contraction principle the operator Γ has unique fixed point U^*. According to Lemma 2 then U^* is the unique solution for the Cauchy problem. $\square$

In what follows we will present the Abstract Gronwall Lemma:

Lemma 7. *(Abstract Gronwall Lemma [3]) Let $(X, d, \leq)$ be an ordered L-space and $T : X \to X$ an operator. We suppose that:*

(i) T is PO;

(ii) T is increasing.

Then $(LF)_T \leq x_T^* \leq (UF)_T$, where x_T^* is the unique fixed point of the operator T.

We will apply this abstract lemma to the Cauchy problem (1).

Theorem 8. *Let the Cauchy problem*

$$(1) \quad \begin{cases} D_H U = F(t, U), \ t \in J \\ \qquad\qquad U(t_0) = U^0 \end{cases}$$

where $U^0 \in P_{cp,cv}(\mathbb{R}^n), \ t_0 \geq 0, \ J = [t_0, t_0 + a], \ a > 0.$

Suppose that $F(t, \cdot)$ is an L-Lipschitz increasing monotone operator for all $t \in J$. Then we have:

$$(LS)_{(1)} \leq U^* \leq (US)_{(1)}$$

where U^* is the unique solution for problem (1) and $(LS)_{(1)}$ respectively $(US)_{(1)}$ represents the set of lower solution respectively the set of upper solution for the problem (1).

Proof. Let $\Gamma : C(J, P_{cp,cv}(\mathbb{R}^n)) \longrightarrow C(J, P_{cp,cv}(\mathbb{R}^n))$

$$\Gamma U(t) := U^0 + \int_{t_0}^{t} F(s, U(s))ds, \ t \in J.$$

Then we have:

i) By Theorem 6 we have that Γ as a contraction. We denote by $U^* \in C(J, P_{cp,cv}(\mathbb{R}^n))$ the unique fixed point. According to Lemma 2 we have that U^* is the unique solution for the Cauchy problem.

ii) We proved that Γ is increasing. Let $U, V \in C(J, P_{cp,cv}(\mathbb{R}^n))$ with $U \leq V \Rightarrow U(t) \leq_m V(t), \ \text{for all } t \in J.$

Since $F(t, \cdot)$ is monotone we have $F(t, U(t)) \leq F(t, V(t)), \ \text{for all } t \in J.$

Then

$$U^0 + \int_{t_0}^{t} F(s, U(s)) \leq U^0 + \int_{t_0}^{t} F(s, V(s))ds, \quad \text{for all } t \in J$$

$$\Rightarrow \Gamma U(t) \leq \Gamma V(t), \quad \text{for all } t \in J \Rightarrow \ \Gamma U \leq \Gamma V.$$

So Γ is monotonously increasing and Picard. Be applying Lemma 7 we have:

$$(LF)_\Gamma \leq U^* \leq (UF)_\Gamma.$$

Consequently $(LF)_\Gamma,\ (UF)_\Gamma$ coincide to the set of the lower and upper solutions for problem (1). □

In what follows an abstract comparison lemma will be presented.

Lemma 9. *(Abstract Gronwall- comparison lemma* [3]*) Let $(X, d, \leq)$ be an ordered L-space and $T, \Gamma : X \to X$ two operators. We suppose that:*

(i) T and Γ are POs;

(ii) T is increasing;

(iii) $T \leq \Gamma$.

Then $x \leq Tx \Rightarrow x \leq x^*_\Gamma$.

We have the folowing theorem.

Theorem 10. *Let as consider the following two Cauchy problems:*

$$(1) \quad \begin{cases} D_H U = F(t, U),\ t \in J \\ U(t_0) = U^0 \end{cases}$$

$$(2) \quad \begin{cases} D_H V = G(t, V),\ t \in J \\ V(t_0) = V^0 \end{cases}$$

where $U^0, V^0 \in P_{cp,cv}(\mathbb{R}^n),\ t_0 \geq 0,\ J = [t_0, t_0 + a],\ a > 0$.

Suppose that:

i) F is continuous on $J \times P_{cp,cv}(\mathbb{R}^n)$ and $F(t, \cdot)$ is Lipschitz;

ii) G is continuous on $J \times P_{cp,cv}(\mathbb{R}^n)$, $V^0 \in P_{cp,cv}(\mathbb{R}^n)$ and $G(t, \cdot)$ is Lipschitz;

iii) $F(t, \cdot)$ is increasing for all $t \in J$.

Then $U \leq \Gamma U \Longrightarrow U \leq V^*$ where V^* is the unique solution for the problem (2).

Proof. Since $F(t, \cdot)$ is Lipschitz, there exists $L \geq 0$ such that

$$H(F(t, U), F(t, V)) \leq LH(U, V),\ \text{for all } U, V \in P_{cp,cv}(\mathbb{R}^n),\ t \in J.$$

Since $G(t,\cdot)$ is Lipschitz, there exists $L_G \geq 0$ such that

$$H(G(t,U),G(t,V)) \leq L_G H(U,V), \text{ for all } U,V \in P_{cp,cv}(\mathbb{R}^n),\ t \in J.$$

By Theorem 8, Γ and T satisfy the contraction principle and we have that Γ and T are Picard operators.

By iii) we have $F(t,U) \subset G(t,U),$ for all $U \in P_{cp,cv}(\mathbb{R}^n),\ t \in J,$

$$\text{then } U^0 + \int_{t_0}^{t} F(s,U(s))ds \leq V^0 + \int_{t_0}^{t} G(s,U(s))ds,$$

$$\text{thus } \Gamma U(t) \subseteq TU(t) \Longrightarrow \Gamma U \leq TU \Longrightarrow \Gamma \leq T.$$

By Lemma 9 the proof is complete. □

We recall the following abstract Gronwall lemma for the case of WPO.

Lemma 11. *(Abstract Gronwall lemma* [3]*) Let $(X,d,\leq)$ be an ordered L-space and $T : X \to X$ an operator. We suppose that*

(i) T is WPO;

(ii) T is increasing.

Then

a) $x \leq Tx \Rightarrow x \leq T^{\infty}x$;

b) $x \geq Tx \Rightarrow x \geq T^{\infty}x$.

The basic result in the WPOs theory is the following:

Theorem 12. *(Characterization theorem* [4]*) Let (X,d) be an L-space and $f : X \to X$ be an operator. The operator f is WPO if and only if there exists a partition of X, $X = \bigcup_{\gamma\in\Gamma} X_\gamma$ such that:*

(a) $X_\gamma \in I(A),$ for all $\gamma \in \Gamma$;

(b) f to X_γ, $f_{|X_\gamma} : X_\gamma \to X_\gamma$ is PO for all $\gamma \in \Gamma$.

We will apply the above lemma to the Cauchy problem (1).

Theorem 13. *Let us consider the Cauchy (1)*

We suppose that:

i) $F(t,.)$ is Lipschitz, for all $t \in J$;

ii) $F(t,.)$ is increasing, for all $t \in J$;

iii) F is continuous on $J \times P_{cp,cv}(\mathbb{R}^n)$ and $U^0 \in P_{cp,cv}(\mathbb{R}^n)$.

Then

i) U is a lower solution of the problem (1) $\Rightarrow U \leq U_U^*$;

ii) U is a upper solution of the problem (1) $\Rightarrow U \geq U_U^*$

where U_U^* is a solution for the problem (1) and $U_U^*(t) = \lim\limits_{n\to\infty} U_n(t)$ where $U_n \in C(J, P_{cp,cv}(\mathbb{R}^n))$ is recurrently defined by the relation:

$$\begin{cases} U_{n+1}(t) = U_n(t_0) + \int_{t_0}^t F(s, U_n(s))ds, \ n \in \mathbb{N} \\ \hfill U^0 = U. \end{cases}$$

Proof. Let $T : C(J, P_{cp,cv}(\mathbb{R}^n)) \to C(J, P_{cp,cv}(\mathbb{R}^n))$ defined by

$$TU(t) = U(t_0) + \int_{t_0}^t F(s, U(s))ds, \quad \text{for all } t \in J.$$

According to Lemma 2 we have (1) $\Leftrightarrow$ (2) $\Leftrightarrow U = TU$. Thus $S_{(1)} = FixT$.

Let $Z = C(J, P_{cp,cv}(\mathbb{R}^n))$ and $Z_\gamma = \{U \in C(J, P_{cp,cv}(\mathbb{R}^n)) | U(t_0) = \gamma\}, \ \gamma \in \mathbb{R}$. Then $Z = \bigcup\limits_{\gamma\in\mathbb{R}} Z_\gamma$ is a partition of $C(J, P_{cp,cv}(\mathbb{R}^n))$. Moreover $Z_\gamma \in I(T)$ and Z is a closed subset of $C(J, P_{cp,cv}(\mathbb{R}^n))$ for all $\gamma \in \mathbb{R}$.

Since $F(t, .) : Z \to Z$ is a L-Lipschitz for all $t \in J$. By Theorem 6 the operator $T_{|Z_\gamma}$ is Picard for all $\gamma \in \mathbb{R}$. Hence T is WPO (by the characterization Theorem 12).

In the above conditions the Cauchy problem (1) is equivalent with the fixed point equation, $TU = U$, where the operator T is WPO.

Since $F(t, \cdot)$ is monotone we have $F(t, U(t)) \leq F(t, V(t))$, for all $t \in J$.

Then

$$U^0 + \int_{t_0}^t F(s, U(s)) \leq U^0 + \int_{t_0}^t F(s, V(s))ds, \quad \text{for all } t \in J$$

$$\Rightarrow TU(t) \leq TV(t), \quad \text{for all } t \in J \Rightarrow \ TU \leq TV.$$

Thus T is monotonously increasing and WPO. By applying Lemma 11 the proof is complete. $\square$

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Department of Applied Mathematics
Babeş-Bolyai University
Cluj-Napoca, Romania
E-mail address: ti_camelia@yahoo.com