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FIXED POINT THEOREMS FOR MULTIVALUED WEAK CONTRACTIONS

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Abstract. The purpose of this work is to present some fixed point results for the so-called multivalued weak contractions. Our results are extensions of the theorems given by M. Berinde and V. Berinde in [1] and by C. Chifu and G. Petruşel in [2].

1. **Preliminaries**

Let us recall first some standard notations and terminologies which are used throughout the paper. For the following notions we consider the context of a metric space (X, d).

We denote by $\widetilde{B}(x_0, r)$ the closed ball centered in $x_0 \in X$ with radius $r > 0$, i.e., $\widetilde{B}(x_0, r) = \{x \in X | \ d(x, x_0) \leq r\}$.

Let $\mathcal{P}(X)$ be the set of all nonempty subsets of X. We also denote:

$$P(X) := \{Y \in \mathcal{P}(X) | \ Y \neq \emptyset\}; P_b(X) := \{Y \in P(X) | \ Y \text{ is bounded }\};$$

$$P_{cl}(X) := \{Y \in P(X) | \ Y \text{ is closed }\}.$$

Let us define the gap functional between A and B by

$$D_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}, \ D_d(A, B) = \inf\{d(a, b) \mid a \in A, \ b \in B\}$$

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(in particular, if $x_0 \in X$ then $D_d(x_0, B) := D_d(\{x_0\}, B))$ and the (generalized) Pompeiu-Hausdorff functional

$$H_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}, \quad H_d(A, B) = \max\{\sup_{a \in A} D_d(a, B), \sup_{b \in B} D_d(A, b)\}.$$

If $T : X \to P(X)$ is a multivalued operator, then $x \in X$ is called fixed point for T if and only if $x \in T(x)$ and strict fixed point if and only if $T(x) = \{x\}$. The set $Fix(T) := \{x \in X | x \in T(x)\}$ is called the fixed point set of T and $SFix(T) := \{x \in X | \{x\} = T(x)\}$ is the strict fixed point set of T.

If X is a metric space, then the multivalued operator $T : X \to P(X)$ is said to be closed if and only if its graph $Graph(F) := \{(x, y) \in X \times X : y \in F(x)\}$ is a closed subset of $X \times X$.

Let (X, d) be a metric space and $T : X \to P(X)$ be a multivalued operator. T is said to be a multivalued weak contraction or multivalued (θ, L)-weak contraction (see [1]) if and only if there exists $\theta \in]0, 1[$ and $L \geq 0$ such that

$$H(T(x), T(y)) \leq \theta \cdot d(x, y) + L \cdot D(y, T(x)), \text{ for all } x, y \in X.$$

The aim of this article is to extend some fixed point results for multivalued weak-contractions given by M. Berinde and V. Berinde in [1] and by C. Chifu and G. Petruşel in [2]. Our results are also in connection to some other theorems in this field, see [3], [5].

2. Main results

Our first result is a local one and it extends the theorem given by M. Berinde and V. Berinde in [1], to the case of a metric space endowed with two metrics.

Theorem 1. *Let X be a nonempty set, ρ and d two metrics on X, $x_0 \in X$, $r > 0$ and $T : \widetilde{B}_\rho(x_0, r) \to P(X)$ be a multivalued operator. We suppose that:*

(i) *(X, d) is a complete metric space;*

(ii) *there exists $c > 0$ such that $d(x, y) \leq c \cdot \rho(x, y)$, for each $x, y \in \widetilde{B}_\rho(x_0, r)$;*

(iii) *$T : (\widetilde{B}_\rho(x_0, r), d) \to (P(X), H_d)$ is closed;*

(iv) *T is a multivalued (θ, L)-weak contraction with respect to ρ;*

(v) $D_\rho(x_0, T(x_0)) < (1-\theta)r$.

Then we have:

(a) $Fix(T) \neq \emptyset$;

(b) *there exists a sequence* $(x_n)_{n\in\mathbb{N}} \subset \widetilde{B}_\rho(x_0, r)$ *such that:*

(b1) $x_{n+1} \in T(x_n),\ n \in \mathbb{N}$;

(b2) $x_n \overset{d}{\to} x^* \in Fix(T)$, *as* $n \to \infty$;

(b3) $d(x_n, x^*) \leq c \cdot \theta^n \cdot r$, *for each* $n \in \mathbb{N}$.

Proof. By (v), we have that there exists $x_1 \in T(x_0)$such that

$$\rho(x_0, x_1) < (1-\theta)r. \tag{1}$$

Since T is a (θ, L)-weak contraction we have that

$$H_\rho(T(x_0), T(x_1)) \leq \theta \cdot \rho(x_0, x_1) + L \cdot D_\rho(x_1, T(x_0)) = \theta \cdot \rho(x_0, x_1) < \theta \cdot (1-\theta) \cdot r.$$

Thus, for $x_1 \in T(x_0)$ there exists $x_2 \in T(x_1)$ such that

$$\rho(x_1, x_2) < \theta \cdot (1-\theta) \cdot r. \tag{2}$$

By (1) and (2) we obtain that

$$\rho(x_0, x_2) \leq \rho(x_0, x_1) + \rho(x_1, x_2) < (1-\theta) \cdot r + \theta \cdot (1-\theta) \cdot r = (1-\theta^2)r.$$

Hence $x_2 \in \widetilde{B}_\rho(x_0, r)$.

Proceeding inductively, we can construct a sequence $(x_n)_{n\in\mathbb{N}} \subset \widetilde{B}_\rho(x_0, r)$ having the following properties

$$x_{n+1} \in T(x_n),\ n \in \mathbb{N}, \tag{3}$$

$$\rho(x_n, x_{n+1}) < \theta^n \cdot (1-\theta) \cdot r. \tag{4}$$

We want to prove that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Let $p \in \mathbb{N}$. Then we have

$$\begin{aligned} \rho(x_n, x_{n+p}) &\leq \rho(x_n, x_{n+1}) + ... + \rho(x_{n+p-1}, x_{n+p}) \\ &< \theta^n \cdot (1-\theta) \cdot r \cdot (1 + \theta + ... + \theta^{p-1}) \\ &= \theta^n \cdot r \cdot (1-\theta^p). \end{aligned}$$

Letting $n \to \infty$, since $\theta \in]0,1[$, we have that $\rho(x_n, x_{n+p}) \to 0$. Thus $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to the metric ρ. By (ii) we have that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to the metric d, too. Since (X,d) is a complete metric space, there exists $x^* \in X$ such that $x_n \xrightarrow{d} x^*$ as $n \to \infty$. It remains to show that $x^* \in Fix(T)$. Since $Graph(T)$ is closed with respect to (X,d) we get that $x^* \in Fix(T)$.

We already proved that $\rho(x_n, x_{n+p}) < \theta^n \cdot r \cdot (1-\theta^p)$, By (ii), we have that there exists $c > 0$ such that $d(x_n, x_{n+p}) \leq c \cdot \rho(x_n, x_{n+p}) < c \cdot \theta^n \cdot r \cdot (1-\theta^p)$. Letting $p \to \infty$ we obtain that $d(x_n, x^*) \leq c \cdot \theta^n \cdot r$, for each $n \in \mathbb{N}$. □

We can state the above result on a set endowed with one metric.

Theorem 2. *Let (X,d) be a complete metric space, $x_0 \in X$, $r > 0$ and $T : \widetilde{B}(x_0, r) \to P(X)$ a multivalued (θ, L)-weak contraction. We assume that*

$$D(x_0, T(x_0)) < (1-\theta)r.$$

Then we have:

(a) *$Fix(T) \neq \emptyset$;*

(b) *there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset \widetilde{B}_\rho(x_0, r)$ such that:*

- (b1) *$x_{n+1} \in T(x_n)$, $n \in \mathbb{N}$;*
- (b2) *$x_n \xrightarrow{d} x^* \in Fix(T)$, as $n \to \infty$;*
- (b3) *$d(x_n, x^*) \leq \theta^n \cdot r$, for each $n \in \mathbb{N}$.*

In what follows we continue with a global version of Theorem 1 for multivalued (θ, L)-weak contractions on a set with two metrics.

Theorem 3. *Let X be a nonempty set, ρ and d twp metrics on X and $T : X \to P(X)$ a multivalued operator. We suppose that*

(i) *(X,d) is a complete metric space;*

(ii) *there exists $c > 0$ such that $d(x,y) \leq c \cdot \rho(x,y)$, for each $x, y \in X$;*

(iii) *$T : (X,d) \to (P(X), H_d)$ is closed;*

(iv) *T is a multivalued (θ, L)-weak contraction.*

Then we have:

(a) $Fix(T) \neq \emptyset$;

(b) *there exists a sequence* $(x_n)_{n\in\mathbb{N}} \subset X$ *such that:*

(b1) $x_{n+1} \in T(x_n),\ n \in \mathbb{N}$;

(b2) $x_n \xrightarrow{d} x^* \in Fix(T)$, *as* $n \to \infty$.

Proof. Fix $x_0 \in X$, choose $r > 0$ such that $D_\rho(x_0, T(x_0)) < (1-\theta)r$. The conclusion follows from Theorem 1. □

The following homotopy result extends some results given by M. Berinde, V. Berinde in [1] and C. Chifu, G. Petruşel in [2].

Theorem 4. *Let* (X, d) *be a complete metric space and* U *be an open subset of* X. *Let* $G : \overline{U} \times [0, 1] \to P(X)$ *be a multivalued operator such that the following assumptions are satisfied:*

(i) $x \notin G(x, t)$, *for each* $x \in \partial U$ *and each* $t \in [0, 1]$;

(ii) $G(\cdot, t) : \overline{U} \to P(X)$ *is a* (θ, L)*-weak contraction, for each* $t \in [0, 1]$;

(iii) *there exists a continuous, increasing function* $\psi : [0, 1] \to \mathbb{R}$ *such that*

$$H(G(x, t), G(x, s)) \leq |\psi(t) - \psi(s)|, \text{ for all } x \in \overline{U};$$

(iv) $G : \overline{U} \times [0, 1] \to P(X)$ *is closed.*

Then $G(\cdot, 0)$ *has a fixed point if and only if* $G(\cdot, 1)$ *has a fixed point.*

Proof. Suppose that $z \in Fix(G(\cdot, 0))$. From (i) we have that $z \in U$. We define the following set:

$$E := \{(x, t) \in U \times [0, 1] | x \in G(x, t)\}.$$

Since $(z, 0) \in E$, we have that $E \neq \emptyset$. We introduce a partial order on E defined by:

$$(x, t) \leq (y, s) \text{ if and only if } t \leq s \text{ and } d(x, y) \leq \frac{2}{1-\theta}[\psi(s) - \psi(t)].$$

Let M be a totally ordered subset of E, $t^* := \sup\{t\ |(x, t) \in M\}$ and $(x_n, t_n)_{n\in\mathbb{N}^*} \subset M$ be a sequence such that $(x_n, t_n) \leq (x_{n+1}, t_{n+1})$ and $t_n \to t^*$ as $n \to \infty$. Then

$$d(x_m, x_n) \leq \frac{2}{1-\theta}[\psi(t_m) - \psi(t_n)], \text{ for each } m, n \in \mathbb{N}^*,\ m > n.$$

Letting $m, n \to +\infty$ we obtain that $d(x_m, x_n) \to 0$, thus $(x_n)_{n\in\mathbb{N}^*}$ is a Cauchy sequence. Denote by $x^* \in X$ its limit. Since $x_n \in G(x_n, t_n)$, $n \in \mathbb{N}^*$ and G is closed, we have that $x^* \in G(x^*, t^*)$. From (i) we obtain that $x^* \in U$, so $(x^*, t^*) \in E$.

From the fact that M is totally ordered we have that $(x, t) \leq (x^*, t^*)$, for each $(x, t) \in M$. Thus (x^*, t^*) is an upper bound of M. We can apply Zorn's Lemma, so E admits a maximal element $(x_0, t_0) \in E$. We want to prove that $t_0 = 1$.

Suppose that $t_0 < 1$. Let $r > 0$ and $t \in]t_0, 1]$ such that $B(x_0, r) \subset U$ and $r := \frac{2}{1-\theta}[\psi(t) - \psi(t_0)]$. Then we have

$$\begin{aligned} D(x_0, G(x_0, t)) &\leq D(x_0, G(x_0, t_0)) + H(G(x_0, t_0), G(x_0, t)) \\ &\leq \psi(t) - \psi(t_0) = \frac{(1-\theta)\cdot r}{2} < (1-\theta)\cdot r. \end{aligned}$$

Since $\widetilde{B}(x_0, r) \subset \overline{U}$, the multivalued operator $G(\cdot, t) : \widetilde{B}(x_0, r) \to P_{cl}(X)$ satisfies the assumptions of Theorem 1 for all $t \in [0, 1]$. Hence there exists $x \in \widetilde{B}(x_0, r)$ such that $x \in G(x, t)$. Thus, by (i), we get that $(x, t) \in E$. Since $d(x_0, x) \leq r = \frac{2}{1-\theta}[\psi(t) - \psi(t_0)]$, we have that $(x_0, t_0,) < (x, t)$, which is a contradiction with the maximality of (x_0, t_0). Thus $t_0 = 1$.

Conversely, if $G(\cdot, 1)$ has a fixed point, by a similar approach we can obtain that $G(\cdot, 0)$ has a fixed point too. □

In 2006 A. Petruşel and I. A. Rus (see [4]) extended the notion of well-posed fixed point problem from singlevalued to multivalued operators, as follows.

Definition 1. *(A. Petruşel, I. A. Rus,* [4]*) Let (X, d) be a metric space, $Y \subset P(X)$ and $T : Y \to P_{cl}(X)$ be a multivalued operator. The fixed point problem is well-posed for T with respect to D iff:*

(a) *$Fix(T) = \{x^*\}$;*

(b) *If $x_n \in Y$, $n \in \mathbb{N}$ and $D(x_n, T(x_n)) \to 0$ as $n \to \infty$, then $x_n \to x^*$, as $n \to \infty$.*

The following result is a well-posed fixed point theorem for multivalued (θ, L)-weak contractions on a set endowed with one metric.

Theorem 5. *Let (X,d) be a complete metric space $T : X \to P_{cl}(X)$ is a multivalued (θ, L)-weak contraction with $\theta + L < 1$. Suppose that $SFix(T) \neq \emptyset$. Then the fixed point problem is well-posed for T with respect to D.*

Proof. First we want to prove that $Fix(T) = SFix(T) = \{x^*\}$. Let $x^* \in SFix(T)$. Clearly $SFix(T) \subset Fix(T)$. Thus, we only have to prove that $Fix(T) = \{x^*\}$. Let $x \in Fix(T)$ with $x^* \neq x$. Then

$$\begin{aligned} d(x^*, x) &= D(T(x^*), x) \leq H(T(x^*), T(x)) \\ &\leq \theta \cdot d(x^*, x) + L \cdot D(x, T(x^*)) \\ &= \theta \cdot d(x^*, x) + L \cdot d(x, x^*) = (\theta + L) \cdot d(x, x^*). \end{aligned}$$

Since $\theta + L < 1$ this is a contradiction, which proves that $Fix(T) = \{x^*\}$ and hence $Fix(T) = SFix(T) = \{x^*\}$.

Let $x^* \in SFix(T)$. Suppose $D(x_n, T(x_n)) \to 0$, as $n \to \infty$. Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be two sequences such that $y_n \in T(x_n)$. Then we have

$$\begin{aligned} d(x_n, x^*) &\leq d(x_n, y_n) + d(y_n, x^*) = d(x_n, y_n) + D(y_n, T(x^*)) \\ &\leq d(x_n, y_n) + H(T(x_n), T(x^*)). \end{aligned}$$

Taking the infimum over $y_n \in T(x_n)$ we have

$$\begin{aligned} d(x_n, x^*) &\leq D(x_n, T(x_n)) + H(T(x_n), T(x^*)) \\ &\leq D(x_n, T(x_n)) + \theta d(x_n, x^*) + LD(x_n, T(x^*)) \\ &= D(x_n, T(x_n)) + \theta d(x_n, x^*) + Ld(x_n, x^*). \end{aligned}$$

Thus $(1 - \theta - L)d(x_n, x) \leq D(x_n, T(x_n))$. Since $\theta + L < 1$, we have that

$$d(x_n, x^*) \leq \frac{1}{1 - \theta - L} D(x_n, T(x_n)) \to 0 \text{ as } n \to \infty.$$

□

Remark 1. *The above result give rise to the following open question: in which conditions the fixed point problem for (θ, L)-weak contractions is well-posed with respect to D, where $\theta \in]0, 1[$ and $L \geq 0$ (i.e., for $\theta + L \geq 1$, too).*

Remark 2. *It is also an open problem in the case of* (θ, L)*-weak contraction, in which conditions takes place the following implication*

$$SFix(T) \neq \emptyset \Rightarrow Fix(T) = SFix(T) = \{x^*\}.$$

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