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A NOTE ON A GEOMETRIC CONSTRUCTION OF LARGE CAYLEY GRAPHS OF GIVEN DEGREE AND DIAMETER

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Abstract. An infinite series and some sporadic examples of large Cayley graphs with given degree and diameter are constructed. The graphs arise from arcs, caps and other objects of finite projective spaces.

A simple finite graph Γ is a (Δ, D)*-graph* if it has maximum degree Δ, and diameter at most D. The (Δ, D)*-problem* (or *degree/diameter problem*) is to determine the largest possible number of vertices that Γ can have. Denoted this number by $n(\Delta, D)$, the well-known *Moore bound* states that $n(\Delta, D) \leq \frac{\Delta(\Delta-1)^D-2}{\Delta-2}$. This is known to be attained only if either $D = 1$ and the graph is $K_{\Delta+1}$, or $D = 2$ and $\Delta = 1, 2, 3, 7$ and perhaps 57. If in addition Γ is required to be vertex-transitive, then the only known general lower bound is given as

$$n(\Delta, 2) \geq \left\lfloor \frac{\Delta+2}{2} \right\rfloor \cdot \left\lceil \frac{\Delta+2}{2} \right\rceil. \tag{1}$$

This is obtained by choosing Γ to be the Cayley graph $\mathrm{Cay}(\mathbb{Z}_a \times \mathbb{Z}_b, S)$, where $a = \lfloor \frac{\Delta+2}{2} \rfloor$, $b = \lceil \frac{\Delta+2}{2} \rceil$, and $S = \{\, (x, 0), (0, y) \mid x \in \mathbb{Z}_a \setminus \{0\}, y \in \mathbb{Z}_b \setminus \{0\} \,\}$. If $\Delta = kD+m$, where k, m are integers and $0 \leq m < D$, then a straightforward generalization of this construction results in a Cayley (Δ, D)-graph of order

$$\left\lfloor \frac{\Delta+D}{D} \right\rfloor^{D-m} \cdot \left\lceil \frac{\Delta+D}{D} \right\rceil^{m}. \tag{2}$$

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Throughout this note we will refer these graphs as GCCG-graphs (General Construction from Cyclic Groups). For special values of the parameters, (1) and (2) have been improved using various constructions. For more on the topic, we refer to [1, 8].

In this note we restrict our attention to the class of linear Cayley graphs. We present some constructions where the resulting graphs improve the lower bounds (1) and (2). For small number of vertices these are also compared to the known largest vertex transitive graphs having the same degree and diameter.

Let V denote the n-dimensional vector space over the finite field $\mathbb{F}_q$ of q elements, where $q = p^e$ for a prime p. For $S \subseteq V$ such that $0 \notin S$, and $S = -S := \{-x \mid x \in S\}$, the *Cayley graph* $\mathrm{Cay}(V, S)$ is the graph having vertex-set V, and edges $\{x, x+s\}$, $x \in V$, $s \in S$. To S we also refer as the connection set of the graph. A Cayley graph $\mathrm{Cay}(V, S)$ is said to be *linear*, [6, pp. 243] if $S = \alpha S := \{\alpha x \mid x \in S\}$ for all nonzero scalars $\alpha \in \mathbb{F}_q$. In this case $S \cup \{0\}$ is a union of 1-dimensional subspaces, and therefore, it can also be regarded as a point set in the projective space $\mathrm{PG}(n-1, q)$. Conversely, any point set $\mathcal{P}$ in $\mathrm{PG}(n-1, q)$ gives rise to a linear Cayley graph, namely the one having connection set $\{x \in V \setminus \{0\} \mid \langle x \rangle \in \mathcal{P}\}$. We denote this graph by $\Gamma(\mathcal{P})$. Given an arbitrary point set $\mathcal{P}$ in $\mathrm{PG}(n, q)$, $\langle \mathcal{P} \rangle$ denotes the projective subspace generated by the points in $\mathcal{P}$, and $\binom{\mathcal{P}}{k}$ ($k \in \mathbb{N}$) is the set of all subsets of $\mathcal{P}$ having cardinality k. The degree and diameter of linear Cayley graphs are given in the next proposition.

Proposition 1. *Let $\mathcal{P}$ be a set of k points in $\mathrm{PG}(n, q)$ with $\langle \mathcal{P} \rangle = \mathrm{PG}(n, q)$. Then $\Gamma(\mathcal{P})$ has q^{n+1} vertices, with degree $k(q-1)$, and with diameter*

$$D = \min \left\{ \, d \mid \cup_{\mathcal{X} \in \binom{\mathcal{P}}{d}} \langle \mathcal{X} \rangle = \mathrm{PG}(n, q) \, \right\}. \tag{3}$$

Proof. Let $\Gamma = \Gamma(\mathcal{P})$. It is immediate from its definition that Γ has q^{n+1} vertices and that its degree is equal to $k(q-1)$. Now let V denote the $(n+1)$-dimensional vector space over $\mathbb{F}_q$. Being a Cayley graph, Γ is automatically vertex-transitive, and so its diameter is the maximal distance $\delta_\Gamma(0, x)$ where $0 \in V$, and x runs over V. By δ_Γ we denote the usual distance function of Γ.

Let $x \in V \setminus \{0\}$, and let $P = \langle x \rangle$ be the corresponding point in $\mathrm{PG}(n,q)$. It can be seen that $\delta_\Gamma(0,x) = k$ where k is the minimal number of independent points $P_1, \ldots, P_k \in \mathcal{P}$ such that $P \in \langle P_1, \ldots, P_k \rangle$. Now, (3) shows that $\delta_\Gamma(0,x) \leq D$ for every $x \in V$, in particular, the diameter of Γ is at most D.

On the other hand, by (3), there exists a $Q \in \mathrm{PG}(n,q)$ for which $Q \notin \langle P_1, \ldots, P_{D-1} \rangle$ for any $P_1, \ldots, P_{D-1} \in \mathcal{P}$. Thus if y is an element of V with $\langle y \rangle = Q$, then $\delta_\Gamma(0,y) \geq D$. Therefore, the diameter of Γ cannot be less than D, which completes the proof. □

Once the number of vertices and the diameter for $\Gamma(\mathcal{P})$ are fixed to be q^{n+1} and D, respectively, our task becomes to search for the smallest possible point set $\mathcal{P}$ for which

$$\cup_{\mathcal{X} \in \binom{\mathcal{P}}{D}} \langle \mathcal{X} \rangle = \mathrm{PG}(n,q).$$

A point set having this property is called a *(D-1)-saturating set*.

The constructions

If $D = 2$, then a 1-saturating set $\mathcal{P}$ is a set of points of $\mathrm{PG}(n,q)$ such that the union of lines joining pairs of points of $\mathcal{P}$ covers the whole space. Assume that $n = 2$. If $\mathcal{P}$ contains k points, then the graph has degree $k(q-1)$ and the number of vertices is q^3. Hence this is better than the general lower bound (1) if and only if $q^3 > (k(q-1)+2)^2/4$, which is equivalent to

$$2\sqrt{q} + \frac{2}{\sqrt{q}+1} > k. \tag{4}$$

There are two known general constructions for 1-saturating sets in the plane: complete arcs and double blocking sets of Baer subplanes.

If q is a square, and $\Pi_{\sqrt{q}}$ is a Baer subplane of $\mathrm{PG}(2,q)$, of order $\sqrt{q}$, then each point of $\mathrm{PG}(2,q) \setminus \Pi_{\sqrt{q}}$ is incident with exactly one line of $\Pi_{\sqrt{q}}$. A double blocking set of a plane meets each line of the plane in at least two points. Hence a double blocking set of $\Pi_{\sqrt{q}}$ is a 1-saturating set of $\mathrm{PG}(2,q)$. The cardinality of a double blocking set of $\Pi_{\sqrt{q}}$ is at least $2(\sqrt{q}+\sqrt[4]{q}+1)$. This is greater than the bound given in (4), hence we cannot construct good graphs from these sets.

A complete k-arc $\mathcal{K}$ is a set of k points such that no three of them are collinear, and there is no $(k+1)$-arc containing $\mathcal{K}$. Thus $\mathcal{K}$ is a 1-saturating set, because if a point P would not be covered by the secants of $\mathcal{K}$, then $\mathcal{K}\cup\{P\}$ would be a $(k+1)$-arc. The cardinality of the smallest complete arc in $\mathrm{PG}(2,q)$ is denoted by $t_2(2,q)$. For the known values of $t_2(2,q)$ we refer to [3]. The general lower bounds are $t_2(2,q)>\sqrt{2q}+1$ for arbitrary q and $t_2(2,q)>\sqrt{3q}+1/2$ for $q=p^i$, $i=1,2,3$. But unfortunately the known complete arcs have bigger cardinality. The inequality

$$t_2(2,q)<2\sqrt{q}+\frac{2}{\sqrt{q}+1}$$

is satisfied only for $q=8,9,11$ and 13. Table 1 gives the corresponding values of $t_2(2,q)$ and the parameters of the graphs arising from these arcs.

q	$t_2(2,q)$	D	Δ	number of vertices of Γ	$\lfloor\frac{\Delta+2}{2}\rfloor\cdot\lceil\frac{\Delta+2}{2}\rceil$.
8	6	2	42	512	484
9	6	2	48	729	625
11	7	2	70	1331	1296
13	8	2	96	2197	2116

Table 1

Besides complete arcs and double blocking sets of Baer subplanes another class of small 1-saturating sets in $\mathrm{PG}(2,p)$ was examined by computer. These point sets are contained in 3 concurrent lines. For small prime orders $p=11,13,17,19$, using a simple back-track algorithm we found 1-saturating sets of this type with cardinality $10,11,13$ and 14, respectively. The corresponding graphs do not improve the bound in (1).

Now let $n>2$. Then a set of k points such that no three of them are collinear is called k-cap. A k-cap is complete, if it is not contained in any $(k+1)$-cap. Hence complete caps in $\mathrm{PG}(n,q)$ are 1-saturating sets. For the sizes of the known complete caps we refer to [7]. There is one infinite series which gives better graphs than the GCCG-graphs. Due to Davydov and Drozhzhina-Labinskaya [5], for $n = 2m-1 > 7$ there is a complete $(27 \cdot 2^{m-4} - 1)$-cap in $\mathrm{PG}(n,2)$. This gives a graph of degree $27 \cdot 2^{m-4} - 1$ and of order 2^{2m}. It has much more vertices than the corresponding GCCG-graph, because

$$2^{2m} = 1024 \cdot 2^{2m-10} > 729 \cdot 2^{2m-10} + 27 \cdot 2^{m-5} = \left\lfloor \frac{27 \cdot 2^{m-4}+1}{2} \right\rfloor \cdot \left\lceil \frac{27 \cdot 2^{m-4}+1}{2} \right\rceil.$$

Hence we proved the following theorem.

Theorem 1. *Let* $\Delta = 27 \cdot 2^{m-4} - 1$ *and* $m > 7$. *Then*

$$n(\Delta, 2) \geq \frac{256}{729}(\Delta+1)^2.$$

There are sporadic examples, too. For $n = 3$ and $q = 2$ there is a complete 5-cap in $\mathrm{PG}(3,2)$. The corresponding graph has degree $\Delta = 5$ and the number of vertices is $n = 16$. The best known graph of degree 5 and diameter 2 has 24 vertices, and the best known Cayley graph has 18 vertices [2], so in this case there are bigger graphs. For $q = 3, 4$ and 5 the smallest complete caps in $\mathrm{PG}(3,q)$ have $2(q+1)$ points. The corresponding graphs have the same parameters as the GCCG-graphs.

For $n = 4$ and $q = 2, 3, 4$ there are complete caps in $\mathrm{PG}(4,q)$ with cardinalities 9, 11 and 20, respectively. For $n = 5$ and $q = 2, 3$ there are complete caps in $\mathrm{PG}(5,q)$ with cardinalities 13 and 22. The corresponding graphs have more vertices than the previously known examples. Table 2 gives the parameters of the graphs arising from these caps.

projective space	size of the complete cap	D	Δ	number of vertices of Γ	$\lfloor \frac{\Delta+2}{2} \rfloor \cdot \lceil \frac{\Delta+2}{2} \rceil$.
PG(4, 2)	9	2	9	32	30
PG(4, 3)	11	2	22	243	144
PG(4, 4)	20	2	60	1024	961
PG(5, 2)	13	2	13	64	56
PG(5, 3)	22	2	44	729	529

Table 2

In $\mathrm{PG}(3,q)$, $q > 3$, the smallest known 1-saturating set has $2q+1$ points [4]. Let π be a plane, Ω be an oval in π, P be a point of Ω, for q even let $N \in \pi$ be the nucleus of Ω, for q odd let $N \in \pi$ be a point such that the line NP is the tangent to Ω at P, and finally let ℓ be a line such that $\ell \cap \pi = \{P\}$. Then it is easy to check that $(\Omega \cup \ell \cup \{N\}) \setminus \{P\}$ is a 1-saturating set in $\mathrm{PG}(3,q)$. The corresponding graph has degree $\Delta = 2q^2 - q - 1$, and the number of its vertices is $q^4 > (\Delta + \sqrt{\Delta/2} + 5/4)^2/4$. Hence we proved the following theorem.

Theorem 2. *Let $q > 3$ be a prime power and let $\Delta = 2q^2 - q - 1$. Then*

$$n(\Delta, 2) > \frac{1}{4}\left(\Delta + \sqrt{\frac{\Delta}{2}} + \frac{5}{4}\right)^2.$$

Let ℓ_1 and ℓ_2 be two skew lines in $\mathrm{PG}(3,q)$. If P is any point not on $\ell_1 \cup \ell_2$, then the plane generated by P and ℓ_1 meets ℓ_2 in a unique point T_2, and the line PT_2 meets ℓ_1 in a unique point T_1. Hence the line T_1T_2 contains P, so the set of points of $\ell_1 \cup \ell_2$ is a 1-saturating set in $\mathrm{PG}(3,q)$. The corresponding graph has degree $\Delta = 2(q^2-1)$, and the number of its vertices is $q^4 = \big((\Delta+2)/2\big)^2$. Hence this construction gives graphs having the same parameters as the GCCG-graphs.

A straightforward generalization of the skew line construction is the following. Let $\ell_1, \ell_2, \ldots, \ell_m$ be a set of m lines whose union spans $\mathrm{PG}(2m-1,q)$. Then the set of points of $\cup_{i=1}^{m} \ell_i$ is an $(m-1)$-saturating set and the corresponding graph has parameters $D = m$, $\Delta = 2m(q^2-1)$, and the number of its vertices is q^{2m}. These parameters are the same as the parameters of the GCCG-graphs.

Another class of examples for $(D-1)$-saturating sets in $\mathrm{PG}(D,q)$ is the class of complete arcs. These objects are generalizations of the planar arcs. A point set $\mathcal{K}$ is a complete k-arc in $\mathrm{PG}(D,q)$ if no D points of $\mathcal{K}$ lie in a hyperplane, and there is no $(k+1)$-arc containing $\mathcal{K}$. The corresponding graph has degree $k(q-1)$ and the number of vertices is q^{D+1}. Hence this is better than the known general lower bound if and only if

$$q^{D+1} > \left(\frac{k(q-1)+D}{D}\right)^{D}, \quad \text{that is} \quad k < \frac{D(q\sqrt[D]{q}-1)}{q-1}. \tag{5}$$

The typical examples for complete arcs are the normal rational curves, and almost all of the known complete arcs are normal rational curves, or subsets of these curves. There is only one known complete k-arc which satisfies (5). This is a normal rational curve in $\mathrm{PG}(4,3)$. The corresponding graph has degree $\Delta = 15$, diameter $D = 3$ and the number of its vertices is 256.

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