

STRONG AND CONVERSE FENCHEL DUALITY FOR VECTOR OPTIMIZATION PROBLEMS IN LOCALLY CONVEX SPACES

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Abstract. In relation to the vector optimization problem $v\text{-min}_{x \in X}(f + g \circ A)(x)$, with f, g proper and cone-convex functions and $A : X \rightarrow Y$ a linear continuous operator between separated locally convex spaces, we define a general vector Fenchel-type dual problem. For the primal-dual pair we prove weak, and under appropriate regularity conditions, strong and converse duality. In the particular case when the image space is \mathbb{R}^m we compare the new dual with two other duals, whose definitions were inspired from [9] and [10], respectively. The sets of Pareto efficient elements of the image sets of their feasible sets through the corresponding objective functions prove to be equal, despite the fact that among the image sets of the problems, strict inclusion usually holds. This equality allows us to derive weak, strong and converse duality results for the later two dual problems, from the corresponding results of the first mentioned one. Our results could be implemented in various practical areas, since they provide sufficient conditions for the existence of optimal solutions for vector optimization problems defined on very general spaces. They can be used in medical areas, for example in the study of chronic diseases and in oncology.

Received by the editors: 04.12.2008.

2000 *Mathematics Subject Classification.* 49N15, 32C37, 90C29.

Key words and phrases. conjugate functions, Fenchel duality, vector optimization, weak, strong and converse duality.

This paper was presented at the 7-th Joint Conference on Mathematics and Computer Science, July 3-6, 2008, Cluj-Napoca, Romania.

This paper is supported in the framework of CRONIS. Project number is 11-003/2007, financed by National Programs Management Center through the 4th Programme "Partnerships in Proprietary Domains".

1. Introduction

Vector optimization problems have generated a great deal of interest during the last years, not only from a theoretical point of view, but also from a practical one, due to their applicability in different fields, such as economics, engineering and lately in medical areas. In general, when dealing with scalar optimization problems, the duality theory proves to be an important tool for giving dual characterizations of the optimal solutions of a primal problem. Similar characterizations can also be given for vector optimization problems.

An overview on the literature dedicated to this field shows that the general interest has been centered on vector problems having inequality constraints and on an extension of the classical Lagrange duality approach. We recall in this direction the concepts developed by Mond and Weir in [23], [24] (whose formulation is based on optimality conditions which follow from the scalar Lagrange duality). Tanino, Nakayama and Sawaragi examined in [21] the duality for vector optimization in finite dimensional spaces, using perturbations, which led them also to Lagrange-type duals. They extended Rockafellar's fully developed theory from [19] for scalar optimization to the vector case. In Jahn's paper [16] the Lagrange dual appears explicitly in the formulation of the feasible set of the multiobjective dual.

Another approach is due to Boţ and Wanka, who, in [8] constructed a vector dual using the Fenchel-Lagrange dual for scalar optimization problems, introduced by the authors in [3], [6], [7].

With respect to the vector duality based on Fenchel's duality concept, the bibliography is not very rich. We mention in this direction the works of Breckner and Kolumbán [10] and [11], continued by Breckner in [12], [13], Gerstewitz and Göpfert [15], Malivert [18] as well as the recent paper of Boţ, Dumitru (Grad) and Wanka [9].

In relation to the vector optimization problem $v\text{-min}_{x \in X}(f + g \circ A)(x)$, with f, g proper and cone-convex functions and $A : X \rightarrow Y$ a linear continuous operator between separated locally convex spaces, we define a general vector Fenchel-type dual problem. For this dual pairs of problems we prove weak, and under appropriate,

quite general regularity conditions, strong and converse duality. In the particular case when the image space is \mathbb{R}^m , we compare the new dual with two other duals, whose definitions were inspired from [9] and [10], respectively. Their sets of optimal solutions prove to be equal, despite the fact among the image sets of the problems, strict inclusion usually holds. This equality allows us to derive weak, strong and converse duality results for the later two dual problems, from the corresponding results of the first mentioned one.

The paper is organized as it follows. In Section 2 we recall some elements of convex analysis which are used later on. Using the formulation of the scalarized dual, we define in Section 3, the new vector dual problem. For it, we prove weak, strong and converse duality. In order to be able to understand the position of our dual, among other duals given in the literature, we present another Fenchel-type dual problems inspired by Breckner and Kolumbán's paper [11]. Weak, strong and converse duality for the later problem can be proved, using the corresponding theorems for the initial treated problems. Section 4 contains a further comparison, to a third dual problem, this time inspired from Boţ, Dumitru (Grad) and Wanka, (cf. [9]). The image sets of the three duals are tightly connected, as it is proved. Moreover, we illustrate by some examples that in general these inclusions are strict. Finally, we show that even though this happens, the sets of the maximal elements of the image sets of the feasible sets through the corresponding objective functions coincide.

The practical applicability of our results is vast, since they provide, among others, sufficient conditions for the existence of optimal solutions for a large area of optimization problems, in both finite and infinite dimensional spaces. Such results could be successfully applied in the study of chronic diseases, oncology, economy and the list could continue.

2. Preliminaries

Let X be a real separated locally convex space, and let X^* be its topological dual. By $\langle x^*, x \rangle$ we understand the value of the linear continuous functional $x^* \in X^*$ at $x \in X$.

Given a function $f : X \rightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, its *domain* is the set

$$\text{dom } f := \{x \in X : f(x) < +\infty\}.$$

We call f *proper* if $\text{dom } f \neq \emptyset$ and $f(x) > -\infty$ for all $x \in X$. The *conjugate function* associated with f is $f^* : X^* \rightarrow \overline{\mathbb{R}}$ defined by

$$f^*(x^*) := \sup_{x \in X} \{\langle x^*, x \rangle - f(x)\} \text{ for all } x^* \in X^*.$$

The function f is said to be *convex* if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for all $x, y \in X$ and all $\lambda \in [0, 1]$.

Given a nonempty convex cone $C \subseteq X$, we denote by

$$C^+ := \{x^* \in X^* : \langle x^*, x \rangle \geq 0 \text{ for all } x \in C\}$$

its *dual cone* and by

$$C^{+0} := \{x^* \in X^* : \langle x^*, x \rangle > 0 \text{ for all } x \in C \setminus \{0\}\}$$

the *quasi-interior* of the dual cone. The convex cone C induces on X a partial ordering defined by $x \leq_C y$ (denoted also by $y \geq_C x$) if $y - x \in C$ for all $x, y \in X$. If $y - x \in C \setminus \{0\}$ we use the notation $x \leq_C y$ (denoted also by $y \geq_C x$).

There are notions referring to extended real-valued functions that can be generalized to functions taking values in infinite dimensional spaces. Thus, let Y be another real separated locally convex space partially ordered by the nonempty convex cone K . To Y we attach a greatest element ∞_Y with respect to \leq_K , which does not belong to Y . Moreover, we set $Y^\bullet := Y \cup \{\infty_K\}$ and consider on Y^\bullet the following operations: $y + \infty_K = \infty_K$, $t \cdot \infty_K = \infty_K$ and $\langle \lambda, \infty_K \rangle = +\infty$ for all $y \in Y$, $t \geq 0$ and $\lambda \in K^+$.

For a function $F : X \rightarrow Y^\bullet$ its *domain* is defined by

$$\text{dom } F := \{x \in X : F(x) \in Y\}.$$

If $\text{dom}(F) \neq \emptyset$, then F is said to be *proper*. The most common extension of the classical convexity of an extended real-valued function to a vector-valued function is the notion of cone-convexity. Thus, F is said to be *K-convex* if

$$F(tx + (1-t)y) \leq_K tF(x) + (1-t)F(y)$$

for all $x, y \in X$ and all $t \in [0, 1]$.

For each $\lambda \in K^+$ we consider the function $(\lambda F) : X \rightarrow \overline{\mathbb{R}}$ defined by $(\lambda F)(x) = \langle \lambda, F(x) \rangle$ for all $x \in X$. In literature there are known several generalizations of the lower semicontinuity of extended real-valued functions to vector-valued functions. Here we mention one of them. The function F is said to be *star-K lower semicontinuous* if, for each $\lambda \in K^+$, the function (λF) is lower semicontinuous.

If U is a nonempty subset of X we denote by $\text{lin}U$ its *linear hull* and by $\text{cone}U := \cup_{\lambda \geq 0} \lambda U$ its *conic hull*. The *algebraic interior* associated with U is the set

$$\text{core}U := \{u \in U : \forall x \in X, \exists \delta > 0 \text{ s.t. } \forall \lambda \in [0, \delta] : u + \lambda x \in U\}.$$

When U is a convex set, then $u \in \text{core}U$ if and only if $\text{cone}(U - x) = X$. In general, we have $\text{int}U \subseteq \text{core}U$, where $\text{int}U$ denotes the *interior* of U . When U is convex then $\text{int}U = \text{core}U$ if one of the following conditions is satisfied: $\text{int}U \neq \emptyset$; X is a Banach space and U is closed; X is finite dimensional (cf. [20]). Further, by maintaining the convexity assumption for U , one can define the *strong quasi-relative interior* of U , denoted by $\text{sqri}U$, as

$$\text{sqri}U := \{u \in U : \text{cone}(U - u) \text{ is a closed linear subspace of } X\} \text{ (cf. [1]).}$$

We notice that $\text{core}U \subseteq \text{sqri}U$. If X is finite dimensional, then $\text{sqri}U = \text{ri}U$, where $\text{ri}U$ denotes the *relative interior* of the set U , i.e. the set of the interior points of U relative to the affine hull of U .

3. Fenchel-Type Vector Duality

Let X, Y and V be real separated locally convex spaces, and let V be partially ordered by a nonempty pointed convex cone $K \subseteq V$. We shall study the general vector

optimization problem

$$(P) \quad \underset{x \in X}{\text{v-min}}(f + g \circ A)(x),$$

where $f : X \rightarrow V^\bullet = V \cup \{\infty_K\}$ and $g : Y \rightarrow V^\bullet$ are proper, K -convex functions and $A : X \rightarrow Y$ is a linear continuous operator such that $\text{dom } f \cap A^{-1}(\text{dom } g) \neq \emptyset$.

Due to the fact that the partial order induced on a vector space by a convex cone is not total, several notions of optimal solutions for vector optimization problems have been introduced during the years in the literature. For such definitions and their properties we refer the reader to [17]. In this paper we work with Pareto-efficient and properly efficient solutions. Particularly, for the problem (P) we study the existence of properly efficient solutions.

Definition 1. *An element $\bar{x} \in X$ is a properly efficient solution to (P) if there exists $v^* \in K^{+0}$ such that*

$$\langle v^*, (f + g \circ A)(\bar{x}) \rangle \leq \langle v^*, (f + g \circ A)(x) \rangle \text{ for all } x \in X.$$

Duality is an extremely used procedure in optimization. It consists in associating with a certain optimization problem, called primal problem, a new one, called dual problem, whose solutions may characterize the optimal solutions of the primal problem. In order to ensure strong and converse duality, respectively, certain regularity conditions have to be imposed on the functions and sets involved in the definition of the problems.

In this paper we treat three different types of dual problems associated with the vector optimization problem (P), for which we prove weak, strong and converse duality. Furthermore, we shall compare the image sets of the feasible sets through the corresponding objective functions for the three problems.

The first dual associated with the primal vector optimization problem is

$$(D^{\leq}) \quad \underset{(v^*, y^*, v) \in \mathcal{B}^{\leq}}{\text{v-max}} \quad h^{\leq}(v^*, y^*, v),$$

where the feasible set is

$$\mathcal{B}^{\leq} = \{(v^*, y^*, v) \in K^{+0} \times Y^* \times \mathbb{R} : \langle v^*, v \rangle \leq -(v^* f)(-A^* y^*) - (v^* g)(y^*)\},$$

and the objective function is

$$h^{\leq}(v^*, y^*, v) = v.$$

For this new optimization problem, we are interested in investigating the Pareto efficient solutions, defined below.

Definition 2. *An element $(\bar{v}^*, \bar{y}^*, \bar{v}) \in \mathcal{B}^{\leq}$ is said to be an efficient (Pareto efficient) solution to (D^{\leq}) if there exists no $(v^*, y^*, v) \in \mathcal{B}^{\leq}$ such that*

$$h^{\leq}(\bar{v}^*, \bar{y}^*, \bar{v}) \leq_K h^{\leq}(v^*, y^*, v).$$

As stated above, for a primal-dual pair of optimization problems, weak duality must always hold, under general assumptions. This is the case for our problems, as it is proved in the following theorem.

Theorem 1 (Weak Duality for $(P)-(D^{\leq})$). *There exist no $x \in X$ and no $(v^*, y^*, v) \in \mathcal{B}^{\leq}$ such that*

$$(f + g \circ A)(x) \leq_K h^{\leq}(v^*, y^*, v).$$

Proof. We proceed by contradiction, assuming that there exist $\bar{x} \in X$ and $(\bar{v}^*, \bar{y}^*, \bar{v}) \in \mathcal{B}^{\leq}$ such that $(f + g \circ A)(\bar{x}) \leq_K h^{\leq}(\bar{v}^*, \bar{y}^*, \bar{v})$. This implies obviously that $\bar{x} \in (\text{dom } f) \cap A^{-1}(\text{dom } g)$. Due to the fact that $\bar{v}^* \in K^{+0}$ it follows that

$$\langle \bar{v}^*, \bar{v} \rangle > \langle \bar{v}^*, (f + g \circ A)(\bar{x}) \rangle \geq \inf_{x \in X} \{ \langle \bar{v}^*, f(x) \rangle + \langle \bar{v}^*, (g \circ A)(x) \rangle \}.$$

Moreover, from the weak duality theorem for the scalarized optimization problem on the right hand side of the inequality above and its Fenchel dual, we have

$$\inf_{x \in X} \{ \langle \bar{v}^*, f(x) \rangle + \langle \bar{v}^*, (g \circ A)(x) \rangle \} \geq \sup_{y^* \in Y^*} \{ -(\bar{v}^* f)^*(-A^* y^*) - (\bar{v}^* g)^*(y^*) \}.$$

Combining the relations above, we obtain

$$\langle \bar{v}^*, \bar{v} \rangle > -(\bar{v}^* f)^*(-A^* \bar{y}^*) - (\bar{v}^* g)^*(\bar{y}^*),$$

which contradicts the fact that $(\bar{v}^*, \bar{y}^*, \bar{v}) \in \mathcal{B}^{\leq}$. Hence the conclusion of the theorem holds. \square

In order to ensure strong duality between the previously mentioned problems, a regularity condition has to be fulfilled. It actually ensures the existence of strong duality for the scalar optimization problem

$$(P_{\bar{v}^*}) \inf_{x \in X} \{(\bar{v}^* f)(x) + (\bar{v}^* g)(Ax)\}$$

and its Fenchel dual problem

$$(D_{\bar{v}^*}) \sup_{y^* \in Y^*} \{-(\bar{v}^* f)^*(-A^* y^*) - (\bar{v}^* g)^*(y^*)\}$$

for all $\bar{v}^* \in K^{+0}$. So, we are looking for sufficient conditions that are independent from the choice of $\bar{v}^* \in K^{+0}$.

The first regularity condition, which we mention at this point, is derived from [14]. In the particular case of our problem it has the following formulation:

$$(RC_1) \parallel \exists x_0 \in \text{dom } f \cap A^{-1}(\text{dom } g) \text{ such that } g \text{ is continuous at } A(x_0).$$

When $M \subset Y$ is a given set, we use the following notation:

$$A^{-1}(M) := \{x \in X : Ax \in M\}.$$

In Fréchet spaces one can state the following regularity conditions for the primal-dual pair $(P_{\bar{v}^*}) - (D_{\bar{v}^*})$:

$$(RC_2) \parallel \begin{array}{l} X \text{ and } Y \text{ are Fréchet spaces,} \\ f \text{ and } g \text{ are star-}K \text{ lower-semicontinuous,} \\ \text{and } 0 \in \text{sqri}(\text{dom } g - A(\text{dom } f)) \end{array}$$

along with its stronger versions

$$(RC_{2'}) \parallel \begin{array}{l} X \text{ and } Y \text{ are Fréchet spaces,} \\ f \text{ and } g \text{ are star-}K \text{ lower-semicontinuous,} \\ \text{and } 0 \in \text{core}(\text{dom } g - A(\text{dom } f)) \end{array}$$

and

$$(RC_{2''}) \left\| \begin{array}{l} X \text{ and } Y \text{ are Fréchet spaces,} \\ f \text{ and } g \text{ are star-}K \text{ lower-semicontinuous,} \\ \text{and } 0 \in \text{int}(\text{dom } g - A(\text{dom } f)). \end{array} \right.$$

For more details with respect to these regularity conditions we refer the reader to [2]. In the finite dimensional setting one can use the following regularity condition:

$$(RC_3) \left\| \begin{array}{l} \dim(\text{lin}(\text{dom } g - A(\text{dom } f))) < +\infty \text{ and} \\ \text{ri}(\text{dom } g) \cap \text{ri}(A(\text{dom } f)) \neq \emptyset \end{array} \right.$$

which becomes in case $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$

$$(RC_4) \parallel \exists x' \in \text{ri}(\text{dom } f) \text{ s.t. } Ax' \in \text{ri}(\text{dom } g) .$$

The condition (RC_4) is the classical regularity condition for the scalar Fenchel duality in finite dimensional spaces and has been stated by Rockafellar in [19].

A newly studied approach in giving sufficient conditions for strong duality is the one employing closed cone constraint qualifications which turn out to be weaker than the interior-type ones. For such conditions and their comparison to the interior-type ones specified above, and others, we refer the reader to the paper by Boç and Wanka [5].

Theorem 2 (Strong Duality Theorem for $(P) - (D^{\leq})$). *Assume that one of the regularity conditions $(RC_1) - (RC_3)$ is satisfied. If $\bar{x} \in X$ is a properly efficient solution to (P) , then there exists an efficient solution $(\bar{v}^*, \bar{y}^*, \bar{v}) \in \mathcal{B}^{\leq}$ to (D^{\leq}) such that $(f + g \circ A)(\bar{x}) = h^{\leq}(\bar{v}^*, \bar{y}^*, \bar{v}) = \bar{v}$.*

Proof. Due to the fact that \bar{x} is a properly efficient solution to (D^{\leq}) we obtain that $\bar{x} \in \text{dom}(f) \cap A^{-1}(\text{dom } g)$ and that there exists a $\bar{v}^* \in K^{+0}$ such that

$$\langle \bar{v}^*, (f + g \circ A)(\bar{x}) \rangle = \inf_{x \in X} \{(\bar{v}^* f)(x) + (\bar{v}^* g)(Ax)\}.$$

The functions $(\bar{v}^* f)$ and $(\bar{v}^* g)$ are proper and convex. The regularity assumption guarantees the existence of strong duality for the scalarized optimization problem

$\inf_{x \in X} \{(\bar{v}^* f)(x) + (\bar{v}^* g)(Ax)\}$ and its Fenchel dual. Thus there exists $\bar{y}^* \in Y^*$ such that

$$\begin{aligned} \inf_{x \in X} \{(\bar{v}^* f)(x) + (\bar{v}^* g)(Ax)\} &= \sup_{y^* \in Y^*} \{-(\bar{v}^* f)^*(-A^* y^*) - (\bar{v}^* g)^*(y^*)\} = \\ &= -(\bar{v}^* f)^*(-A^* \bar{y}^*) - (\bar{v}^* g)^*(\bar{y}^*). \end{aligned}$$

By defining $\bar{v} := (f + g \circ A)(\bar{x}) \in V$ we obtain that $(\bar{v}^*, \bar{y}^*, \bar{v}) \in \mathcal{B}^\leq$. Now we prove that it is an efficient solution to (D^\leq) .

Let us assume by contradiction that this is not the case. This implies the existence of $(v^*, y^*, v) \in \mathcal{B}^\leq$ such that $\bar{v} = (f + g \circ A)(\bar{x}) \leq_K v = h^\leq(v^*, y^*, v)$ which is a contradiction to the weak duality theorem, Theorem 1. \square

The forthcoming result plays a crucial role in proving the converse duality theorem.

Theorem 3. *Assume that one of the regularity conditions $(RC_1) - (RC_3)$ is satisfied and that $B^\leq \neq \emptyset$. Then*

$$V \setminus \text{cl} \{(f + g \circ A)(\text{dom } f \cap A^{-1}(\text{dom } g)) + K\} \subseteq \text{core } h^\leq(B^\leq).$$

Proof. Let $\eta \in V \setminus \text{cl} \{(f + g \circ A)((\text{dom } f \cap A^{-1}(\text{dom } g)) + K)\}$ be arbitrarily chosen. Due to the fact that f and g are K -convex functions, A is a linear continuous operator and K is a convex cone, we see that the set

$$(f + g \circ A)(\text{dom } f \cap A^{-1}(\text{dom } g)) + K$$

is convex, thus $\text{cl} \{(f + g \circ A)(\text{dom } f \cap A^{-1}(\text{dom } g)) + K\}$ is a closed and convex set. According to a separation theorem (see [25]), we obtain the existence of $\eta^* \in V^* \setminus \{0\}$ and $\alpha \in \mathbb{R}$ such that

$$\langle \eta^*, \eta \rangle < \alpha < \langle \eta^*, b \rangle, \forall b \in \text{cl} \{(f + g \circ A)(\text{dom } f \cap A^{-1}(\text{dom } g)) + K\}. \quad (1)$$

We prove that $\eta^* \in K^+ \setminus \{0\}$. Let us suppose by contradiction that there exists a $k \in K$ such that $\langle \eta^*, k \rangle < 0$. This means that for a fixed $x_0 \in \text{dom}(f) \cap A^{-1}(\text{dom } g)$ the inequality

$$\alpha < \langle \eta^*, (f + g \circ A)(x_0) \rangle + \langle \eta^*, tk \rangle$$

holds for all $t \geq 0$. Allowing now $t \rightarrow +\infty$, we obtain that $\alpha < -\infty$, which is obviously a contradiction. Therefore, $\eta^* \in K^+ \setminus \{0\}$.

Due to the fact that $\mathcal{B}^{\leq} \neq \emptyset$, there exists $(v^*, y^*, v) \in \mathcal{B}^{\leq}$, hence

$$\langle v^*, v \rangle \leq -(v^* f)^*(-A^* y^*) - (v^* g)^*(y^*).$$

Applying the weak duality theorem for the scalarized problem

$$(P_{v^*}) \inf_{x \in X} \langle v^*, (f + g \circ A)(x) \rangle$$

and its Fenchel dual, we have that

$$-(v^* f)^*(-A^* y^*) - (v^* g)^*(y^*) \leq \inf_{x \in X} \langle v^*, (f + g \circ A)(x) \rangle,$$

and hence

$$\langle v^*, v \rangle \leq \inf_{x \in X} \langle v^*, (f + g \circ A)(x) \rangle. \quad (2)$$

For each $s \in (0, 1)$ we have that

$$\langle sv^* + (1-s)\eta^*, \eta \rangle = \langle \eta^*, \eta \rangle + s(\langle v^*, \eta \rangle - \langle \eta^*, \eta \rangle) = \alpha - \gamma + s(\langle v^*, v \rangle - \alpha + \gamma) \quad (3)$$

with $\gamma := \alpha - \langle \eta^*, \eta \rangle > 0$. Furthermore, from (1) and (2) we obtain

$$\langle sv^* + (1-s)\eta^*, b \rangle \geq s\langle v^*, v \rangle + (1-s)\alpha = \alpha + s(\langle v^*, v \rangle - \alpha) \quad (4)$$

for all $b \in (f + g \circ A)(\text{dom } f \cap A^{-1}(\text{dom } g))$. Thus there exists $\bar{s} \in (0, 1)$, close enough to 0, such that $\bar{s}(\langle v^*, v \rangle - \alpha + \gamma) < \frac{1}{2}\gamma$ and $\bar{s}(\langle v^*, v \rangle - \alpha) > -\frac{1}{2}\gamma$. For the convex combination obtained with the help of \bar{s} it holds

$$v_{\bar{s}}^* := sv^* + (1-s)\eta^* \in sK^{+0} + (1-s)(K^+ \setminus \{0\}) \subseteq K^{+0} + K^+ \subseteq K^{+0}.$$

Thus, using (3) and (4), we obtain

$$\langle v_{\bar{s}}^*, \eta \rangle < \alpha - \frac{1}{2}\gamma < \langle v_{\bar{s}}^*, b \rangle, \forall b \in (f + g \circ A)(\text{dom } f \cap A^{-1}(\text{dom } g)).$$

From the hypothesis we know that one of the regularity conditions holds. Thus, from the strong duality for the scalar optimization problems $(P_{v^*}) - (D_{v^*})$ there exists an

optimal solution y^* of the dual (D_{v^*}) , therefore

$$\begin{aligned} \langle v_{\bar{s}}^*, \eta \rangle &< \inf_{x \in X} \langle v_{\bar{s}}^*, (f + g \circ A)(x) \rangle = \sup_{z^* \in Y^*} \{-(v_{\bar{s}}^* f)^*(-A^* z^*) - (v_{\bar{s}}^* g)^*(z^*)\} \\ &= -(v_{\bar{s}}^* f)^*(-A^* y^*) - (v_{\bar{s}}^* g)^*(y^*). \end{aligned}$$

This means that there exists $\varepsilon > 0$ such that

$$\langle v_{\bar{s}}^*, \eta \rangle + \varepsilon < -(v_{\bar{s}}^* f)^*(-A^* y^*) - (v_{\bar{s}}^* g)^*(y^*).$$

For all $p \in V$ there exists $\delta_p > 0$ such that $\langle v_{\bar{s}}^*, \delta_p p \rangle < \varepsilon$, and thus

$$\langle v_{\bar{s}}^*, \eta + \lambda p \rangle \leq \langle v_{\bar{s}}^*, \eta \rangle + \varepsilon < \{-(v_{\bar{s}}^* f)^*(-A^* y^*) - (v_{\bar{s}}^* g)^*(y^*)\}, \forall \lambda \in [0, \delta_p].$$

Hence $(v^*, y^*, \eta + \lambda p) \in \mathcal{B}^{\leq}$ for all $\lambda \in [0, \delta_p]$, and further $\eta + \lambda p \in h^{\leq}(\mathcal{B}^{\leq})$, guaranteeing that $\eta \in \text{core}(\mathcal{B}^{\leq})$. \square

Theorem 4 (Converse Duality Theorem for $(P) - (D^{\leq})$). *Assume that one of the regularity conditions $(RC_1) - (RC_3)$ is satisfied and the set*

$$(f + g \circ A)(\text{dom } f \cap A^{-1}(\text{dom } g)) + K$$

is closed. Then for each efficient solution $(\bar{v}^, \bar{y}^*, \bar{v}) \in \mathcal{B}^{\leq}$ to (D^{\leq}) there exists a properly efficient solution $\bar{x} \in X$ to (P) , such that*

$$(f + g \circ A)(\bar{x}) = h^{\leq}(\bar{v}^*, \bar{y}^*, \bar{v}) = \bar{v}.$$

Proof. First we show that $\bar{v} \in (f + g \circ A)(\text{dom } f \cap A^{-1}(\text{dom } g)) + K$. Let us proceed by contradiction. This would mean, by using Theorem 3, that $\bar{v} \in \text{core } h^{\leq}(\mathcal{B}^{\leq})$. Thus for a $k \in K \setminus \{0\}$ there exists $\lambda > 0$ such that $\bar{v}_\lambda := \bar{v} + \lambda k \in h^{\leq}(\mathcal{B}^{\leq})$. Furthermore, $\bar{v}_\lambda - \bar{v} = \lambda k \in K \setminus \{0\}$ and hence $\bar{v}_\lambda \geq_K \bar{v}$, a contradiction to the efficiency of $\bar{v} \in h^{\leq}(\mathcal{B}^{\leq})$.

Thus $\bar{v} \in (f + g \circ A)(\text{dom } f \cap A^{-1}(\text{dom } g)) + K$. But this means that there exist $\bar{x} \in (\text{dom } f \cap A^{-1}(\text{dom } g))$ and $\bar{k} \in K$ such that

$$\bar{v} = (f + g \circ A)(\bar{x}) + \bar{k}.$$

By assuming that $\bar{k} \neq 0$ we would obtain that $h^{\leq}(\bar{v}^*, \bar{y}^*, \bar{v}) = \bar{v} \geq_K (f + g \circ A)(\bar{x})$, a contradiction to the weak duality statement of Theorem 1. Hence $\bar{k} = 0$ and thus $\bar{v} = (f + g \circ A)(\bar{x})$. Employing now the definition of \mathcal{B}^{\leq} and the weak duality theorem which holds for the scalarized optimization problem $(P_{\bar{v}^*})$, we obtain

$$\begin{aligned} \langle \bar{v}^*, (f + g \circ A)(\bar{x}) \rangle &= \langle \bar{v}^*, \bar{v} \rangle \leq -(v^* f)^*(-A^* y^*) - (v^* g)^*(y^*) \\ &\leq \inf_{x \in X} \langle \bar{v}^*, (f + g \circ A)(x) \rangle. \end{aligned}$$

Therefore, \bar{x} is a properly efficient solution to (P) . \square

The scalar Fenchel duality was involved for the first time in the definition of a vector dual problem by Breckner and Kolumban, in [10], in a very general framework. Inspired by the approach introduced in this work, one gets the following dual vector optimization problem associated with (P)

$$(D^{BK}) \quad \text{v-max}_{(v^*, y^*, v) \in \mathcal{B}^{BK}} h^{BK}(v^*, y^*, v),$$

where

$$\mathcal{B}^{BK} := \{(v^*, y^*, v) \in K^{+0} \times Y^* \times V : \langle v^*, v \rangle = -(v^* f)^*(-A^* y^*) - (v^* g)^*(y^*)\}$$

and

$$h^{BK}(v^*, y^*, v) = v.$$

Remark 1. *As it can be easily observed from the definition, without any other additional assumptions, the following inclusion holds:*

$$h^{BK}(\mathcal{B}^{BK}) \subseteq h^{\leq}(\mathcal{B}^{\leq}).$$

Theorem 5. *The following equality holds:*

$$\text{v-max } h^{BK}(\mathcal{B}^{BK}) = \text{v-max } h^{\leq}(\mathcal{B}^{\leq}).$$

Proof. " \subseteq " Let $(v^*, y^*, v) \in \mathcal{B}^{BK}$ be such that $v \in \text{v-max}(h^{BK}(\mathcal{B}^{BK}))$. Then $v \in h^{\leq}(\mathcal{B}^{\leq})$. We suppose that $v \notin \text{v-max}(h^{\leq}(\mathcal{B}^{\leq}))$. This means that there exists

$(v_0^*, y_0^*, v_0) \in \mathcal{B}^{\leq}$ such that $v_0 \geq_K v$. Due to the maximality of v in $h^{BK}(\mathcal{B}^{BK})$ we have that $(v_0^*, y_0^*, v_0) \notin \mathcal{B}^{BK}$, therefore

$$\langle v_0^*, v_0 \rangle < -(v_0^* f)^*(-A^* y_0^*) - (v_0^* g)^*(y_0^*).$$

Consequently there exists a $k \in K \setminus \{0\}$ and $v_k := v_0 + k$ such that

$$\langle v_0^*, v_k \rangle = -(v_0^* f)^*(-A^* y_0^*) - (v_0^* g)^*(y_0^*),$$

which means that $(v_0^*, y_0^*, v_k) \in \mathcal{B}^{BK}$ and $v_k \geq_K v_0$. Since this is a contradiction to the maximality of v_0 , $v\text{-max } h^{BK}(\mathcal{B}^{BK}) \subseteq v\text{-max } h^{\leq}(\mathcal{B}^{\leq})$.

" \supseteq " By taking $(v^*, y^*, v) \in \mathcal{B}^{\leq}$ such that $v \in v\text{-max } h^{\leq}(\mathcal{B}^{\leq})$ we prove that it belongs to $v\text{-max } h^{BK}(\mathcal{B}^{BK})$. The first step is to prove that $(v^*, y^*, v) \in \mathcal{B}^{BK}$. Assuming the contrary, one has

$$\langle v^*, v \rangle < -(v^* f)^*(-A^* y^*) - (v^* g)^*(y^*)$$

and there exists $k \in K \setminus \{0\}$ such that $v_k := v + k$ satisfies

$$\langle v^*, v \rangle < \langle v^*, v_k \rangle = -(v^* f)^*(-A^* y^*) - (v^* g)^*(y^*).$$

Since $(v^*, y^*, v) \in \mathcal{B}^{\leq}$ and $v_k \geq_K v$, we have a contradiction to the maximality of v . Hence $(v^*, y^*, v) \in \mathcal{B}^{BK}$.

We further suppose that $v \notin v\text{-max } h^{BK}(\mathcal{B}^{BK})$. This means actually that there exists $(\bar{v}^*, \bar{y}^*, \bar{v}) \in \mathcal{B}^{BK} \subseteq \mathcal{B}^{\leq}$ such that $\bar{v} \geq_K v$, which is actually a contradiction to the maximality of $v \in h^{\leq}(\mathcal{B}^{\leq})$. Therefore, $v \in v\text{-max } h^{BK}(\mathcal{B}^{BK})$. \square

Remark 2. *In the proof of the previous theorem, no assumptions regarding the nature of the functions and sets involved in the formulation of (P) were made. This means that the sets of efficient elements of $h^{\leq}(\mathcal{B}^{\leq})$ and $h^{BK}(\mathcal{B}^{BK})$ are always identical.*

Using the weak, strong and converse duality theorems between the dual pair of vector optimization problems (P) and (D^{\leq}) , similar results can be proved for the primal-dual pair (P) – (D^{BK}) .

Theorem 6. *The following statements are true:*

- a) (*Weak duality*) There exist no $x \in X$ and no $(v^*, y^*, v) \in \mathcal{B}^{BK}$ such that $(f + g \circ A)(x) \leq_K h^{BK}(v^*, y^*, v)$.
- b) (*Strong duality*) Let one of the regularity conditions $(RC_1) - (RC_3)$ be satisfied. If $\bar{x} \in X$ is a properly efficient solution to (P) , then there exists an efficient solution $(\bar{v}^*, \bar{y}^*, \bar{v}) \in \mathcal{B}^{BK}$ to (D^{BK}) such that

$$(f + g \circ A)(\bar{x}) = h^{BK}(\bar{v}^*, \bar{y}^*, \bar{v}) = \bar{v}.$$

- c) (*Converse duality*) Let one of the regularity conditions $(RC_1) - (RC_3)$ be satisfied and let $(f + g \circ A)(\text{dom } f \cap A^{-1}(\text{dom } g)) + K$ be a closed set. Then for each efficient solution $(\bar{v}^*, \bar{y}^*, \bar{v}) \in \mathcal{B}^{BK}$ to (D^{BK}) , there exists a properly efficient solution $\bar{x} \in X$ to (P) , such that

$$(f + g \circ A)(\bar{x}) = h^{BK}(\bar{v}^*, \bar{y}^*, \bar{v}) = \bar{v}.$$

Proof. a) It follows from Remark 1 and Theorem 1.

b) It follows from Theorem 5 and Theorem 2.

c) It follows from Theorem 5 and Theorem 4. \square

When $V := \mathbb{R}$ and $K := \mathbb{R}_+$, one can identify V^\bullet with $\mathbb{R} \cup \{+\infty\}$. Assuming that $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ and $g : Y \rightarrow \mathbb{R} \cup \{+\infty\}$ are proper and convex functions, the primal problem becomes

$$(P) \inf_{x \in X} (f + g \circ A)(x).$$

An element $(v^*, y^*, v) \in \mathcal{B}^{\leq}$ if and only if $v^* > 0$, $y^* \in Y^*$ and $v \in \mathbb{R}$ fulfill

$$v^* v \leq -(v^* f)^*(-A^* y^*) - (v^* g)(y^*).$$

Using the characterization of the conjugate functions, we get that

$$(v^* f)^*(-A^* y^*) = v^* f^*\left(-\frac{1}{v^*} A^* y^*\right) \text{ and } (v^* g)(y^*) = v^* g^*\left(\frac{1}{v^*} y^*\right).$$

Thus

$$v^* v \leq -v^* f^*\left(-\frac{1}{v^*} A^* y^*\right) - v^* g^*\left(\frac{1}{v^*} y^*\right) \iff v \leq f^*\left(-\frac{1}{v^*} A^* y^*\right) - g^*\left(\frac{1}{v^*} y^*\right).$$

The dual problem becomes

$$(D_{\leq}) \quad \sup_{v^* > 0, y^* \in Y^*} \left\{ -f^*\left(-\frac{1}{v^*}A^*y^*\right) - g^*\left(\frac{1}{v^*}y^*\right) \right\} = \sup_{y^* \in Y^*} \{f^*(-A^*y^*) - g^*(y^*)\}$$

which is exactly the classical scalar Fenchel dual problem to (PV) . The same conclusion applies when particularizing in an analogous manner the vector dual problem (DV_{BK}) .

4. The case when $V := \mathbb{R}^m$

In this section we focus our attention on the special case when $V := \mathbb{R}^m$ and $K := \mathbb{R}_+^m$. In addition to the two dual problems studied before, we introduce a new one, whose formulation was inspired from [9]. Nevertheless, a more particular case was treated there, namely the one when $X := \mathbb{R}^n$ and $Y := \mathbb{R}^k$.

The primal problem turns into

$$(P) \quad \text{v-min}_{x \in X} (f(x) + (g \circ A)(x)),$$

where f and g are two vector functions such that

$$f = (f_1, f_2, \dots, f_m)^T \quad \text{and} \quad g = (g_1, g_2, \dots, g_m)^T$$

with $f_i : X \rightarrow \overline{\mathbb{R}}$, $g_i : Y \rightarrow \overline{\mathbb{R}}$ proper and convex functions for each $i \in \{1, \dots, m\}$, and $A : X \rightarrow Y$ is a linear continuous operator.

Furthermore, we assume that the following regularity condition is satisfied

$$(RC_m) \quad \left\| \begin{array}{l} \exists x' \in \bigcap_{i=1}^m \text{dom } f_i \cap A^{-1} \left(\bigcap_{i=1}^m \text{dom } g_i \right) \text{ such that} \\ f_i \text{ and } g_i \text{ are continuous at } x' \text{ for all } i \in \{1, \dots, m\}. \end{array} \right.$$

We consider the following dual optimization problem associated with (P) :

$$(D^{BGW}) \quad \text{v-max}_{(p,q,\lambda,t) \in B^{BGW}} h^{BGW}(p, q, \lambda, t),$$

where

$$\mathcal{B}^{\mathcal{B}GW} = \left\{ \begin{array}{l} (p, q, \lambda, t) : p = (p_1, \dots, p_m) \in (X^*)^m, \\ q = (q_1, \dots, q_m) \in (Y^*)^m \\ \lambda = (\lambda_1, \dots, \lambda_m) \in \text{int } \mathbb{R}_+^m, \\ t = (t_1, \dots, t_m) \in \mathbb{R}^m, \\ \sum_{i=1}^m \lambda_i (p_i + A^* q_i) = 0, \sum_{i=1}^m \lambda_i t_i = 0 \end{array} \right\},$$

and h is defined by

$$h(p, q, \lambda, t) = (h_1(p, q, \lambda, t), \dots, h_m(p, q, \lambda, t)),$$

with

$$h_i(p, q, \lambda, t) = -f_i^*(p_i) - g_i^*(q_i) + t_i \text{ for all } i \in \{1, \dots, m\}.$$

Proposition 7. *The following relations referring to the image sets of the three dual problems hold:*

- a) $h^{BK}(\mathcal{B}^{BK}) \subseteq h^{BGW}(\mathcal{B}^{BGW}) \cap \mathbb{R}^m;$
- b) $h^{BGW}(\mathcal{B}^{BGW}) \cap \mathbb{R}^m \subseteq h^{\leq}(\mathcal{B}^{\leq}).$

Proof. a) Let $v \in h^{BK}(\mathcal{B}^{BK})$. Then there exist $v^* \in \text{int } \mathbb{R}_+^m$ and $y^* \in Y^*$ such that $(v^*, y^*, v) \in \mathcal{B}^{BK}$. Furthermore,

$$\sum_{i=1}^m v_i^* v_i = - \left(\sum_{i=1}^m v_i^* f_i \right)^* (-A^* y^*) - \left(\sum_{i=1}^m v_i^* g_i \right)^* (y^*).$$

Since (RC_m) is fulfilled, we can apply the infimal convolution formula and obtain the existence of $p_i \in X^*, q_i \in Y^*, i \in \{1, \dots, m\}$, such that

$$\sum_{i=1}^m v_i^* p_i = -A^* y^*, \sum_{i=1}^m v_i^* q_i = y^*,$$

$$\left(\sum_{i=1}^m v_i^* f_i \right)^* (-A^* y^*) = \sum_{i=1}^m v_i^* f_i^*(p_i) \text{ and } \left(\sum_{i=1}^m v_i^* g_i \right)^* (y^*) = \sum_{i=1}^m v_i^* g_i^*(q_i).$$

Moreover, $\sum_{i=1}^m v_i^* (p_i + A^* q_i) = 0$. For more details on the the infimal convolution formula and on the regularity conditions that ensure the equalities above we refer the reader to [19] and [4]. Returning to our problem, we have that

$$\sum_{i=1}^m v_i^* v_i = - \sum_{i=1}^m v_i^* f_i^*(p_i) - \sum_{i=1}^m v_i^* g_i^*(q_i).$$

For

$$t_i := v_i + f_i^*(p_i) + g_i^*(q_i) \quad \forall i \in \{1, \dots, m\},$$

it holds

$$\sum_{i=1}^m v_i^* t_i = \sum_{i=1}^m v_i^* v_i + \sum_{i=1}^m v_i^* f_i^*(p_i) + \sum_{i=1}^m v_i^* g_i^*(q_i) = 0.$$

Then $(p, q, v^*, t) \in \mathcal{B}^{BGW}$ and for all $i \in \{1, \dots, m\}$, $h_i(p, q, v^*, t) = v_i$, thus $v = h(p, q, v^*, t) \in h(\mathcal{B}^{BGW}) \cap \mathbb{R}^m$. Hence

$$h^{BK}(\mathcal{B}^{BK}) \subseteq h^{BGW}(\mathcal{B}^{BGW}) \cap \mathbb{R}^m.$$

b) Let $(p, q, \lambda, t) \in \mathcal{B}^{BGW}$ be such that $h(p, q, \lambda, t) \in h(\mathcal{B}^{BGW}) \cap \mathbb{R}^m$. For $y^* := \sum_{i=1}^m \lambda_i q_i$ and $v := h^{BGW}(p, q, \lambda, t)$ we have

$$\begin{aligned} \sum_{i=1}^m \lambda_i v_i &= \sum_{i=1}^m \lambda_i h_i(p, q, \lambda, t) = \sum_{i=1}^m \lambda_i \left(-f_i^*(p_i) - g_i^*(q_i) + t_i \right) \\ &= \sum_{i=1}^m \lambda_i \left(-f_i^*(p_i) - g_i^*(q_i) \right) \\ &\leq \sup \left\{ - \sum_{i=1}^m \lambda_i f_i^*(p_i) : \sum_{i=1}^m \lambda_i p_i = -A^* y^* \right\} \\ &\quad + \sup \left\{ - \sum_{i=1}^m \lambda_i g_i^*(q_i) : \sum_{i=1}^m \lambda_i q_i = y^* \right\} \\ &\leq - \left(\sum_{i=1}^m \lambda_i f_i \right)^* (-A^* y^*) - \left(\sum_{i=1}^m \lambda_i g_i \right)^* (y^*). \end{aligned}$$

Hence $(\lambda, y^*, v) \in \mathcal{B}^{\leq}$ and $h^{BGW}(p, q, \lambda, t) = v \in h^{\leq}(\mathcal{B}^{\leq})$. Thus

$$h^{BGW}(\mathcal{B}^{BGW}) \cap \mathbb{R}^m \subseteq h^{\leq}(\mathcal{B}^{\leq}).$$

□

In the following we give some examples which prove that the inclusions in Proposition 7 are in general strict, i.e.

$$h^{BK}(\mathcal{B}^{BK}) \subsetneq h^{BGW}(\mathcal{B}^{BGW}) \cap \mathbb{R}^m \subsetneq h^{\leq}(\mathcal{B}^{\leq}).$$

Example 8. Consider $X = Y = \mathbb{R}$, $A(x) = x$ for all $x \in \mathbb{R}$, and the functions $f, g : \mathbb{R} \rightarrow \mathbb{R}^2$ given by

$$f(x) = (x-1, -x-1)^T \quad \text{and} \quad g(x) = (x, -x)^T \quad \text{for all } x \in \mathbb{R}.$$

We prove that $h^{BGW}(\mathcal{B}^{BGW}) \cap \mathbb{R}^m \subsetneq h^{\leq}(\mathcal{B}^{\leq})$.

Since

$$f_1^*(p) = \begin{cases} 1, & \text{if } p = 1, \\ +\infty, & \text{otherwise,} \end{cases}, \quad f_2^*(p) = \begin{cases} 1, & \text{if } p = -1, \\ +\infty, & \text{otherwise,} \end{cases}$$

$$g_1^*(p) = \begin{cases} 0, & \text{if } p = 1, \\ +\infty, & \text{otherwise,} \end{cases}, \quad g_2^*(p) = \begin{cases} 0, & \text{if } p = -1, \\ +\infty, & \text{otherwise,} \end{cases}$$

one has

$$(f_1 + f_2)^*(p) = \inf \{f_1^*(p_1) + f_2^*(p_2) : p_1 + p_2 = p\} = \begin{cases} 2, & \text{if } p = 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$(g_1 + g_2)^*(p) = \inf \{g_1^*(p_1) + g_2^*(p_2) : p_1 + p_2 = p\} = \begin{cases} 0, & \text{if } p = 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

For $\lambda = (1, 1)^T$, $p = 0$ and $d = (-2, -2)^T$ we have

$$(\lambda, p, d) \in \mathcal{B}^{\leq} \quad \text{and} \quad d \in h^{\leq}(\mathcal{B}^{\leq})$$

due to the fact that

$$\lambda^T d = -2 - 2 = -4 < -2 = -(f_1 + f_2)^*(p) - (g_1 + g_2)^*(p).$$

Next we show that $d \notin h^{BGW}(\mathcal{B}^{BGW})$. Let us suppose by contradiction that there exists $(p', q', \lambda', t') \in \mathcal{B}^{BGW}$ such that $h^{BGW}(p', q', \lambda', t') = d$. This means

$$-f_i^*(p'_i) - g_i^*(q'_i) + t'_i = 0 \quad \text{for } i \in \{1, 2\}.$$

Taking into account the values we got for the conjugate of the functions involved, the equalities above hold only if

$$p'_1 = 1, p'_2 = -1, q'_1 = 1 \text{ and } q'_2 = -1.$$

In this case, $\sum_{i=1}^2 \lambda'_i (p'_i + q'_i) = 0$, which means that with this choice, we are still within the set \mathcal{B}^{BGW} . We obtain thus

$$-1 + t'_i = 0 \text{ for } i \in \{1, 2\}, \text{ meaning that } t'_1 = t'_2 = 1.$$

Since we have supposed that $(p', q', \lambda', t') \in \mathcal{B}^{BGW}$, $\lambda'_1 + \lambda'_2 = 0$ must hold. This is a contradiction due to the fact that $\lambda' \in \text{int } \mathbb{R}_+^2$.

Thus, for $d = (-2, -2)^T \in h^\leq(\mathcal{B}^\leq)$, there exists no $(p', q', \lambda', t') \in \mathcal{B}^{BGW}$ such that $h^{BGW}(p', q', \lambda', t') = d$, which shows that $h(\mathcal{B}^{BGW}) \cap \mathbb{R}^m \subsetneq h^\leq(\mathcal{B}^\leq)$.

Example 9. Consider $X = Y = \mathbb{R}$, $A(x) = x$ for all $x \in X$, and the functions $f, g : \mathbb{R} \rightarrow \mathbb{R}^2$ given by

$$f(x) = (2x^2 - 1, x^2)^T \text{ and } g(x) = (-2x, -x + 1)^T \text{ for all } x \in \mathbb{R}.$$

We prove that $h^{BK}(\mathcal{B}^{BK}) \subsetneq h^{BGW}(\mathcal{B}^{BGW}) \cap \mathbb{R}^m$.

For $p = (3, 0), q = (-2, -1)$, we have $\lambda = (1, 1)^T, t = (\frac{3}{8}, -\frac{3}{8})^T$, $\sum_{i=1}^2 \lambda_i (p_i + q_i) = 0$, and $\sum_{i=1}^2 \lambda_i t_i = 0$. Thus $(p, q, \lambda, t) \in \mathcal{B}^{BGW}$. Applying the definition of the conjugate function, we calculate the following values:

$$f_1^*(3) = \sup_{x \in \mathbb{R}} \{3x - 2x^2 + 1\} = \frac{17}{8}, \quad f_2^*(0) = \sup_{x \in \mathbb{R}} \{-x^2\} = 0,$$

$$g_1^*(-2) = \sup_{x \in \mathbb{R}} \{-2x + 2x\} = 0, \quad g_2^*(-1) = \sup_{x \in \mathbb{R}} \{-x + x - 1\} = -1.$$

Hence

$$h_1^{BGW}(p, q, \lambda, t) = -\frac{17}{8} - 0 + \frac{3}{8} = -\frac{14}{8}, \quad h_2^{BGW}(p, q, \lambda, t) = 0 + 1 - \frac{3}{8} = \frac{5}{8}.$$

Now suppose that there exists $(\lambda', p', d') \in \mathcal{B}^{BK}$ such that

$$d' = h^{BGW}(p, q, \lambda, t) = \left(-\frac{14}{8}, \frac{5}{8}\right)^T.$$

Then

$$\lambda'^T d' = - \left(\sum_{i=1}^2 \lambda'_i f_i \right)^* (p') - \left(\sum_{i=1}^2 \lambda'_i g_i \right)^* (-p'). \quad (5)$$

But calculating the values of the conjugate functions we reach the conclusion that

$$\begin{aligned} & - \left(\sum_{i=1}^2 \lambda'_i f_i \right)^* (p') - \left(\sum_{i=1}^2 \lambda'_i g_i \right)^* (-p') = \\ & = \inf_{x \in \mathbb{R}} \{ -p'x + x^2 (2\lambda'_1 + \lambda'_2) - \lambda_1 \} + \inf_{x \in \mathbb{R}} \{ x(p' - 2\lambda'_1 - \lambda'_2) + \lambda'_2 \} \\ & = \inf_{x \in \mathbb{R}} \{ -(2\lambda'_1 + \lambda'_2)x + x^2 (2\lambda'_1 + \lambda'_2) \} - \lambda_1 + \lambda'_2 \\ & = -\frac{2\lambda'_1 + \lambda'_2}{4} - \lambda_1 + \lambda'_2. \end{aligned}$$

By (3) we obtain that

$$-\frac{14}{8}\lambda'_1 + \frac{5}{8}\lambda'_2 = -\frac{2\lambda'_1 + \lambda'_2}{4} - \lambda_1 + \lambda'_2$$

which is equivalent to

$$-\frac{3(2\lambda'_1 + \lambda'_2)}{8} = -\frac{2\lambda'_1 + \lambda'_2}{4}, \text{ i.e. } 2\lambda'_1 + \lambda'_2 = 0,$$

obviously a contradiction to $\lambda' \in \text{int } \mathbb{R}_+^2$. Therefore, for (p, q, λ, t) chosen as in the beginning of the example, there exists no $(\lambda', p', d') \in \mathcal{B}^{BK}$ such that $d' = h^{BGW}(p, q, \lambda, t)$. Hence $h^{BK}(\mathcal{B}_{BK}) \cap \mathbb{R}^m \subsetneq h^{BGW}(\mathcal{B}^{BGW})$.

Below we prove that the sets of optimal solutions to (D^{BGW}) and (D^\leq) coincide.

Theorem 10. *The following equality holds:*

$$\text{v-max } h^{BGW}(\mathcal{B}^{BGW}) = \text{v-max } h^\leq(\mathcal{B}^\leq).$$

Proof. "v-max $h^{BGW}(\mathcal{B}^{BGW}) \subseteq \text{v-max } h^\leq(\mathcal{B}^\leq)$ " Let $\bar{v} \in \text{v-max } h^{BGW}(\mathcal{B}^{BGW})$. Since $h^{BGW}(\mathcal{B}^{BGW}) \cap \mathbb{R}^m \subseteq h^\leq(\mathcal{B}^\leq)$, one has $\bar{v} \in h^\leq(\mathcal{B}^\leq)$. Let us suppose by contradiction, that $\bar{v} \notin \text{v-max } h^\leq(\mathcal{B}^\leq)$. Then there exists $v \in h^\leq(\mathcal{B}^\leq)$, with $(v^*, y^*, v) \in \mathcal{B}^\leq$, such that $\bar{v} \leq_{\mathbb{R}_+^m} v$. Then we have

$$\langle v^*, \bar{v} \rangle < \langle v^*, v \rangle \leq - (v^* f)^* (-A^* y^*) - (v^* g)^* (y^*).$$

So, there exists \tilde{v} such that $v \leq_{\mathbb{R}_+^m} \tilde{v}$ (obviously, $\bar{v} \leq_{\mathbb{R}_+^m} \tilde{v}$) for which

$$\langle v^*, \tilde{v} \rangle = -(v^* f)^* (-A^* y^*) - (v^* g)^* (y^*).$$

Thus we have obtained an element $(v^*, y^*, \tilde{v}) \in \mathcal{B}^{BK}$. Since

$$h^{BK}(\mathcal{B}^{BK}) \subseteq h^{BGW}(\mathcal{B}^{BGW}) \cap \mathbb{R}^m,$$

it follows that $\tilde{v} \in h^{BGW}(\mathcal{B}^{BGW})$, which contradicts the maximality of \bar{v} in $h^{BGW}(\mathcal{B}^{BGW})$. Therefore,

$$\text{v-max } h^{BGW}(\mathcal{B}^{BGW}) \subseteq \text{v-max } h^{\leq}(\mathcal{B}^{\leq}).$$

" $\text{v-max } h^{\leq}(\mathcal{B}^{\leq}) \subseteq \text{v-max } h^{BGW}(\mathcal{B}^{BGW})$ " Let $\bar{v} \in \text{v-max } h^{\leq}(\mathcal{B}^{\leq})$. By Theorem 5 it follows that $\bar{v} \in \text{v-max } h^{BK}(\mathcal{B}^{BK})$. Since $h^{BK}(\mathcal{B}^{BK}) \subseteq h^{BGW}(\mathcal{B}^{BGW}) \cap \mathbb{R}^m$, we have further $\bar{v} \in h^{BGW}(\mathcal{B}^{BGW})$. Let us suppose by contradiction that there exists $(p, q, \lambda, t) \in \mathcal{B}^{BGW}$ such that $\bar{v} \leq_{\mathbb{R}_+^m} d := h^{BGW}(p, q, \lambda, t)$. Since $h^{BGW}(\mathcal{B}^{BGW}) \subseteq h^{\leq}(\mathcal{B}^{\leq})$, one has $d \in h^{\leq}(\mathcal{B}^{\leq})$, but $\bar{v} \leq_{\mathbb{R}_+^m} d$ which is a contradiction to the maximality of \bar{v} . Therefore

$$\text{v-max } h^{\leq}(\mathcal{B}^{\leq}) \subseteq \text{v-max } h^{BGW}(\mathcal{B}^{BGW}).$$

□

As one can easily notice from Theorems 5 and 10 along with examples 8 and 9, the following equalities hold:

$$\text{v-max } h^{BK}(\mathcal{B}^{BK}) = \text{v-max } h^{BGW}(\mathcal{B}^{BGW}) = \text{v-max } h^{\leq}(\mathcal{B}^{\leq}),$$

even though

$$h^{BK}(\mathcal{B}^{BK}) \underset{\neq}{\subset} h^{BGW}(\mathcal{B}^{BGW}) \cap \mathbb{R}^m \underset{\neq}{\subset} h^{\leq}(\mathcal{B}^{\leq}).$$

Using the weak, strong and converse duality theorems between the dual pair of the vector optimization problems (P) and (D^{\leq}) , similar results can be proved for the dual pair (P) and (D^{BGW}) . Thus

Theorem 11. *The following statements are true:*

- a) (Weak duality) There exist no $x \in X$ and no $(p, q, \lambda, t) \in \mathcal{B}^{BGW}$ such that $(f + g \circ A)(x) \leq_K h^{BGW}(p, q, \lambda, t)$.
- b) (Strong duality) If $\bar{x} \in X$ is a properly efficient solution to (P), then there exists an efficient solution $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t}) \in \mathcal{B}^{BGW}$ to (D^{BGW}) such that

$$(f + g \circ A)(\bar{x}) = h^{BGW}(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t}).$$

- c) (Converse duality) If the set

$$(f + g \circ A)(\text{dom } f \cap A^{-1}(\text{dom } g)) + K$$

is closed, then for each efficient solution $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t}) \in \mathcal{B}^{BGW}$ to (D^{BGW}) there exists a properly efficient solution $\bar{x} \in X$ to (P) such that

$$(f + g \circ A)(\bar{x}) = h^{BGW}(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t}).$$

Proof. a) It follows from Proposition 7 b) and Theorem 1.

b) It follows from Theorem 10 and Theorem 2.

c) It follows from Theorem 10 and Theorem 4. \square

As it will be seen in the following example, Theorem 4, which was important in the proof of the converse duality for dual (D^{\leq}) , does not hold for the more particular dual problems (D^{BK}) and (D^{BGW}) .

Example 12. Let $X = Y = \mathbb{R}$ and $V := \mathbb{R}^2$. Put $A(x) = x$ for all $x \in \mathbb{R}$ and define the functions $f, g : \mathbb{R} \rightarrow \mathbb{R}^2$ by

$$f(x) = (-3x + 7, 2x) \text{ and } g(x) = (3x - 7, -2x), \forall x \in \mathbb{R}.$$

We show that $\mathbb{R}^2 \setminus \text{cl}((f + g)(\mathbb{R}) + \mathbb{R}_+^2) \not\subseteq \text{core } h^{BGW}(\mathcal{B}^{BGW}) \cap \mathbb{R}^2$.

Under the above specified framework, $\text{dom } f = \text{dom } g = \mathbb{R}$ and the feasible solution set of (D^{BGW}) is

$$\mathcal{B}^{BGW} = \left\{ \begin{array}{l} (p, q, \lambda, d) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \text{int } \mathbb{R}_+^2 \times \mathbb{R}^2 : \\ \lambda_1(p_1 + q_1) + \lambda_2(p_2 + q_2) = 0, \lambda_1 t_1 + \lambda_2 t_2 = 0 \end{array} \right\}$$

and

$$h^{BGW}(\mathcal{B}^{BGW}) = \left\{ \begin{array}{l} (-f_1^*(p_1) - g_1^*(q_1) + t_1, -f_2^*(p_2) - g_2^*(q_2) + t_2) : \\ (\lambda_1, \lambda_2) \in \text{int } \mathbb{R}_+^2, \\ \lambda_1(p_1 + q_1) + \lambda_2(p_2 + q_2) = 0, \lambda_1 t_1 + \lambda_2 t_2 = 0. \end{array} \right\}$$

We start by noticing that

$$(f + g)(\mathbb{R}) + \mathbb{R}_+^2 = \mathbb{R}_+^2 = \text{cl}((f + g)(\mathbb{R}) + \mathbb{R}_+^2).$$

Furthermore

$$f_1^*(p) = \begin{cases} -7, & \text{if } p = 3, \\ +\infty, & \text{otherwise,} \end{cases} \quad \text{and } f_2^*(p) = \begin{cases} 0, & \text{if } p = 2, \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$g_1^*(q) = \begin{cases} 7, & \text{if } q = 3, \\ +\infty, & \text{otherwise,} \end{cases} \quad \text{and } g_2^*(q) = \begin{cases} 0, & \text{if } q = -2, \\ +\infty, & \text{otherwise.} \end{cases}$$

Therefore

$$-f_1^*(p) - g_1^*(q) + t = \begin{cases} t, & \text{if } p = -3, q = 3, \\ -\infty, & \text{otherwise,} \end{cases}$$

and

$$-f_2^*(p) - g_2^*(q) + t = \begin{cases} t, & \text{if } p = 2, q = -21, \\ -\infty, & \text{otherwise.} \end{cases}$$

Hence

$$\begin{aligned} h^{BGW}(\mathcal{B}^{BGW}) &= \left\{ \begin{array}{l} (t_1, t_2) \in \mathbb{R}^2 : (\lambda_1, \lambda_2) \in \text{int } \mathbb{R}^2, \\ \lambda_1(-3 + 3) + \lambda_2(2 - 2) = 0, \lambda_1 t_1 + \lambda_2 t_2 = 0 \end{array} \right\} \\ &= \{(t_1, t_2) \in \mathbb{R}^2 : \lambda_1 t_1 + \lambda_2 t_2 = 0, \lambda_1 > 0, \lambda_2 > 0\}. \end{aligned}$$

Now let us fix $\bar{v} := (-1, -1)$, for which we have that $\bar{v} \in \mathbb{R}^2 \setminus \text{cl}((f + g)(\mathbb{R}) + \mathbb{R}^2)$.

We notice that $\bar{v} \notin h^{BGW}(\mathcal{B}^{BGW}) \cap \mathbb{R}^2$. We prove this by contradiction. Assuming that $\bar{v} \in h^{BGW}(\mathcal{B}^{BGW}) \cap \mathbb{R}^2$ it follows that $\lambda_1(-1) + \lambda_2(-1) = 0$ with $\lambda_1 > 0, \lambda_2 > 0$, which is obviously a contradiction.

So $\bar{v} \notin h^{BGW}(\mathcal{B}^{BGW}) \cap \mathbb{R}^2$, and hence it follows from Proposition 7 a) that $\bar{v} \notin h^{BK}(\mathcal{B}^{BK})$. Nevertheless, from Theorem 3 it follows that $\bar{v} \in \text{core } h^{\leq}(\mathcal{B}^{\leq})$.

The conclusion is that a direct converse duality proof for the case of problem (D^{BGW}) would be more difficult, unless embedded in (D^{\leq}) .

Acknowledgment. The author would like to gratefully thank Dr. Radu Ioan Boț for his help, comments and for the improvements suggested, which have essentially upgraded the quality of the paper.

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