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EXISTENCE AND DATA DEPENDENCE FOR MULTIVALUED WEAKLY CONTRACTIVE OPERATORS

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Abstract. The purpose of this paper is to study the data dependence for the fixed point set of a multivalued weakly contractive operator with respect to a w-distance in the sense of T. Suzuki and W. Takahashi. We also give a fixed point result for a multivalued weakly φ-contraction on a metric space endowed with a w-distance.

1. **Introduction**

Let (X, d) be a metric space. A singlevalued operator T from X into itself is called r-contractive (see [2]) if there exists a real number $r \in [0, 1)$ such that $d(T(x), T(y)) \leq rd(x, y)$ for every $x, y \in X$. It is well know that if X is a complete metric space then a contractive operator from x into itself has a unique fixed point in X.

In 1996, the Japanese mathematicians O. Kada, T. Suzuki and W. Takahashi introduced the concept of w-distance (see[2]) and discussed some properties of this functional. Later on, T. Suzuki and W. Takahashi gave some fixed points results for a new class of nonlinear operators, namely the so-called weakly contractive operators (see[3]).

The purpose of this paper is to study the data dependence for the fixed point set of a multivalued weakly contractive operator with respect to a w-distance in the sense of T. Suzuki and W. Takahashi, see [3]. We also give a fixed point result for a

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multivalued weakly φ-contraction on a metric space endowed with a w-distance. For connected results see [6], [4].

2. **Preliminaries**

Let (X, d) be a complete metric space. We will use the following notations (see also [1], [5]).

$P(X)$ - *the set of all nonempty subsets of* X;

$\mathcal{P}(X) = P(X) \bigcup \emptyset$

$P_{cl}(X)$ - *the set of all nonempty closed subsets of* X;

$P_b(X)$ - *the set of all nonempty bounded subsets of* X;

$P_{b,cl}(X)$ - *the set of all nonempty bounded and closed subsets of* X;

We introduce now the following generalized functionals on a b-metric space (X, d).

The gap functional:

$$(1) \quad D : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$$

$$D(A,B) = \begin{cases} \inf\{d(a,b)|\ a \in A,\ b \in B\}, & A \neq \emptyset \neq B \\ 0, & A = \emptyset = B \\ +\infty, & \text{otherwise} \end{cases}$$

In particular, if $x_0 \in X$ then $D(x_0, B) := D(\{x_0\}, B)$.

The excess generalized functional:

$$(2) \quad \rho : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$$

$$\rho(A,B) = \begin{cases} \sup\{D(a,B)|\ a \in A\}, & A \neq \emptyset \neq B \\ 0, & A = \emptyset \\ +\infty, & B = \emptyset \neq A \end{cases}$$

Pompeiu-Hausdorff generalized functional:

$$(3) \quad H : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$$

$$H(A,B) = \begin{cases} \max\{\rho(A,B), \rho(B,A)\}, & A \neq \emptyset \neq B \\ 0, & A = \emptyset = B \\ +\infty, & \text{othewise} \end{cases}$$

Delta functional:

$$(4)\ \ \delta : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$$

$$\delta(A,B) = \begin{cases} \sup\{d(a,b) : a \in A, b \in B\}, & A \neq \emptyset \neq B \\ 0, & A = \emptyset = B \\ +\infty, & \text{othewise} \end{cases}$$

In particular, $\delta(A) := \delta(A, A)$ is the diameter of the set A.

It is known that $(P_{b,cl}(X), H)$ is a complete metric space provided (X, d) is a complete metric space.

We will denote by $FixF := \{x \in X \mid x \in F(x)\}$, the set of the fixed points of F.

The concept of w-distance was introduced by O. Kada, T. Suzuki and W. Takahashi (see[2]) as follows:

Let (X,d) be a metric space. Then, the functional $w : X \times X \to [0, \infty)$ is called w-distance on X if the following axioms are satisfied :

1. $w(x, z) \leq w(x, y) + w(y, z)$, for any $x, y, z \in X$;
2. for any $x \in X : w(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous;
3. for any $\varepsilon > 0$, exists $\delta > 0$ such that $w(z, x) \leq \delta$ and $w(z, y) \leq \delta$ implies $d(x, y) \leq \varepsilon$.

Let us give some examples of w-distance (see [2])

Example 2.1. Let (X, d) be a metric space . Then the metric "d" is a w-distance on X.

Example 2.2. Let X be a normed linear space with norm $||\cdot||$. Then the function $w : X \times X \to [0, \infty)$ defined by $w(x, y) = ||x|| + ||y||$ for every $x, y \in X$ is a w-distance.

Example 2.3. Let (X,d) be a metric space and let $g : X \to X$ a continuous mapping. Then the function $w : X \times Y \to [0, \infty)$ defined by:

$$w(x, y) = max\{d(g(x), y), d(g(x), g(y))\}$$

for every $x, y \in X$ is a w-distance.

For the proof of the main results we need the following crucial result for w-distance (see[3]).

Lemma 2.4._Let (X, d) be a metric space, and let w be a w-distance on X. Let (x_n) and (y_n) be two sequences in X, let (α_n), (β_n) be sequences in $[0, +\infty[$ converging to zero and let $x, y, z \in X$. Then the following hold:_

1. _If $w(x_n, y) \leq \alpha_n$ and $w(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$._
2. _If $w(x_n, y_n) \leq \alpha_n$ and $w(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then (y_n) converges to z._
3. _If $w(x_n, x_m) \leq \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then (x_n) is a Cauchy sequence._
4. _If $w(y, x_n) \leq \alpha_n$ for any $n \in \mathbb{N}$, then (x_n) is a Cauchy sequence._

3. **Data dependence for w-contractive multivalued operators**

In [3]. the definition of a weakly contractive multivalued operator is given, as follows.

Definition 3.1. Let X be a metric space with metric d. A multivalued operator $T : X \to P(X)$ is called weakly contractive or w-contractive if there exists a w-distance w on X and $r \in [0, 1)$ such that for any $x_1, x_2 \in X$ and $y_1 \in T(x_1)$ there is $y_2 \in T(x_2)$ with $w(y_1, y_2) \leq rw(x_1, x_2)$.

Then, in the same paper, T. Suzuki and W. Takahashi gave the following fixed point result for a multivalued weakly contractive operator (see Theorem 1, [3]).

Theorem 3.2. _Let X be a complete metric space and let $T : X \to P(X)$ be a w-contractive multivalued operator such that for any $x \in X$, $T(x)$ is a nonempty closed subset of X. Then there exists $x_0 \in X$ such that $x_0 \in T(x_0)$ and $w(x_0, x_0) = 0$._

The main result of this section is the following data dependence theorem with respect to the fixed point set of the above class of operators.

Theorem 3.3. *Let (X, d) be a complete metric space, $T_1, T_2 : X \to P_{cl}(X)$ be two w-contractive multivalued operators with $r_i \in [0, 1)$ with $i = \{1, 2\}$. Then the following are true:*

1. $FixT_1 \neq \emptyset \neq FixT_2$;
2. *We suppose that there exists $\eta > 0$ such that for every $u \in T_1(x)$ there exists $v \in T_2(x)$ such that $w(u, v) \leq \eta$, (respectively for every $v \in T_2(x)$ there exists $u \in T_1(x)$ such that $w(v, u) \leq \eta$).*

Then for every $u^* \in FixT_1$ there exists $v^* \in FixT_2$ such that

$$w(u^*, v^*) \leq \frac{\eta}{1-r}, \text{ where } r = r_i \text{ for } i = \{1, 2\};$$

(respectively for every $v^* \in FixT_2$ there exists $u^* \in FixT_1$ such that

$$w(v^*, u^*) \leq \frac{\eta}{1-r}, \text{ where } r = r_i \text{ for } i = \{1, 2\})$$

Proof. Let $u_0 \in FixT_1$, then $u_0 \in T_1(u_0)$. Using the hypothesis 2. we have that there exists $u_1 \in T_2(u_0)$ such that $w(u_0, u_1) \leq \eta$.

Since T_1, T_2 are weakly contractive with $r_i \in [0, 1)$ and $i = \{1, 2\}$ we have that for every $u_0, u_1 \in X$ with $u_1 \in T_2(u_0)$ there exists $u_2 \in T_2(u_1)$ such that

$$w(u_1, u_2) \leq rw(u_0, u_1)$$

For $u_1 \in X$ and $u_2 \in T_2(u_1)$ there exists $u_3 \in T_2(u_2)$ such that

$$w(u_2, u_3) \leq rw(u_1, u_2) \leq r^2 w(u_0, u_1)$$

By induction we obtain a sequence $(u_n)_{n \in \mathbb{N}} \in X$ such that

(1) $u_{n+1} \in T_2(u_n)$, for every $n \in \mathbb{N}$;

(2) $w(u_n, u_{n+1}) \leq r^n w(u_0, u_1)$

For $n, p \in \mathbb{N}$ we have the inequality

$$\begin{aligned} w(u_n, u_{n+p}) &\leq w(u_n, u_{n+1}) + w(u_{n+1}, u_{n+2}) + \cdots + w(u_{n+p-1}, u_{n+p}) \leq \\ &< r^n w(u_0, u_1) + r^{n+1} w(u_0, u_1) + \cdots + r^{n+p-1} w(u_0, u_1) \leq \\ &\leq \frac{r^n}{1-r} w(u_0, u_1) \end{aligned}$$

By the Lemma 2.4.(3) we have that the sequence $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since (X,d) is a complete metric space we have that there exists $v^* \in X$ such that $u_n \xrightarrow{d} v^*$.

By the lower semicontinuity of $w(x,\cdot) : X \to [0,\infty)$ we have

$$w(u_n, v^*) \le \lim_{p\to\infty} \inf w(u_n, u_{n+p}) \le \frac{r^n}{1-r} w(u_0, u_1) \tag{1}$$

For $u_{n-1}, v^* \in X$ and $u_n \in T_2(u_{n-1})$ there exists $z_n \in T_2(v^*)$ such that, using relation (1), we have

$$w(u_n, z_n) \le r w(u_{n-1}, v^*) \le \frac{r^{n-1}}{1-r} w(u_0, u_1) \tag{2}$$

Applying Lemma 2.4.(2), from relations (1) and (2) we have that $z_n \xrightarrow{d} v^*$.

Then, we know that $z_n \in T_2(v^*)$ and $z_n \xrightarrow{d} v^*$. In this case, by the closure of T_2 result that $v^* \in T_2(v^*)$. Then, by $w(u_n, v^*) \le \frac{r^n}{1-r} w(u_0, u_1)$, with $n \in \mathbb{N}$, for $n = 0$ we obtain

$$w(u_0, v^*) \le \frac{1}{1-r} w(u_0, u_1) \le \frac{\eta}{1-r}$$

which completes the proof. □

4. **Existence of fixed points for multivalued weakly φ-contractive operators**

Let us define first, the notion of multivalued weakly φ-contractive operator.

Definition 4.1. Let (X,d) be a metric space and $T : X \to P(X)$ be a multivalued operator. Then T is called weakly φ-contractive if there exists a w-distance on X and a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that for every x_1, x_2 and $y_1 \in T(x_1)$ there is $y_2 \in T(x_2)$ with $w(y_1, y_2) \le \varphi(w(x_1, x_2))$.

The main result is the following result for weakly φ-contractive operators.

Theorem 4.2. *Let (X,d) be a complete metric space, $w : X \times X \to \mathbb{R}_+$ a w-distance on X, $T : X \to P_{cl}(X)$ be a multivalued operator and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ a function such that are accomplish the following conditions:*

1. *T are weakly φ-contractive operator;*

2. *The function* φ *is a monotone increasing function such that*

$$\sigma(t) := \sum_{n=0}^{\infty} \varphi^n(t) < \infty, \text{ for every } t \in \mathbb{R}_+ \setminus \{0\}.$$

Then there exists $x^* \in X$ *such that* $x^* \in T(x^*)$ *and* $w(x^*, x^*) = 0$.

Proof. First, we remark that condition (2) from hypothesis implies that $\varphi(t) < t$ for $t < 0$.

Fix $x_0 \in x$; for $x_1 \in T(x_0)$ there exists $x_2 \in T(x_1)$ such that

$$w(x_1, x_2) \leq \varphi(w(x_0, x_1)).$$

For $x_1 \in X$ and $x_2 \in T(x_1)$ there exists $x_3 \in T(x_2)$ such that

$$w(x_2, x_3) \leq \varphi(w(x_1, x_2)) \leq \varphi(\varphi(w(x_0, x_1))) = \varphi^2(w(x_0, x_1)).$$

By induction we obtain a sequence $(x_n)_{n\in\mathbb{N}} \in X$ such that

(i) $x_{n+1} \in T(x_n)$, for $n \in \mathbb{N}$;

(ii) $w(x_n, x_{n+1}) \leq \varphi^n(w(x_0, x_1))$, for $n \in \mathbb{N}$.

For $n, p \in \mathbb{N}$ we have

$$\begin{aligned} w(x_n, x_{n+p}) &\leq w(x_n, x_{n+1}) + w(x_{n+1}, x_{n+2}) + \cdots + w(x_{n+p-1}, x_{n+p}) \leq \\ &< \varphi^n(w(x_0, x_1)) + \varphi^{n+1}(w(x_0, x_1)) + \cdots + \varphi^{n+p-1}(w(x_0, x_1)) \leq \\ &\leq \sum_{n=k}^{\infty} \varphi^k(w(x_0, x_1)) \leq \sigma(w(x_0, x_1)). \end{aligned}$$

Letting $n \to \infty$ we have

$$\lim_{n\to\infty} w(x_n, x_{n+p}) \leq \lim_{n\to\infty} \sigma(\varphi^n(w(x_0, x_1))) = 0.$$

By the Lemma 2.4.(3) we have that the sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since (X, d) is a complete metric space then there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$.

For $n, m \in \mathbb{N}$ with $m > n$ from the above inequality we have

$$w(x_n, x_m) \leq \sigma(\varphi^n(w(x_0, x_1))).$$

Since $(x_m)_{m\in\mathbb{N}}$ converge to x^* and $w(x_n, \cdot)$ is lower semicontinuous we have

$$w(x_n, x^*) \leq \lim_{m\to\infty} \inf w(x_n, x_m) \leq \lim_{m\to\infty} \sigma(\varphi^n(w(x_0, x_1))) \leq \sigma(\varphi^n(w(x_0, x_1))).$$

So, for every $n \in \mathbb{N}$, $w(x_n, x^*) \le \sigma(\varphi^n(w(x_0, x_1)))$

For $x^* \in X$ and $x_n \in T(x_{n-1})$ there exists $u_n \in T(x^*)$ such that

$$w(x_n, u_n) \le \varphi(w(x_{n-1}, x^*)) \le \varphi(\sigma(\varphi^{n-1}(w(x_0, x_1)))) < \sigma(\varphi^{n-1}(w(x_0, x_1)))$$

So, we know that:

$$w(x_n, u_n) \le \sigma(\varphi^{n-1}(w(x_0, x_1)))$$

$$w(x_n, x^*) \le \sigma(\varphi^n(w(x_0, x_1)))$$

Then, by the Lemma 2.4.(2), we obtain that $u_n \stackrel{d}{\to} x^*$. As $u_n \in T(x^*)$ and using the closure of T result that $x^* \in T(x^*)$.

For $x^* \in X$ and $x^* \in T(x^*)$, using the hypothesis (1), there exists $z_1 \in T(x^*)$ such that

$$w(x^*, z_1) \le \varphi(w(x^*, x^*)).$$

For $x^*, z_1 \in X$ and $x^* \in T(x^*)$ there exists $z_2 \in T(z_1)$ such that

$$w(x^*, z_2) \le \varphi(x^*, z_1).$$

By induction we get a sequence $(z_n)_{n\in\mathbb{N}} \in X$ such that

(i) $z_{n+1} \in T(z_n)$, for every $n \in \mathbb{N}$;

(ii) $w(x^*, z_n) \le \varphi(w(x^*, z_{n-1}))$, for every $n \in \mathbb{N} \setminus \{0\}$.

Therefore we have

$$w(x^*, z_n) \le \varphi(w(x^*, z_{n-1})) \le \varphi(\varphi(w(x^*, z_{n-2}))) = \varphi^2(w(x^*, z_{n-2})) \le \cdots \le$$

$$\le \varphi^n(w(x^*, z_1)) \le \varphi^n(w(x^*, x^*)).$$

Thus $w(x^*, z_n) \le \varphi^n(w(x^*, x^*))$.

When $n \to \infty$, $\varphi^n(w(x^*, x^*))$ converge to 0. Thus, by the Lemma 2.4.(4) we obtain that $(z_n)_{n\in\mathbb{N}} \in X$ is a Cauchy sequence in (X, d) and there exists $z^* \in X$ such that $z_n \stackrel{d}{\to} z^*$.

Since $w(x^*, \cdot)$ is lower semicontinuous we have

$$0 \le w(x^*, z^*) \le \lim_{n\to\infty} \inf w(x^*, z_n) \le \lim_{n\to\infty} \varphi^n(w(x^*, x^*)) = 0.$$

Then $w(x^*, z^*) = 0$.

So, by triangle inequality we have

$$w(x_n, z^*) \leq w(x_n, x^*) + w(x^*, z^*) \leq \sigma(\varphi^n(w(x_0, x_1))).$$

Since $\sigma(\varphi^n(w(x_0, x_1)))$ converge to 0 when $n \to \infty$ we have

$$w(x_n, z^*) \leq \sigma(\varphi^n(w(x_0, x_1)))$$

$$w(x_n, x^*) \leq \sigma(\varphi^n(w(x_0, x_1)))$$

Using Lemma 2.4.(1) result that $z^* = x^*$, then $w(x^*, x^*) = 0$. □

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