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EXTRACTING FUZZY IF-THEN RULE BY USING THE INFORMATION MATRIX TECHNIQUE WITH QUASI-TRIANGULAR FUZZY NUMBERS

ZOLTÁN MAKÓ

Abstract. In the paper [7] C. Huang and C. Moraga suggested a new method to extract fuzzy if-then rules from training data based on information matrix technique with Gaussian membership function. In this paper, we extend this method to the Archimedean t-normed space of quasi-triangular fuzzy numbers.

1. Introduction

The core of a fuzzy controller is its set of fuzzy if-then rules. Today, fuzzy control is increasingly seen as a universal approximator (H. B. Verbruggen and P. M. Brujin, 1997) by the control community, and thus is strongly used for approximating functions (D. Dubois and H. Prade, 1997).

A fuzzy system is a set of if-then fuzzy rules that maps inputs to outputs. Each fuzzy rule defines a fuzzy patch in the input-output state space of the function. A fuzzy patch is a fuzzy Cartesian product of if-part fuzzy set and then-part fuzzy set. An additive fuzzy system approximates the function by covering its graph with fuzzy patches (see Figure 1). C. Huang and C. Moraga in 2005 suggested a new method to extract fuzzy if-then rules from training data based on information matrix technique with Gaussian membership function. In this paper, we extend this method to the

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Archimedean t-normed space of quasi-triangular fuzzy numbers and we show that the additive fuzzy system with quasi triangular fuzzy numbers is a function approximator.

2. The Archimedean fuzzy normed space of quasi-triangular fuzzy numbers

Triangular norms and co-norms were introduced by K. Menger (1942) and studied first by B. Schweizer and A. Sklar (1961, 1963, 1983) to model distances in probabilistic metric spaces. In fuzzy sets theory triangular norms and co-norms are extensively used to model logical connection *and* and *or*. In fuzzy literatures, these concepts were studied e. g. by E. Creţu (2001), J. Dombi (1982), D. Dubois and H. Prade (1985), J. Fodor (1991, 1999), S. Jenei (1998, 1999, 2000, 2001, 2004), V. Radu (1974, 1984, 1992).

Definition 2.1. *The function* $N:[0,1]\to[0,1]$ *is a* negation operation *if:*

(i) $N(1)=0$ *and* $N(0)=1$*;*

(ii) N *is continuous and strictly decreasing;*

(iii) $N(N(x))=x$, *for all* $x\in[0,1]$.

Definition 2.2. *Let* N *be a negation operation. The mapping* $T:[0,1]\times[0,1]\to[0,1]$ *is a* triangular norm *(briefly* t-norm*) if satisfies the properties:*

$$\textit{Symmetry}: T(x,y)=T(y,x),\ \forall x,y\in[0,1];$$

$$\textit{Associativity}: T(T(x,y),z)=T(x,T(y,z)),\ \forall x,y,z\in[0,1];$$

$$\textit{Monotonicity}: T(x_1,y_1)\le T(x_2,y_2)\ \ if\, x_1\le x_2\ \textit{and}\ y_1\le y_2;$$

$$\textit{One identity}:\ T(x,1)=x,\ \forall x\in[0,1]$$

and the mapping $S:[0,1]\times[0,1]\to[0,1]$,

$$S(x,y)=\ N(T(N(x),N(y)))$$

is a triangular co-norm (the dual of T *given by* N*).*

Definition 2.3. *The t-norm* T *is* Archimedean *if* T *is continuous and* $T(x,x)<x$, *for all* $x\in(0,1)$.

Definition 2.4. *The t-norm T is called* strict *if T is strictly increasing in both arguments.*

Theorem 2.5 (C. H. Ling, 1965)**.** *Every Archimedean t-norm T is representable by a continuous and decreasing function $g : [0,1] \to [0,+\infty]$ with $g(1) = 0$ and*

$$T(x,y) = g^{[-1]}(g(x) + g(y)),$$

where

$$g^{[-1]}(x) = \begin{cases} g^{-1}(x) & \text{if } 0 \le x < g(0), \\ 0 & \text{if } x \ge g(0). \end{cases}$$

If g_1 and g_2 are the generator function of T, then there exist $c > 0$ such that $g_1 = cg_2$.

Remark 2.6. *If the Archimedean t-norm T is strict, then $g(0) = +\infty$ otherwise $g(0) = g_0 < \infty$.*

Theorem 2.7 (E. Trillas, 1979)**.** *An application $N : [0,1] \to [0,1]$ is a negation if and only if an increasing and continuous function $e : [0,1] \to [0,1]$ exists, with $e(0) = 0$, $e(1) = 1$ such that $N(x) = e^{-1}(1 - e(x))$, for all $x \in [0,1]$.*

Remark 2.8. *The generator function of negation $N(x) = 1 - x$ is $e(x) = x$. Another negation generator function is*

$$e_\lambda(x) = \frac{\ln(1 + \lambda x)}{\ln(1 + \lambda)},$$

where $\lambda > -1$, $\lambda \ne 0$.

Remark 2.9. *Examples to t-norm are following:*

- *minim:* $\min(x,y) = \min\{x,y\}$;
- *product:* $P(x,y) = xy$, *the generator function is* $g(x) = -\ln x$;
- *weak:* $W(x,y) = \begin{cases} \min\{x,y\} & \text{if } \max\{x,y\} = 1, \\ 0 & \text{otherwise.} \end{cases}$

If the negation operation is $N(x) = 1 - x$, then the dual of these t-norms are:

- *maxim:* $\max(x,y) = \max\{x,y\}$;
- *probability:* $S_P(x,y) = x + y - xy$;
- *strong:* $S_W(x,y) = \begin{cases} \max\{x,y\} & \text{if } \min\{x,y\} = 0, \\ 1 & \text{otherwise.} \end{cases}$

Proposition 2.10. *If T is a t-norm and S is the dual of T, then*

$$W(x,y) \leq T(x,y) \leq \min\{x,y\},$$
$$\max\{x,y\} \leq S(x,y) \leq S_W(x,y),$$

for all $x, y \in [0,1]$.

The fuzzy set concept was introduced in mathematics by K. Menger in 1942 and reintroduced in the system theory by L. A. Zadeh in 1965. L. A. Zadeh has introduced this notion to measure quantitatively the vagueness of the linguistic variable. The basic idea was: if X is a set, then all A subsets of X can be identified with its characteristic function $\chi_A : X \to \{0,1\}$, $\chi_A(x) = 1 \Leftrightarrow x \in A$ and $\chi_A(x) = 0 \Leftrightarrow x \notin A$.

The notion of fuzzy set is another approach of the subset notion. There exist continue and transitory situations in which we have to sugest that an element belongs to a set at different levels. We indicate this fact with membership degree.

Definition 2.11. *Let X be a set. A mapping $\mu : X \to [0,1]$ is called* membership function*, and the set $A = \{(x, \mu(x)) \ / x \in X\}$ is called* fuzzy set *on X. The membership function of A is denoted by μ_A. The collection of all fuzzy sets on X we will denote by $\mathcal{F}(X)$.*

In order to use fuzzy sets and relations in any intelligent system we must be able to perform set and arithmetic operations. In fuzzy theory the extension of arithmetic operations to fuzzy sets was formulated by L.A. Zadeh in 1965.

The operations on $\mathcal{F}(X)$ are uniquely determined by T, N and the corresponding operations of X by using the generalized t-norm based extension principle (Z. Makó, 2006).

Definition 2.12. *The triplet $(\mathcal{F}(X), T, N)$ will be called* fuzzy t-normed space.

By using t-norm based extension principle the Cartesian product of fuzzy sets may be defined in the following way.

Definition 2.13. *The* T-Cartesian product*'s membership function of fuzzy sets* $A_i \in \mathcal{F}(X_i)$, $i = 1, ..., n$ *is defined as*

$$\mu_{\bar{A}}(x_1, x_2, ..., x_n) = \\ T\left(\mu_{A_1}(x_1), T\left(\mu_{A_2}(x_2), T\left(...T\left(\mu_{A_{n-1}}(x_{n-1}), \mu_{A_n}(x_n)\right)...\right)\right)\right),$$

for all $(x_1, x_2, ..., x_n) \in X_1 \times X_2 \times ... \times X_n$.

The construction of membership function of fuzzy sets is an important problem in vagueness modeling. Theoretically, the shape of fuzzy sets must depend on the applied triangular norm.

We noticed that, if the model constructed on the computer does not comply the requests of the given problem, then we choose another norm. The membership function must be defined in such a way that the change of the t-norm modifies the shape of the fuzzy sets, but the calculus with them remains valid. This desideratum is satisfied, for instance if the quasi-triangular fuzzy numbers introduced by M. Kovacs in 1992 are used.

Let $p \in [1, +\infty]$ and $g : [0, 1] \to [0, \infty]$ be a continuous, strictly decreasing function with the boundary properties $g(1) = 0$ and $\lim_{t \to 0} g(t) = g_0 \leq \infty$. We define the quasi-triangular fuzzy number in fuzzy t-normed space $(\mathcal{F}(\mathbb{R}), T_{gp}, N)$, where

$$T_{gp}(x, y) = g^{[-1]}\left((g^p(x) + g^p(y))^{\frac{1}{p}}\right) \tag{1}$$

is an Archimedean t-norm generated by g and

$$N(x) = \begin{cases} 1 - x & \text{if} \quad g_0 = +\infty, \\ g^{-1}(g_0 - g(x)) & \text{if} \quad g_0 \in \mathbb{R}. \end{cases} \tag{2}$$

is a negation operation.

Definition 2.14. *The* set of quasi-triangular fuzzy numbers *is*

$$\mathcal{N}_g = \{A \in \mathcal{F}(\mathbb{R}) \ / \textit{ there is } a \in \mathbb{R}, d > 0 \textit{ such that} \tag{3}$$
$$\left.\mu_A(x) = g^{[-1]}\left(\frac{|x-a|}{d}\right) \textit{ for all } x \in \mathbb{R}\right\} \bigcup$$
$$\{A \in \mathcal{F}(\mathbb{R}) \ / \textit{ there is } a \in \mathbb{R} \textit{ such that}$$
$$\left.\mu_A(x) = \chi_{\{a\}}(x) \textit{ for all } x \in \mathbb{R}\right\},$$

where χ_A is characteristic function of the set A. The element of $\mathcal{N}_g$ will be called quasi-triangular fuzzy number *generated by g with center λ and spread d and we will denote them with $<\lambda, d>$. The triplet $(\mathcal{N}_g, T_{gp}, N)$ is the* Archimedean t-normed space of quasi-triangular fuzzy numbers.

3. **The information matrix**

Let (x_i, y_i), $i = 1, 2, \ldots, n$ be observations of a given sample X. Let A_j, $j = 1, 2, \ldots p$ and B_k, $k = 1, 2, \ldots, q$ be fuzzy sets with membership functions μ_{A_j} and μ_{B_k}, respectively. Let $U = \{A_j |\ j = 1, 2, \ldots p\}$; $V = \{B_k |\ k = 1, 2, \ldots, q\}$. By using the definition of Cartesian product (2.13) we get, that the membership value of sample (x_i, y_i) in fuzzy set $A_j \times B_k$ is

$$r_{j,k}(x_i, y_i) = T\left(\mu_{A_j}(x_i), \mu_{B_k}(y_i)\right).$$

This value is called information gain of (x_i, y_i) at $A_j \times B_k$ with respect to the t-norm T.

The $R = (R_{j,k})_{j=1,2,\ldots\ p;k=1,2,\ldots,q}$ is called an information matrix of X on $U \times V$, where

$$R_{j,k} = \sum_{i=1}^{n} r_{j,k}(x_i, y_i).$$

4. **Extracting fuzzy if–then rules**

We extract fuzzy if–then rules according to the centre of the rows of an information matrix. The method consists of the following steps:

Step 1. We choose a number $p \in [1, +\infty]$ and a derivable generator function $g : [0, 1] \to [0, +\infty]$ of the T_{gp} norm with $\int\limits_0^1 g(x)\,dx = \omega \in \mathbb{R}$.

Step 2. We divide the illustrating space $[a, b] \times [m, M]$ in $p \times q$ square with grid points (u_j, v_k).

Step 3. Let $A_j =< u_j, h_u >$ and $B_k =< v_k, h_v >$ be quasi-triangular fuzzy numbers with spread $h_u = 2 * \frac{b-a}{p-1}$ and $h_v = 2 * \frac{M-m}{q-1}$, for all $j = 0, 1, ..., p-1$, $k = 0, 1, ..., q-1$.

Step 4. We calculate the information gains:

$$r'_{j,k,i} = T_{gp}\left(\mu_{A_j}(x_i), \mu_{B_k}(y_i)\right) = g^{[-1]}\left(\left[\left(\frac{|x_i - u_j|}{h_u}\right)^p + \left(\frac{|y_i - v_k|}{h_v}\right)^p\right]^{1/p}\right).$$

Step 5. We calculate the normalized information gains:

$$r_{j,k,i} = \frac{r'_{j,k,i}}{\sum\limits_{j=1}^{p} \sum\limits_{k=1}^{q} r'_{j,k,i}}. \tag{4}$$

Step 6. We calculate the normalized information matrix:

$$R = \begin{bmatrix} R_{1,1} & R_{1,2} & \cdots & R_{1,q} \\ R_{2,1} & R_{2,2} & \cdots & R_{2,q} \\ \vdots & \vdots & \ddots & \vdots \\ R_{p,1} & R_{p,2} & \cdots & R_{p,q} \end{bmatrix}, \tag{5}$$

where

$$R_{j,k} = \sum_{i=1}^{n} r_{j,k,i}. \tag{6}$$

Step 7. We determine the centre of all rows in the normalized information matrix R:

$$c_j = \frac{\sum\limits_{k=1}^{q} R_{j,k} \cdot v_k}{\sum\limits_{k=1}^{q} R_{j,k}}. \tag{7}$$

Step 8. Because each B_k is quasi-triangular fuzzy number, the then-part of fuzzy if-then rules would have the same shape, hence the rule consequent is $\bar{B}_j =<$

$c_j, h_v >$ with membership function

$$\mu_{\bar{B}_j}(v) = g^{[-1]}\left(\frac{|v - c_j|}{h_v}\right).$$

Therefore, we obtain following fuzzy if-then rules:

If x is $< u_1, h_u >$ then y is $< c_1, h_v >$,

If x is $< u_2, h_u >$ then y is $< c_2, h_v >$,

$\vdots$

If x is $< u_p, h_u >$ then y is $< c_p, h_v >$.

An "If x is $< u_j, h_u >$ then y is $< c_j, h_v >$" rule is equivalent to the fuzzy set $A_j \times \bar{B}_j$ with membership function

$$\begin{aligned} \mu_{A_j \times \bar{B}_j}(x, y) &= T_{gp}\left(\mu_{A_j}(x), \mu_{\bar{B}_j}(y)\right) \\ &= g^{[-1]}\left(\left[\left(\frac{|x - u_j|}{h_u}\right)^p + \left(\frac{|y - c_j|}{h_v}\right)^p\right]^{1/p}\right) \text{ for all } (x, y) \in \mathbb{R}^2. \end{aligned}$$

The graph of an $A_j \times \bar{B}_j$ fuzzy set is a fuzzy patch. The size of the patch reflects the rule's vagueness or uncertainty and cover the graph of the approximand function f (See figure 1).

Step 9.The approximator function is

$$F(u) = \frac{\sum_{j=1}^{p} c_j \cdot g^{[-1]}\left(\frac{|u - u_j|}{h_u}\right)}{\sum_{j=1}^{p} g^{[-1]}\left(\frac{|u - u_j|}{h_v}\right)}.$$

Theorem 4.1. *If $f : [a, b] \to \mathbb{R}$ is continuous then F uniformly approximates the f on $[a, b]$.*

Proof. A standard additive system is a function system $G : \mathbb{R}^t \to \mathbb{R}^l$ with p fuzzy rules "If x is A_j then y is B_j" or the patch form $A_j \times B_j$. The if-part fuzzy sets A_j and the then-part fuzzy sets have membership function μ_{A_j} and μ_{B_j}. B. Kosko in

1994 proof that the function

$$G(u) = \frac{\sum_{j=1}^{p} w_j \mu_{A_j}(u) S_j c_j}{\sum_{j=1}^{p} w_j \mu_{A_j}(u) S_j}$$

uniformly approximates the function $f : X \to \mathbb{R}^l$ if $X \subset \mathbb{R}^t$ is compact and f continuous, where w_j is the rule weight, S_j is the volume (or area) of subgraph of B_j, and c_j is the centroid of B_j.

Generally, the centroid of the quasi-triangular fuzzy number $< \lambda, \delta >$ is

$$c = \frac{\int_{-\infty}^{\infty} y\mu_{<\lambda,\delta>}(y)\, dy}{\int_{-\infty}^{\infty} \mu_{<\lambda,\delta>}(y)\, dy} = \lambda,$$

where

$$\int_{-\infty}^{\infty} y\mu_{<\lambda,\delta>}(y)\, dy = 2\lambda\delta\omega \text{ and area is } S = \int_{-\infty}^{\infty} \mu_{<\lambda,\delta>}(y)\, dy = 2\delta\omega.$$

An additive fuzzy system with the same rule weight ($w_1 = w_2 = ... = w_p$) and with fuzzy sets $A_j =< u_j, h_u >$, $\bar{B}_j =< c_j, h_v >$ is the following function approximator:

$$\begin{aligned} G(u) &= \frac{\sum_{j=1}^{p} w_j g^{[-1]}\left(\frac{|u-u_j|}{h_X}\right) 2h_v\omega c_j}{\sum_{j=1}^{p} w_j g^{[-1]}\left(\frac{|u-u_j|}{h_X}\right) 2h_v\omega} \\ &= \frac{\sum_{j=1}^{p} c_j g^{[-1]}\left(\frac{|u-u_j|}{h_X}\right)}{\sum_{j=1}^{p} g^{[-1]}\left(\frac{|u-u_j|}{h_X}\right)} = F(u). \end{aligned}$$

□

Remark 4.2. 1. *If we choose* $g : (0,1] \to [0,\infty)$, $g(t) = \sqrt{-\ln t}$ *and* $p = 2$, *then the membership function of quasi-triangular fuzzy numbers* $< a, d >$ *is*

$$\mu(t) = e^{-\frac{(t-a)^2}{d^2}} \quad \text{if } d > 0, \quad \text{and} \quad \mu(t) = \begin{cases} 1 \text{ if } t = a, \\ 0 \text{ if } t \neq a \end{cases} \quad \text{if } d = 0.$$

and $r'_{j,k,i} = \mu_{A_j}(x_i) \cdot \mu_{B_k}(y_i)$. *This is the information matrix technique model with the normal diffusion function elaborated by C. Huang - C. Moraga in 2005.*

2. *If we choose* $g : (0,1] \to [0,\infty)$, $g(t) = -\ln t$ *and* $p = 1$, *then the membership functions of quasi-triangular fuzzy numbers* $< a, d >$ *is*

$$\mu(t) = e^{-\frac{|t-a|}{d}} \quad \text{if } d > 0, \quad \text{and} \quad \mu(t) = \begin{cases} 1 \text{ if } t = a, \\ 0 \text{ if } t \neq a \end{cases} \quad \text{if } d = 0.$$

and $r'_{j,k,i} = \mu_{A_j}(x_i) \cdot \mu_{B_k}(y_i)$. *This is the model with Laplace membership function studied by S. Mitaim and B. Kosko in 2001.*

Example 4.3. *Let us use the information matrix technique to approach the following nonlinear function:*

$$f : [-6,6] \to \mathbb{R}, \ f(x) = x \sin x$$

by consider a sample with 121 values from $[-6,6] \times [-6,6]$ *with uniform distribution to be input values:*

$$X = \{(x_i, y_i) : \ x_i = -6 + 0.1 * i, \ y_i = f(x_i), \ i = 0, 1, ..., 120\}.$$

Step 1. Let $g : [0,1] \to [0,1]$, $g(t) = 1 - t^2$ be the generator function and $p = 3$ and

$$g^{[-1]}(t) = \begin{cases} \sqrt{1-t} \text{ if } t \in [0,1], \\ 0 \text{ else.} \end{cases}$$

Step 2. We divide the illustrating space $[-6,6] \times [-6,6]$ in 50×100 square with grid points (u_j, v_k), where $u_j = -6 + j \cdot \frac{h_u}{2}$ and $v_k = -6 + k \cdot \frac{h_v}{2}$, $h_u = 0.488$, $h_v = 0.242$, $j = 0, ..., 49$, $k = 0, ..., 99$.

Step 3. In this case $A_j =< u_j, h_u >$ and $B_k =< v_k, h_v >$ with membership functions

$$\mu_{A_j}(t) = \begin{cases} \sqrt{1 - \frac{|t-u_j|}{h_u}} \text{ if } t \in [u_j - h_u, u_j + h_u], \\ 0 \text{ else,} \end{cases}$$

$$\mu_{B_k}(t) = \begin{cases} \sqrt{1 - \frac{|t-v_k|}{h_v}} \text{ if } t \in [v_k - h_v, v_k + h_v], \\ 0 \text{ else.} \end{cases}$$

Step 4. The information gains are:

$$r'_{j,k,i} = \begin{cases} \sqrt{1-\left[\left(\frac{|x_i-u_j|}{h_u}\right)^3+\left(\frac{|y_i-v_k|}{h_v}\right)^3\right]^{1/3}} \text{ if } \left(\frac{|x_i-u_j|}{h_u}\right)^3+\left(\frac{|y_i-v_k|}{h_v}\right)^3 \le 1, \\ 0 \text{ else.} \end{cases}$$

We calculate the normalized information matrix by using the formulas (4), (5) and (6).

Step 7. We calculate the centres of all rows in the normalized information matrix by formula (7):

$$\begin{aligned} c = {} & (-2.37, -2.90, -3.69, -4.33, -4.65, -4.64, -4.34, -3.83, -3.11, -2.28, \\ & -1.42, -0.56, 0.19, 0.83, 0.30, 0.61, 0.74, 0.71, 0.55, 0.28, \\ & 0.97, 0.67, 0.38, 0.17, 0.06, 0.064, 0.17, 0.38, 0.67, 0.97, \\ & 1.28, 1.55, 1.71, 1.74, 1.61, 1.30, 0.83, 0.19, -0.56, -1.42, \\ & -2.28, -3.11, -3.83, -4.34, -4.64, -4.65, -4.33, -3.69, -2.90, -2.37)^T . \end{aligned}$$

Step 8. The membership function of the then-part of fuzzy if-then rules are

$$\mu_{\bar{B}_j}(v) = \begin{cases} \sqrt{1-\frac{|v-c_j|}{h_v}} \text{ if } t \in [c_j - h_v, c_j + h_v], \\ 0 \text{ else.} \end{cases}$$

The membership function of $A_j \times \bar{B}_j$ fuzzy sets are

$$\mu_{A_j \times \bar{B}_j}(x,y) = \begin{cases} \sqrt{1-\left[\left(\frac{|x-u_j|}{h_u}\right)^3+\left(\frac{|y-v_k|}{h_v}\right)^3\right]^{1/3}} \text{ if } \left(\frac{|x-u_j|}{h_u}\right)^3+\left(\frac{|y-v_k|}{h_v}\right)^3 \le 1, \\ 0 \text{ else.} \end{cases}$$

for all $(x,y) \in \mathbb{R}^2$. The graphs of these functions are patches on the figure 1.

Step 9. The approximator function of f is

$$F(x) = \frac{\sum_{j=1}^{p} c_j \cdot g^{[-1]}\left(\frac{|x-u_j|}{h_u}\right)}{\sum_{j=1}^{p} g^{[-1]}\left(\frac{|x-u_j|}{h_u}\right)},$$

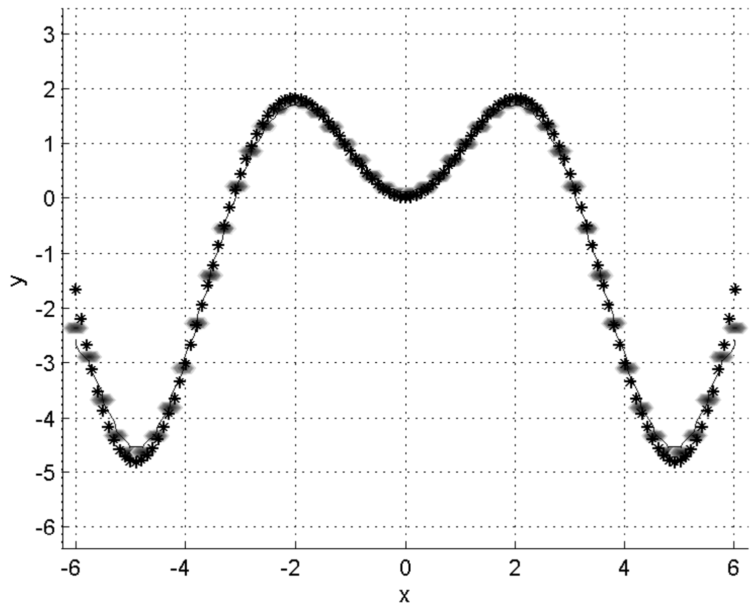

FIGURE 1. The additive fuzzy system F approximates the function f given in the example 4.3 by covering its graph with fuzzy patches. On the figure the stars are the elemets of sample, the curve is the graph of F and the pathces are the graph of fuzzy sets $A_j \times \bar{B}_j$.

for all $x \in [-6, 6]$, where

$$g^{[-1]}\left(\frac{|x-u_j|}{h_u}\right) = \begin{cases} \sqrt{1-\frac{|x-u_j|}{h_u}} \text{ if } x \in [u_j - h_u, u_j + h_u], \\ 0 \text{ else.} \end{cases}$$

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Department of Mathematics and Computer Science
Sapientia University, Miercurea Ciuc, Romania
E-mail address: makozoltan@sapientia.siculorum.ro