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A PIEZOELECTRIC FRICTIONLESS CONTACT PROBLEM WITH ADHESION

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Abstract. We consider a quasistatic frictionless contact problem for a piezoelectric body. The contact is modelled with normal compliance. The adhesion of the contact surfaces is taken into account and modelled by a surface variable, the bonding field. We provide variational formulation for the mechanical problem and prove the existence of a unique weak solution to the problem. The proofs are based on arguments of time-dependent variational inequalities, differential equations and fixed point.

1. **Introduction**

A deformable material which undergoes piezoelectric effects is called a piezoelectric material. However, there are very few mathematical results concerning contact problems involving piezoelectric materials and therefore there is a need to extend the results on models for contact with deformable bodies which include coupling between mechanical and electrical properties. General models for elastic materials with piezoelectric effects can be found in [8, 9, 10, 18, 19] and more recently in [1, 17]. The importance of this paper is to make the coupling of the piezoelectric problem and a frictionless contact problem with adhesion. The adhesive contact between deformable bodies, when a glue is added to prevent relative motion of the surfaces, has received recently increased attention in the mathematical literature. Analysis of models for adhesive contact can be found in [3, 4, 6, 7, 12, 13, 14] and recently in the monographs

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[15, 16] . The novelty in all these papers is the introduction of a surface internal variable, the bonding field, denoted in this paper by α, it describes the pointwise fractional density of adhesion of active bonds on the contact surface, and sometimes referred to as the intensity of adhesion. Following [6, 7], the bonding field satisfies the restriction $0 \le \alpha \le 1$, when $\alpha = 1$ at a point of the contact surface, the adhesion is complete and all the bonds are active, when $\alpha = 0$ all the bonds are inactive, severed, and there is no adhesion, when $0 < \alpha < 1$ the adhesion is partial and only a fraction α of the bonds is active.

In this paper we describe a model of frictionless, adhesive contact between a piezoelectric body and a foundation. We provide a variational formulation of the model and, using arguments of evolutionary equations in Banach spaces, we prove that the model has a unique weak solution.

The paper is structured as follows. In section 2 we present notations and some preliminaries. The model is described in section 3 where the variational formulation is given. In section 4, we present our main result stated in Theorem 4.1 and its proof which is based on the construction of mappings between appropriate Banach spaces and a fixed point arguments.

2. Notation and preliminaries

In this short section, we present the notation we shall use and some preliminary material. For more details, we refer the reader to [2, 5, 11] . We denote by S^d the space of second order symmetric tensors on $\mathbb{R}^d$ $(d = 2, 3)$, while "." and $|\,.\,|$ represent the inner product and the Euclidean norm on S^d and $\mathbb{R}^d$, respectively. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a regular boundary Γ and let ν denote the unit outer normal on Γ. We shall use the notation

$$H = L^2(\Omega)^d = \{\mathbf{u} = (u_i) \;/\; u_i \in L^2(\Omega)\;\},$$

$$H^1(\Omega)^d = \{\mathbf{u} = (u_i) \;/\; u_i \in H^1(\Omega)\;\},$$

$$\mathcal{H} = \{\sigma = (\sigma_{ij}) \;/\; \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\;\},$$

$$\mathcal{H}_1 = \{\sigma \in \mathcal{H} \;/\; Div\ \sigma \in H\},$$

where $\varepsilon : H^1(\Omega)^d \to \mathcal{H}$ and $Div : \mathcal{H}_1 \to H$ are the deformation and divergence operators, respectively, defined by

$$\varepsilon(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \; Div\, \sigma = (\sigma_{i\,j,\,j}).$$

Here and below, the indices i and j run between 1 to d, the summation convention over repeated indices is used and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable. The spaces H, $H^1(\Omega)^d$, $\mathcal{H}$ and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$(\mathbf{u}, \mathbf{v})_H = \int_\Omega \mathbf{u} \, . \, \mathbf{v} \; dx \qquad \forall \, \mathbf{u}, \mathbf{v} \in H,$$

$$(\mathbf{u}, \mathbf{v})_{H^1(\Omega)^d} = \int_\Omega \mathbf{u} \, . \, \mathbf{v} \; dx + \int_\Omega \nabla \mathbf{u} \, . \, \nabla \mathbf{v} \; dx \qquad \forall \, \mathbf{u}, \mathbf{v} \in H^1(\Omega)^d,$$

where

$$\nabla \mathbf{v} = (v_{i,j}) \; \forall \, \mathbf{v} \in H^1(\Omega)^d,$$

$$(\sigma, \tau)_{\mathcal{H}} = \int_\Omega \sigma \, . \, \tau \; dx \qquad \forall \, \sigma, \tau \in \mathcal{H},$$

$$(\sigma, \tau)_{\mathcal{H}_1} = (\sigma, \tau)_{\mathcal{H}} + (Div\; \sigma, Div\; \tau)_H \;\; \forall \; \sigma, \tau \in \mathcal{H}_1.$$

The associated norms on the spaces H, $H^1(\Omega)^d$, $\mathcal{H}$ and $\mathcal{H}_1$ are denoted by $| \, . \, |_H$, $| \, . \, |_{H^1(\Omega)^d}, | \, . \, |_{\mathcal{H}}$ and $| \, . \, |_{\mathcal{H}_1}$respectively. Let $H_\Gamma = H^{\frac{1}{2}}(\Gamma)^d$ and let $\gamma : H^1(\Omega)^d \to H_\Gamma$ be the trace map. For every element $\mathbf{v} \in H^1(\Omega)^d$, we also use the notation $\mathbf{v}$ to denote the trace $\gamma\mathbf{v}$ of $\mathbf{v}$ on Γ and we denote by v_ν and $\mathbf{v}_\tau$ the normal and the tangential components of $\mathbf{v}$ on the boundary Γ given by

$$v_\nu = \mathbf{v} \, . \, \nu, \;\; \mathbf{v}_\tau = \mathbf{v} - v_\nu \nu. \tag{2.1}$$

Similarly, for a regular (say C^1) tensor field $\sigma : \Omega \to S^d$ we define its normal and tangential components by

$$\sigma_\nu = (\sigma\nu) \, . \, \nu, \;\; \sigma_\tau = \sigma\nu - \sigma_\nu \nu, \tag{2.2}$$

and we recall that the following Green's formulas hold:

$$(\sigma, \varepsilon(\mathbf{v}))_{\mathcal{H}} + (Div\; \sigma, \mathbf{v})_H = \int_\Gamma \sigma\nu \, . \, \mathbf{v} \; da \qquad \forall \, \mathbf{v} \in H^1(\Omega)^d. \tag{2.3}$$

$$(\mathbf{D}, \nabla\varphi)_H + (div\ \mathbf{D}, \varphi)_{L^2(\Omega)} = \int_\Gamma \mathbf{D}\ .\ \nu\ \varphi\ da \qquad \forall\, \varphi \in H^1(\Omega). \tag{2.4}$$

Finally, for any real Hilbert space X, we use the classical notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$, where $1 \leq p \leq +\infty$ and $k \geq 1$. We denote by $C(0,T;X)$ and $C^1(0,T;X)$ the space of continuous and continuously differentiable functions from $[0,T]$ to X, respectively, with the norms

$$\mid \mathbf{f} \mid_{C(0,T;X)} = \max_{t\in[0,T]} \mid \mathbf{f}(t) \mid_X ,$$

$$\mid \mathbf{f} \mid_{C^1(0,T;X)} = \max_{t\in[0,T]} \mid \mathbf{f}(t) \mid_X + \max_{t\in[0,T]} \mid \dot{\mathbf{f}}(t) \mid_X ,$$

respectively. Moreover, we use the dot above to indicate the derivative with respect to the time variable and, for a real number r, we use r_+ to represent its positive part, that is $r_+ = \max\{0, r\}$. For the convenience of the reader, we recall the following version of the classical theorem of Cauchy-Lipschitz (see, e.g., [20]).

Theorem 2.1. *Assume that* $(X, \mid . \mid_X)$ *is a real Banach space and* $T > 0$. *Let* $F(t, .) : X \to X$ *be an operator defined a.e. on* $(0,T)$ *satisfying the following conditions:*

1- There exists a constant $L_F > 0$ such that

$$\mid F(t,x) - F(t,y) \mid_X \leq L_F \mid x - y \mid_X \ \forall\, x, y \in X, \text{ a.e. } t \in (0,T)\,.$$

2- There exists $p \geq 1$ such that $t \longmapsto F(t,x) \in L^p(0,T;X) \quad \forall\, x \in X$.

Then for any $x_0 \in X$, there exists a unique function $x \in W^{1,\,p}(0,T;X)$ such that

$$\dot{x}(t) = F(t, x(t)) \qquad \text{a.e. } t \in (0,T)\,,$$

$$x(0) = x_0.$$

Theorem 2.1 will be used in section 4 to prove the unique solvability of the intermediate problem involving the bonding field.

Moreover, if X_1 and X_2 are real Hilbert spaces then $X_1 \times X_2$ denotes the product Hilbert space endowed with the canonical inner product $(.,.)_{X_1\times X_2}$.

3. Mechanical and variational formulations

We describe the model for the process, we present its variational formulation. The physical setting is the following. An electro-elastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with outer Lipschitz surface Γ. The body is submitted to the action of body forces of density $\mathbf{f}_0$ and volume electric charges of density q_0. It is also submitted to mechanical and electric constraint on the boundary. We consider a partition of Γ into three disjoint measurable parts Γ_1, Γ_2 and Γ_3, on one hand, and on two measurable parts Γ_a and Γ_b, on the other hand, such that $meas\ (\Gamma_1) > 0$ and $meas\ (\Gamma_a) > 0$. Let $T > 0$ and let $[0, T]$ be the time interval of interest. The body is clamped on $\Gamma_1 \times (0, T)$, so the displacement field vanishes there. A surface tractions of density $\mathbf{f}_2$ act on $\Gamma_2 \times (0, T)$ and a body force of density $\mathbf{f}_0$ acts in $\Omega \times (0, T)$. We also assume that the electrical potential vanishes on $\Gamma_a \times (0, T)$ and a surface electric charge of density q_2 is prescribed on $\Gamma_b \times (0, T)$. The body is in adhesive contact with an obstacle, or foundation, over the contact surface Γ_3. We suppose that the body forces and tractions vary slowly in time, and therefore, the accelerations in the system may be neglected. Neglecting the inertial terms in the equation of motion leads to a quasistatic approach of the process. We denote by $\mathbf{u}$ the displacement field, by σ the stress tensor field and by $\varepsilon(\mathbf{u})$ the linearized strain tensor. We use a piezoelectric constitutive law given by

$$\sigma = \mathcal{A}(\varepsilon(\mathbf{u})) - \mathcal{E}^*\mathbf{E}(\varphi),$$

$$\mathbf{D} = \mathcal{E}\varepsilon(\mathbf{u}) + B\mathbf{E}(\varphi),$$

these relations represent the electro-viscoelastic constitutive law of the material which $\mathcal{A}$ is a given nonlinear function, $\mathbf{E}(\varphi) = -\nabla\varphi$ is the electric field, $\mathcal{E} = (e_{ijk})$ represents the third order piezoelectric tensor, $\mathcal{E}^*$ is its transposed and is given by $\mathcal{E}^* = (e^*_{ijk})$, where $e^*_{ijk} = e_{kij}$ and B denotes the electric permitivitty tensor.

To simplify the notation, we do not indicate explicitely the dependence of various functions on the variables $\mathbf{x} \in \Omega \cup \Gamma$ and $t \in [0, T]$. Then, the classical formulation of the mechanical problem of piezoelectric material, frictionless, adhesive contact may be stated as follows.

Problem P. *Find a displacement field* $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$ *and a stress field* $\sigma : \Omega \times [0,T] \to S^d$ *, an electric potential field* $\varphi : \Omega \times [0,T] \to \mathbb{R}$*, an electric displacement field* $\mathbf{D} : \Omega \times [0,T] \to \mathbb{R}^d$ and *a bonding field* $\alpha : \Gamma_3 \times [0,T] \to \mathbb{R}$ *such that*

$$\sigma = \mathcal{A}(\varepsilon(\mathbf{u})) + \mathcal{E}^* \nabla\varphi \ \text{ in } \Omega \times (0,T), \tag{3.1}$$

$$\mathbf{D} = \mathcal{E}\varepsilon(\mathbf{u}) - B\ \nabla\varphi \ \text{ in } \Omega \times (0,T), \tag{3.2}$$

$$Div\ \sigma + \mathbf{f}_0 = 0 \quad \text{ in } \Omega \times (0,T), \tag{3.3}$$

$$div\ \mathbf{D} = q_0 \ \text{ in } \Omega \times (0,T), \tag{3.4}$$

$$\mathbf{u} = 0 \quad \text{ on } \Gamma_1 \times (0,T), \tag{3.5}$$

$$\sigma\nu = \mathbf{f}_2 \quad \text{ on } \Gamma_2 \times (0,T), \tag{3.6}$$

$$-\sigma_\nu = p_\nu(u_\nu) - \gamma_\nu \alpha^2 R_\nu(u_\nu) \quad \text{ on } \Gamma_3 \times (0,T), \tag{3.7}$$

$$-\sigma_\tau = p_\tau(\alpha)\mathbf{R}_\tau(\mathbf{u}_\tau) \ \text{ on } \Gamma_3 \times (0,T), \tag{3.8}$$

$$\dot{\alpha} = -(\alpha(\gamma_\nu(R_\nu(u_\nu))^2 + \gamma_\tau \mid \mathbf{R}_\tau(\mathbf{u}_\tau) \mid^2 \ ) - \varepsilon_a)_+ \ \text{ on } \Gamma_3 \times (0,T), \tag{3.9}$$

$$\alpha(0) = \alpha_0 \text{ on } \Gamma_3, \tag{3.10}$$

$$\varphi = 0 \quad \text{ on } \Gamma_a \times (0,T), \tag{3.11}$$

$$\mathbf{D}\ .\ \nu = q_2 \quad \text{ on } \Gamma_b \times (0,T). \tag{3.12}$$

First, (3.1) and (3.2) represent the electro-elastic constitutive law described above. Equations (3.3) and (3.4) represent the equilibrium equations for the stress and electric-displacement fields while (3.5) and (3.6) are the displacement and traction boundary condition, respectively. Condition (3.7) represents the normal compliance conditions with adhesion where γ_ν is a given adhesion coefficient and p_ν is a given positive function which will be described below. In this condition the interpenetrability between the body and the foundation is allowed, that is u_ν can be positive on Γ_3. The contribution of the adhesive to the normal traction is represented by the term $\gamma_\nu \alpha^2 R_\nu(u_\nu)$, the adhesive traction is tensile and is proportional, with proportionality coefficient γ_ν, to the square of the intensity of adhesion and to the normal displacement, but as long as it does not exceed the bond length L. The maximal tensile traction is $\gamma_\nu L$. R_ν is the truncation operator defined by

$$R_\nu(s) = \begin{cases} L & \text{if } s < -L, \\ -s & \text{if } -L \leq s \leq 0, \\ 0 & \text{if } s > 0. \end{cases}$$

Here $L > 0$ is the characteristic length of the bond, beyond which it does not offer any additional traction. The introduction of the operator R_ν, together with the operator $\mathbf{R}_\tau$ defined below, is motivated by mathematical arguments but it is not restrictive for physical point of view, since no restriction on the size of the parameter L is made in what follows. Condition (3.8) represents the adhesive contact condition on the tangential plane, in which p_τ is a given function and $\mathbf{R}_\tau$ is the truncation operator given by

$$\mathbf{R}_\tau(\mathbf{v}) = \begin{cases} \mathbf{v} & \text{if} \quad |\,\mathbf{v}\,| \leq L, \\ L\frac{\mathbf{v}}{|\mathbf{v}|} & \text{if} \quad |\,\mathbf{v}\,| > L. \end{cases}$$

This condition shows that the shear on the contact surface depends on the bonding field and on the tangential displacement, but as long as it does not exceed the bond length L. The frictional tangential traction is assumed to be much smaller than the adhesive one and, therefore, omitted.

Next, the equation (3.9) represents the ordinary differential equation which describes the evolution of the bonding field and it was already used in [3], see also [15, 16] for more details. Here, besides γ_ν, two new adhesion coefficients are involved, γ_τ and ε_a. Notice that in this model once debonding occurs bonding cannot be reestablished since, as it follows from (3.9), $\dot{\alpha} \leq 0$. Finally, (3.10) represents the initial condition in which α_0 is the given initial bonding field, (3.11) and (3.12) represent the electric boundary conditions. To obtain the variational formulation of the problem (3.1)-(3.12), we introduce for the bonding field the set

$$Z = \{\theta \in L^\infty(\Gamma_3) \;/\; 0 \leq \theta \leq 1 \text{ a.e. on } \Gamma_3\},$$

and for the displacement field we need the closed subspace of $H^1(\Omega)^d$ defined by

$$V = \left\{ \mathbf{v} \in H^1(\Omega)^d \ / \ \mathbf{v} = 0 \text{ on } \Gamma_1 \right\}.$$

Since $meas\ (\Gamma_1) > 0$, Korn's inequality holds and there exists a constant $C_k > 0$, that depends only on Ω and Γ_1, such that

$$\mid \varepsilon(\mathbf{v}) \mid_{\mathcal{H}} \geq C_k \mid \mathbf{v} \mid_{H^1(\Omega)^d} \quad \forall \mathbf{v} \in V.$$

A proof of Korn's inequality may be found in [11, p.79]. On the space V we consider the inner product and the associated norm given by

$$(\mathbf{u}, \mathbf{v})_V = (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}}, \quad \mid \mathbf{v} \mid_V = \mid \varepsilon(\mathbf{v}) \mid_{\mathcal{H}} \quad \forall \mathbf{u}, \mathbf{v} \in V. \tag{3.13}$$

It follows that $\mid . \mid_{H^1(\Omega)^d}$ and $\mid . \mid_V$ are equivalent norms on V and therefore $(V, \mid . \mid_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem and (3.13), there exists a constant $C_0 > 0$, depending only on Ω, Γ_1 and Γ_3 such that

$$\mid \mathbf{v} \mid_{L^2(\Gamma_3)^d} \leq C_0 \mid \mathbf{v} \mid_V \qquad \forall \mathbf{v} \in V. \tag{3.14}$$

We also introduce the spaces

$$W = \left\{ \phi \in H^1(\Omega) \ / \ \phi = 0 \text{ on } \Gamma_a \right\},$$

$$\mathcal{W} = \left\{ \mathbf{D} = (D_i) \ / \ D_i \in L^2(\Omega),\ div\ \mathbf{D} \in L^2(\Omega) \right\},$$

where $div\ \mathbf{D} = (D_{i,i})$. The spaces W and $\mathcal{W}$ are real Hilbert spaces with the inner products

$$(\varphi, \phi)_W = \int_\Omega \nabla\varphi . \nabla\phi \ dx,$$

$$(\mathbf{D}, \mathbf{E})_{\mathcal{W}} = \int_\Omega \mathbf{D}.\mathbf{E}\ dx + \int_\Omega div\ \mathbf{D}.div\ \mathbf{E}\ dx.$$

The associated norms will be denoted by $\mid . \mid_W$ and $\mid . \mid_{\mathcal{W}}$, respectively. Notice also that, since $meas(\Gamma_a) > 0$, the following Friedrichs-Poincaré inequality holds:

$$\mid \nabla\phi \mid_{L^2(\Omega)^d} \geq C_F \mid \phi \mid_{H^1(\Omega)} \quad \forall \phi \in W, \tag{3.15}$$

where $C_F > 0$ is a constant which depends only on Ω and Γ_a.

In the study of the mechanical problem (3.1)-(3.12), we assume that the constitutive function $\mathcal{A}: \Omega \times S^d \to S^d$ satisfies

$$\begin{cases} (a) \text{ There exists a constant } L_{\mathcal{A}} > 0 \text{ Such that} \\ \mid \mathcal{A}(\mathbf{x}, \varepsilon_1) - \mathcal{A}(\mathbf{x}, \varepsilon_2) \mid \leq L_{\mathcal{A}} \mid \varepsilon_1 - \varepsilon_2 \mid \;\; \forall \varepsilon_1, \varepsilon_2 \in S^d, \text{ a.e. } \mathbf{x} \in \Omega. \\ (b) \text{ There exists a constant } m_{\mathcal{A}} > 0 \text{ Such that} \\ (\mathcal{A}(\mathbf{x}, \varepsilon_1) - \mathcal{A}(\mathbf{x}, \varepsilon_2)).(\varepsilon_1 - \varepsilon_2) \geq m_{\mathcal{A}} \mid \varepsilon_1 - \varepsilon_2 \mid^2 \;\; \forall \varepsilon_1, \varepsilon_2 \in S^d, \text{ a.e. } \mathbf{x} \in \Omega. \\ (c) \text{ The mapping } \mathbf{x} \to \mathcal{A}(\mathbf{x}, \varepsilon) \text{ is Lebesgue measurable on } \Omega \text{ for any } \varepsilon \in S^d. \\ (d) \text{ The mapping } \mathbf{x} \to \mathcal{A}(\mathbf{x}, \mathbf{0}) \text{ belongs to } \mathcal{H}. \end{cases} \tag{3.16}$$

The operator $B = (B_{ij})$: $\Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies

$$\begin{cases} (a)\ B(\mathbf{x}, \mathbf{E}) = (b_{ij}(\mathbf{x}) E_j)\ \forall \mathbf{E} = (E_i) \in \mathbb{R}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\ (b)\ b_{ij} = b_{ji}, \;\; b_{ij} \in L^\infty(\Omega), \quad 1 \leq i, j \leq d. \\ (c) \text{ There exists a constant } m_B > 0 \text{ such that } B\mathbf{E}.\mathbf{E} \geq m_B \mid \mathbf{E} \mid^2 \\ \forall \mathbf{E} = (E_i) \in \mathbb{R}^d, \text{ a.e. in } \Omega. \end{cases} \tag{3.17}$$

The operator $\mathcal{E}: \Omega \times S^d \to \mathbb{R}^d$ satisfies

$$\begin{cases} (a)\ \mathcal{E} = (e_{ijk}), e_{ijk} \in L^\infty(\Omega), \quad 1 \leq i, j, k \leq d. \\ (b)\ \mathcal{E}(\mathbf{x})\sigma.\tau = \sigma.\ \mathcal{E}^*(\mathbf{x})\tau \quad \forall \sigma, \tau \in S^d, \text{ a.e. in } \Omega. \end{cases} \tag{3.18}$$

The normal compliance function $p_\nu: \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies

$$\begin{cases} (a) \text{ There exists a constant } L_\nu > 0 \text{ such that} \\ \mid p_\nu(\mathbf{x}, r_1) - p_\nu(\mathbf{x}, r_2) \mid \leq L_\tau \mid r_1 - r_2 \mid \;\; \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_3. \\ (b)\ (p_\nu(\mathbf{x}, r_1) - p_\nu(\mathbf{x}, r_2))(r_1 - r_2) \geq 0\ \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_3. \\ (c) \text{ The mapping } \mathbf{x} \to p_\nu(\mathbf{x}, r) \text{ is measurable on } \Gamma_3, \text{ for any } r \in \mathbb{R}. \\ (d)\ p_\nu(\mathbf{x}, r) = 0 \text{ for all } r \leq 0, \text{ a.e. } \mathbf{x} \in \Gamma_3. \end{cases} \tag{3.19}$$

The tangential contact function $p_\tau : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies

$$\begin{cases} (a) \text{ There exists a constant } L_\tau > 0 \text{ such that} \\ \mid p_\tau(\mathbf{x}, d_1) - p_\tau(\mathbf{x}, d_2) \mid \leq L_\tau \mid d_1 - d_2 \mid \ \forall d_1, d_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_3. \\ (b) \text{ There exists } M_\tau > 0 \text{ such that } \mid p_\tau(\mathbf{x}, d) \mid \leq M_\tau \ \forall d \in \mathbb{R}, \text{a.e. } \mathbf{x} \in \Gamma_3. \\ (c) \text{ The mapping } \mathbf{x} \to p_\tau(\mathbf{x}, d) \text{ is measurable on } \Gamma_3, \text{ for any } d \in \mathbb{R}. \\ (d) \text{ The mapping } \mathbf{x} \to p_\tau(\mathbf{x}, 0) \in L^2(\Gamma_3). \end{cases} \tag{3.20}$$

We also suppose that the body forces and surface tractions have the regularity

$$\mathbf{f}_0 \in W^{1,\infty}(0, T; L^2(\Omega)^d), \quad \mathbf{f}_2 \in W^{1,\infty}(0, T; L^2(\Gamma_2)^d), \tag{3.21}$$

$$q_0 \in W^{1,\infty}(0, T; L^2(\Omega)), \quad q_2 \in W^{1,\infty}(0, T; L^2(\Gamma_b)). \tag{3.22}$$

The adhesion coefficients satisfy

$$\gamma_\nu, \gamma_\tau, \varepsilon_a \in L^\infty(\Gamma_3), \gamma_\nu, \gamma_\tau, \varepsilon_a \geq 0 \text{ a.e. on } \Gamma_3. \tag{3.23}$$

The initial bonding field satisfies

$$\alpha_0 \in Z. \tag{3.24}$$

Next, we denote by $\mathbf{f} : [0, T] \to V$ the function defined by

$$(\mathbf{f}(t), \mathbf{v})_V = \int_\Omega \mathbf{f}_0(t) \, . \, \mathbf{v} \, dx \ + \ \int_{\Gamma_2} \mathbf{f}_2(t) \, . \, \mathbf{v} \, da \ \ \forall \mathbf{v} \in V, \ t \in [0, T] \,, \tag{3.25}$$

and we denote by q: $[0, T] \to W$ the function defined by

$$(q(t), \phi)_W = \int_\Omega q_0(t) \, . \, \phi \, dx \ - \ \int_{\Gamma_b} q_2(t) \, . \, \phi \, da \ \ \forall \phi \in W, \ t \in [0, T] \,. \tag{3.26}$$

Next, we denote by $j_{ad} : L^\infty(\Gamma_3) \times V \times V \to \mathbb{R}$ the adhesion functional defined by

$$j_{ad}(\alpha, \mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} (-\gamma_\nu \alpha^2 R_\nu(u_\nu) \, v_\nu + p_\tau(\alpha) \mathbf{R}_\tau(\mathbf{u}_\tau) \, .\mathbf{v}_\tau) \, da. \tag{3.27}$$

In addition to the functional (3.27), we need the normal compliance functional

$$j_{nc}(\mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} p_\nu(u_\nu) v_\nu \, da. \tag{3.28}$$

Keeping in mind (3.19)-(3.20), we observe that the integrals (3.27) and (3.28) are well defined and we note that conditions (3.21)-(3.22) imply

$$\mathbf{f} \in W^{1,\infty}(0,T;V), \;\; q \in W^{1,\infty}(0,T;W). \tag{3.29}$$

Using standard arguments we obtain the variational formulation of the mechanical problem (3.1)-(3.12).

Problem PV. Find a displacement field $\mathbf{u} : [0,T] \to V$, an electric potential field $\varphi : [0,T] \to W$ and a bonding field $\alpha : [0,T] \to L^{\infty}\ (\Gamma_3)$ such that

$$(\mathcal{A}\varepsilon(\mathbf{u}(t)), \varepsilon(\mathbf{v}))_{\mathcal{H}} + (\mathcal{E}^{*}\nabla\varphi(t), \varepsilon(\mathbf{v}))_{\mathcal{H}} + j_{ad}(\alpha(t), \mathbf{u}(t), \mathbf{v})$$

$$+ j_{nc}(\mathbf{u}(t), \mathbf{v}) = (\mathbf{f}(t), \mathbf{v})_V \;\; \forall \mathbf{v} \in V, t \in (0,T)\,, \tag{3.30}$$

$$(B\nabla\varphi(t), \nabla\phi)_{L^2(\Omega)^d} - (\mathcal{E}\varepsilon(\mathbf{u}(t)), \nabla\phi)_{L^2(\Omega)^d} = (q(t), \phi)_W \quad \forall \phi \in W, t \in (0,T)\,, \tag{3.31}$$

$$\dot{\alpha}(t) = -(\alpha(t)(\gamma_\nu(R_{\nu}(u_\nu(t)))^2 + \gamma_\tau \mid \mathbf{R}_{\tau}(\mathbf{u}_\tau(t)) \mid^2 \;\;) - \varepsilon_a)_+ \text{ a.e. } t \in (0,T)\,, \tag{3.32}$$

$$\alpha(0) = \alpha_0. \tag{3.33}$$

The existence of the unique solution of problem PV is stated and proved in the next section.

Remark 3.1. *We note that, in problem P and in problem PV, we do not need to impose explicitly the restriction $0 \leq \alpha \leq 1$. Indeed, equation (3.32) guarantees that $\alpha(\mathbf{x},t) \leq \alpha_0(\mathbf{x})$ and, therefore, assumption (3.24) shows that $\alpha(\mathbf{x},t) \leq 1$ for $t \geq 0$, a.e. $\mathbf{x} \in \Gamma_3$. On the other hand, if $\alpha(\mathbf{x},t_0) = 0$ at time t_0, then it follows from (3.32) that $\dot{\alpha}(\mathbf{x},t) = 0$ for all $t \geq t_0$ and therefore, $\alpha(\mathbf{x},t) = 0$ for all $t \geq t_0$, a.e. $\mathbf{x} \in \Gamma_3$. We conclude that $0 \leq \alpha(\mathbf{x},t) \leq 1$ for all $t \in [0,T]$, a.e. $\mathbf{x} \in \Gamma_3$.*

Below in this section $\alpha, \alpha_1, \alpha_2$ denote elements of $L^2(\Gamma_3)$ such that $0 \leq \alpha, \alpha_1, \alpha_2 \leq 1$ a.e. $\mathbf{x} \in \Gamma_3$, $\mathbf{u}_1, \mathbf{u}_2$ and $\mathbf{v}$ represent elements of V and $C > 0$ represents generic constants which may depend on $\Omega, \Gamma_3, \Gamma_3, p_\nu, p_\tau, \gamma_\nu, \gamma_\tau$ and L.

First, we note that the functional j_{ad} and j_{nc} are linear with respect to the last argument and, therefore,

$$j_{ad}(\alpha, \mathbf{u}, -\mathbf{v}) = -j_{ad}(\alpha, \mathbf{u}, \mathbf{v}),\; j_{nc}(\mathbf{u}, -\mathbf{v}) = -j_{nc}(\mathbf{u}, \mathbf{v}). \tag{3.34}$$

Next, using (3.27), the properties of the truncation operators R_ν and $\mathbf{R}_\tau$ as well as assumption (3.20) on the function p_τ, after some calculus we find

$$j_{ad}(\alpha_1, \mathbf{u}_1, \mathbf{u}_2 - \mathbf{u}_1) + j_{ad}(\alpha_2, \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2)$$
$$\leq C \int_{\Gamma_3} |\alpha_1 - \alpha_2 || \mathbf{u}_1 - \mathbf{u}_2 | \, da,$$

and, by (3.14), we obtain

$$j_{ad}(\alpha_1, \mathbf{u}_1, \mathbf{u}_2 - \mathbf{u}_1) + j_{ad}(\alpha_2, \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2)$$
$$\leq C|\alpha_1 - \alpha_2 |_{L^2(\Gamma_3)}| \mathbf{u}_1 - \mathbf{u}_2 |_V \, . \tag{3.35}$$

Similar computations, based on the Lipschitz continuity of R_ν, $\mathbf{R}_\tau$ and p_τ show that the following inequality also holds

$$| j_{ad}(\alpha, \mathbf{u}_1, \mathbf{v}) - j_{ad}(\alpha, \mathbf{u}_2, \mathbf{v}) | \leq C \mid \mathbf{u}_1 - \mathbf{u}_2 |_V | \mathbf{v} |_V \, . \tag{3.36}$$

We take now $\alpha_1 = \alpha_2 = \alpha$ in (3.35) to deduce

$$j_{ad}(\alpha, \mathbf{u}_1, \mathbf{u}_2 - \mathbf{u}_1) + j_{ad}(\alpha, \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) \leq 0. \tag{3.37}$$

Also, we take $\mathbf{u}_1 = \mathbf{v}$ and $\mathbf{u}_2 = 0$ in (3.36) then we use the equalities $R_\nu(0) = 0$, $\mathbf{R}_\tau(\mathbf{0}) = \mathbf{0}$ and (3.34) to obtain

$$j_{ad}(\alpha, \mathbf{v}, \mathbf{v}) \geq 0. \tag{3.38}$$

Now, we use (3.28) to see that

$$| j_{nc}(\mathbf{u}_1, \mathbf{v}) - j_{nc}(\mathbf{u}_2, \mathbf{v}) | \leq \int_{\Gamma_3} | p_\nu(u_{1\nu}) - p_\nu(u_{2\nu}) || v_\nu | \, da,$$

and therefore (3.19) (a) and (3.14) imply

$$| j_{nc}(\mathbf{u}_1, \mathbf{v}) - j_{nc}(\mathbf{u}_2, \mathbf{v}) | \leq C \mid \mathbf{u}_1 - \mathbf{u}_2 |_V | \mathbf{v} |_V \, . \tag{3.39}$$

We use again (3.28) to see that

$$j_{nc}(\mathbf{u}_1, \mathbf{u}_2 - \mathbf{u}_1) + j_{nc}(\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) \leq \int_{\Gamma_3} (p_\nu(u_{1\nu}) - p_\nu(u_{2\nu}))(u_{2\nu} - u_{1\nu}) da,$$

and therefore (3.19) (b) implies

$$j_{nc}(\mathbf{u}_1, \mathbf{u}_2 - \mathbf{u}_1) + j_{nc}(\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) \leq 0. \tag{3.40}$$

We take $\mathbf{u}_1 = \mathbf{v}$ and $\mathbf{u}_2 = 0$ in the previous inequality and use (3.19) (d) and (3.40) to obtain

$$j_{nc}(\mathbf{v}, \mathbf{v}) \geq 0. \tag{3.41}$$

Inequalities (3.35)-(3.41) and equality (3.34) will be used in various places in the rest of the paper.

4. **An existence and uniqueness result**

Now, we propose our existence and uniqueness result.

Theorem 4.1. *Assume that (3.16)-(3.24) hold. Then there exists a unique solution* $\{\mathbf{u}, \varphi, \alpha\}$ *to problem* PV*. Moreover, the solution satisfies*

$$\mathbf{u} \in W^{1,\infty}(0, T; V), \tag{4.1}$$

$$\varphi \in W^{1,\infty}(0, T; W), \tag{4.2}$$

$$\alpha \in W^{1,\infty}(0, T; L^\infty(\Gamma_3)). \tag{4.3}$$

The functions $\mathbf{u}, \varphi, \sigma, \mathbf{D}$ and α which satisfy (3.1)-(3.2) and (3.30)-(3.33) are called a weak solution of the contact problem P.

We conclude that, under the assumptions (3.16)-(3.24), the mechanical problem (3.1)-(3.12) has a unique weak solution satisfying (4.1)-(4.3). The regularity of the weak solution is given by (4.1)-(4.3) and, in term of stresses,

$$\sigma \in W^{1,\infty}(0, T; \mathcal{H}_1), \tag{4.4}$$

$$\mathbf{D} \in W^{1,\infty}(0, T; \mathcal{W}). \tag{4.5}$$

Indeed, it follows from (3.30) and (3.31) that $Div\ \sigma(t) + \mathbf{f}_0(t) = 0$, $div\ \mathbf{D} = q_0(t)$ for all $t \in [0, T]$ and therefore the regularity (4.1) and (4.2) of $\mathbf{u}$ and φ, combined with (3.16)-(3.22) implies (4.4) and (4.5).

The proof of Theorem 4.1 is carried out in several steps that we prove in what follows, everywhere in this section we suppose that assumptions of Theorem 4.1 hold, and we consider that C is a generic positive constant which depends on

$\Omega, \Gamma_1, \Gamma_3, p_\nu, p_\tau, \gamma_\nu, \gamma_\tau$ and L and may change from place to place. Let $\mathcal{Z}$ denote the closed subset of $C(0,T;L^2(\Gamma_3))$ defined by

$$\mathcal{Z} = \left\{ \theta \in C(0,T;L^2(\Gamma_3)) \ / \ \theta(t) \in Z \ \forall t \in [0,T], \theta(0) = \alpha_0 \right\}. \tag{4.6}$$

Let $\alpha \in \mathcal{Z}$ be given. In the first step we consider the following variational problem.

Problem PV_α. Find a displacement field $\mathbf{u}_\alpha : [0,T] \to V$, an electric potential field $\varphi_\alpha : [0,T] \to W$ *such that*

$$(\mathcal{A}\varepsilon(\mathbf{u}_\alpha(t)), \varepsilon(\mathbf{v}))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi_\alpha(t), \varepsilon(\mathbf{v}))_{\mathcal{H}} + j_{ad}(\alpha(t), \mathbf{u}_\alpha(t), \mathbf{v})$$
$$+ j_{nc}(\mathbf{u}_\alpha(t), \mathbf{v}) = (\mathbf{f}(t), \mathbf{v})_V \quad \forall \mathbf{v} \in V, \ t \in [0,T], \tag{4.7}$$
$$(B\nabla\varphi_\alpha(t), \nabla\phi)_{L^2(\Omega)^d} - (\mathcal{E}\varepsilon(\mathbf{u}_\alpha(t)), \nabla\phi)_{L^2(\Omega)^d}$$
$$= (q(t), \phi)_W \quad \forall \phi \in W, t \in (0,T). \tag{4.8}$$

We have the following result for the problem.

Lemma 4.2. *There exists a unique solution to problem PV_α. The solution satisfies* $(\mathbf{u}_\alpha, \varphi_\alpha) \in C(0,T;V) \times C(0,T;W)$.

Proof. Let $t \in [0,T]$ we consider the product space $X = V \times W$ with the inner product:

$$(x,y)_X = (\mathbf{u}, \mathbf{v})_V + (\varphi, \phi)_W \quad \forall x = (\mathbf{u}, \varphi), \ y = (\mathbf{v}, \phi) \in X, \tag{4.9}$$

and the associated norm $|\ .\ |_X$. Let $A_t : X \to X$ be the operator given by

$$(A_t x, y)_X = (\mathcal{A}\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}} + (B\nabla\varphi, \nabla\phi)_{L^2(\Omega)^d} + (\mathcal{E}^*\nabla\varphi, \varepsilon(\mathbf{v}))_{\mathcal{H}}$$
$$-(\mathcal{E}\varepsilon(\mathbf{u}), \nabla\phi)_{L^2(\Omega)^d} + j_{nc}(\mathbf{u}, \mathbf{v}) + j_{ad}(\alpha(t), \mathbf{u}, \mathbf{v})$$
$$\forall x = (\mathbf{u}, \varphi), \ y = (\mathbf{v}, \phi) \in X. \tag{4.10}$$

We consider the element $F \in X$ given by

$$F = (\mathbf{f}, q) \in X. \tag{4.11}$$

We consider the following equivalence result the couple $x_\alpha = (\mathbf{u}_\alpha, \varphi_\alpha)$ is a solution to problem PV_α if and only if

$$(A_t x_\alpha(t), y)_X = (F(t), y)_X, \ \forall y \in X, \ t \in [0,T]. \tag{4.12}$$

Indeed, let $x_\alpha(t) = (\mathbf{u}_\alpha(t), \varphi_\alpha(t)) \in X$ be a solution to problem PV_α and let $y = (\mathbf{v}, \phi) \in X$. We add the equality (4.7) to (4.8) and we use (4.9)-(4.11) to obtain (4.12). Conversely, let $x_\alpha(t) = (\mathbf{u}_\alpha(t), \varphi_\alpha(t)) \in X$ be a solution to the quasivariational inequality (4.12). We take $y = (\mathbf{v}, 0) \in X$ in (4.12) where $\mathbf{v}$ is an arbitrary element of V and obtain (4.7), then we take $y = (\mathbf{0}, \phi)$ in (4.12), where ϕ is an arbitrary element of W, as a result we obtain (4.8). We use (3.13), (3.15), (3.16)-(3.18), (3.36) and (3.39) to see that the operator A_t is strongly monotone and Lipschitz continuous, it follows by standard results on elliptic variational inequalities that there exists a unique element $(\mathbf{u}_\alpha(t), \varphi_\alpha(t)) \in X$ which solves (4.7)-(4.8).

Now let us show that $(\mathbf{u}_\alpha, \varphi_\alpha) \in C(0,T;V) \times C(0,T;W)$. We let $t_1, t_2 \in [0,T]$ and use the notation $\mathbf{u}_\alpha(t_i) = \mathbf{u}_i$, $\alpha(t_i) = \alpha_i$, $\varphi_\alpha(t_i) = \varphi_i$, $\mathbf{f}(t_i) = \mathbf{f}_i$, $q(t_i) = q_i$ and $x_\alpha(t_i) = (\mathbf{u}_\alpha(t_i), \varphi_\alpha(t_i)) = x_i$ for $i = 1, 2$. We use standard arguments in (4.7) and (4.8) to find

$$
\begin{aligned}
&(\mathcal{A}\varepsilon(\mathbf{u}_1 - \mathbf{u}_2), \varepsilon(\mathbf{u}_1 - \mathbf{u}_2))_{\mathcal{H}} + (\mathcal{E}^*\nabla(\varphi_1 - \varphi_2), \varepsilon(\mathbf{u}_1 - \mathbf{u}_2))_{\mathcal{H}} \\
&= (\mathbf{f}_1 - \mathbf{f}_2, \mathbf{u}_1 - \mathbf{u}_2)_V + j_{nc}(\mathbf{u}_1, \mathbf{u}_2 - \mathbf{u}_1) + j_{nc}(\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) \\
&\quad + j_{ad}(\alpha_1, \mathbf{u}_1, \mathbf{u}_2 - \mathbf{u}_1) + j_{ad}(\alpha_2, \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2), \qquad (4.13)
\end{aligned}
$$

$$
\begin{aligned}
&(B\nabla(\varphi_1 - \varphi_2), \nabla(\varphi_1 - \varphi_2))_{L^2(\Omega)^d} - (\mathcal{E}\varepsilon(\mathbf{u}_1 - \mathbf{u}_2), \nabla(\varphi_1 - \varphi_2))_{L^2(\Omega)^d} \\
&= (q_1 - q_2, \varphi_1 - \varphi_2)_W, \qquad (4.14)
\end{aligned}
$$

and, by using the assumption (3.16)-(3.18) on $\mathcal{A}$, B and $\mathcal{E}$, the properties (3.35) and (3.39) on the functional j_{ad} and j_{nc} respectively and (3.14)-(3.15), we obtain

$$
\mid \mathbf{u}_1 - \mathbf{u}_2 \mid_V \leq C(\mid \mathbf{f}_1 - \mathbf{f}_2 \mid_V + \mid q_1 - q_2 \mid_W + \mid \alpha_1 - \alpha_2 \mid_{L^2(\Gamma_3)}). \qquad (4.15)
$$

$$
\mid \varphi_1 - \varphi_2 \mid_W \leq C(\mid \mathbf{u}_1 - \mathbf{u}_2 \mid_V + \mid q_1 - q_2 \mid_W). \qquad (4.16)
$$

The inequality (4.15) and the regularity of the functions $\mathbf{f}$, q and α show that $\mathbf{u}_\alpha \in C(0,T;V)$. We use (4.16) and the regularity of the functions $\mathbf{u}_\alpha$, q to show that $\varphi_\alpha \in C(0,T;W)$. Thus we conclude the existence part in lemma 4.2 and we note that the uniqueness of the solution follows from the unique solvability of (4.7) and (4.8) at any $t \in [0,T]$. □

In the next step, we use the displacement field $\mathbf{u}_\alpha$ obtained in lemma 4.2 and we consider the following initial-value problem.

Problem PV_θ. Find the adhesion field $\theta_\alpha : [0,T] \to L^\infty(\Gamma_3)$ such that for a.e. $t \in (0,T)$

$$\dot{\theta}_\alpha(t) = -(\theta_\alpha(t)(\gamma_\nu(R_\nu(u_{\alpha\nu}(t)))^2 + \gamma_\tau \mid \mathbf{R}_\tau(\mathbf{u}_{\alpha\tau}(t)) \mid^2 \ ) - \varepsilon_a)_+, \tag{4.17}$$

$$\theta_\alpha(0) = \alpha_0. \tag{4.18}$$

We have the following result.

Lemma 4.3. *There exists a unique solution $\theta_\alpha \in W^{1,\infty}(0,T;L^\infty(\Gamma_3))$ to problem PV_θ. Moreover, $\theta_\alpha(t) \in Z$ for all $t \in [0,T]$.*

Proof. For the simplicity we suppress the dependence of various functions on Γ_3, and note that the equalities and inequalities below are valid a.e. on Γ_3. Consider the mapping $F_\alpha : [0,T] \times L^\infty(\Gamma_3) \to L^\infty(\Gamma_3)$ defined by

$$F_\alpha(t,\theta) = -(\theta(\gamma_\nu(R_\nu(u_{\alpha\nu}(t)))^2 + \gamma_\tau \mid \mathbf{R}_\tau(\mathbf{u}_{\alpha\tau}(t)) \mid^2 \ ) - \varepsilon_a)_+, \tag{4.19}$$

for all $t \in [0,T]$ and $\theta \in L^\infty(\Gamma_3)$. It follows from the properties of the truncation operator R_ν and $\mathbf{R}_\tau$ that F_α is Lipschitz continuous with respect to the second variable, uniformly in time. Moreover, for all $\theta \in L^\infty(\Gamma_3)$, the mapping $t \to F_\alpha(t,\theta)$ belongs to $L^\infty(0,T;L^\infty(\Gamma_3))$. Thus using a version of Cauchy-Lipschitz theorem given in Theorem 2.1 we deduce that there exists a unique function $\theta_\alpha \in W^{1,\infty}(0,T;L^\infty(\Gamma_3))$ solution to the problem PV_θ. Also, the arguments used in Remark 3.1 show that $0 \le \theta_\alpha(t) \le 1$ for all $t \in [0,T]$, a.e. on Γ_3. Therefore, from the definition of the set Z, we find that $\theta_\alpha(t) \in Z$, which concludes the proof of the lemma. □

It follows from lemma 4.3 that for all $\alpha \in \mathcal{Z}$ the solution θ_α of problem PV_θ belongs to $\mathcal{Z}$. Therefore, we may consider the operator $\Lambda : \mathcal{Z} \to \mathcal{Z}$ given by

$$\Lambda\alpha = \theta_\alpha. \tag{4.20}$$

We have the following result.

Lemma 4.4. *There exists a unique element $\alpha^* \in \mathcal{Z}$ such that $\Lambda\alpha^* = \alpha^*$.*

Proof. We show that, for a positive integer m, the mapping Λ^m is a contraction on $\mathcal{Z}$. To this end, we suppose that α_1 and α_2 are two functions in $\mathcal{Z}$ and denote $\mathbf{u}_{\alpha_i} = \mathbf{u}_i$, and $\theta_{\alpha_i} = \theta_i$ the functions obtained in lemmas 4.4 and 4.5, respectively, for $\alpha = \alpha_i$, $i = 1, 2$. Let $t \in [0, T]$. We use (4.7) and (4.8) and arguments similar to those used in the proof of (4.15) to deduce that

$$| \mathbf{u}_1(t) - \mathbf{u}_2(t) |_V \leq C \mid \alpha_1(t) - \alpha_2(t) |_{L^2(\Gamma_3)}, \tag{4.21}$$

which implies

$$\int_0^t | \mathbf{u}_1(s) - \mathbf{u}_2(s) |_V \, ds \leq C \int_0^t | \alpha_1(s) - \alpha_2(s) |_{L^2(\Gamma_3)} \, ds. \tag{4.22}$$

On the other hand, from the Cauchy problem (4.17)-(4.18) we can write

$$\theta_i(t) = \alpha_0 - \int_0^t (\theta_i(s)(\gamma_\nu (R_\nu(u_{i\nu}(s)))^2 + \gamma_\tau \mid \mathbf{R}_\tau(\mathbf{u}_{i\tau}(s)) \mid^2) - \ \varepsilon_a)_+ \, ds, \tag{4.23}$$

and then

$$\begin{aligned} &| \theta_1(t) - \theta_2(t) |_{L^2(\Gamma_3)} \\ &\leq C \int_0^t | \theta_1(s)(R_\nu(u_{1\nu}(s)))^2 - \theta_2(s)(R_\nu(u_{2\nu}(s)))^2 |_{L^2(\Gamma_3)} \, ds \\ &\leq C \int_0^t | \theta_1(s) \mid \mathbf{R}_\tau(\mathbf{u}_{1\tau}(s)) \mid^2 - \theta_2(s) \mid \mathbf{R}_\tau(\mathbf{u}_{2\tau}(s)) \mid^2 |_{L^2(\Gamma_3)} \, ds. \end{aligned}$$

Using the definition of R_ν and $\mathbf{R}_\tau$ and writing $\theta_1 = \theta_1 - \theta_2 + \theta_2$, we get

$$\begin{aligned} &| \theta_1(t) - \theta_2(t) |_{L^2(\Gamma_3)} \\ &\leq C\Big(\int_0^t | \theta_1(s) - \theta_2(s) |_{L^2(\Gamma_3)} \, ds + \int_0^t | \mathbf{u}_1(s) - \mathbf{u}_2(s) |_{L^2(\Gamma_3)^d} \, ds\Big). \end{aligned} \tag{4.24}$$

Next, we apply Gronwall's inequality to deduce

$$| \theta_1(t) - \theta_2(t) |_{L^2(\Gamma_3)} \leq C \int_0^t | \mathbf{u}_1(s) - \mathbf{u}_2(s) |_{L^2(\Gamma_3)^d} \, ds. \tag{4.25}$$

The relation (4.20), the estimate (4.25) and the relation (3.14) lead to

$$| \Lambda\alpha_1(t) - \Lambda\alpha_2(t) |_{L^2(\Gamma_3)} \leq C \int_0^t | \mathbf{u}_1(s) - \mathbf{u}_2(s) |_V \, ds. \tag{4.26}$$

We now combine (4.22) and (4.26) and see that

$$| \Lambda\alpha_1(t) - \Lambda\alpha_2(t) |_{L^2(\Gamma_3)} \leq C \int_0^t | \alpha_1(s) - \alpha_2(s) |_{L^2(\Gamma_3)} \, ds, \tag{4.27}$$

and reiterating this inequality m times we obtain

$$| \Lambda^m\alpha_1 - \Lambda^m\alpha_2 |_{C(0,T;L^2(\Gamma_3))} \leq \frac{C^mT^m}{m!} | \alpha_1 - \alpha_2 |_{C(0,T;L^2(\Gamma_3))} . \tag{4.28}$$

Recall that $\mathcal{Z}$ is a nonempty closed set in the Banach space $C(0,T;L^2(\Gamma_3))$ and note that (4.28) shows that for m sufficiently large the operator $\Lambda^m : \mathcal{Z} \to \mathcal{Z}$ is a contraction. Then by the Banach fixed point theorem (see [16]) it follows that Λ has a fixed point $\alpha^* \in \mathcal{Z}$. □

Now, we have all the ingredients to prove Theorem 4.1.

Proof. Existence. Let $\alpha^* \in \mathcal{Z}$ be the fixed point of Λ and let $(\mathbf{u}^*, \varphi^*)$ be the solution of problem PV_α for $\alpha = \alpha^*$, i.e. $\mathbf{u}^* = \mathbf{u}_{\alpha^*}$, $\varphi^* = \varphi_{\alpha^*}$. Arguments similar to those used in the proof of (4.15) lead to

$$| \mathbf{u}^*(t_1) - \mathbf{u}^*(t_2) |_V \leq C(| q(t_1) - q(t_2) |_W$$

$$+ | \mathbf{f}(t_1) - \mathbf{f}(t_2) |_V + | \alpha^*(t_1) - \alpha^*(t_2) |_{L^2(\Gamma_3)}), \tag{4.29}$$

$$| \varphi^*(t_1) - \varphi^*(t_2) |_W \leq C(| \mathbf{u}^*(t_1) - \mathbf{u}^*(t_2) |_V + | q(t_1) - q(t_2) |_W), \tag{4.30}$$

for all $t_1, t_2 \in [0,T]$. Since $\alpha^* = \theta_{\alpha^*}$ it follows from lemma 4.3 that $\alpha^* \in W^{1,\infty}(0,T;L^\infty(\Gamma_3))$, the regularity of $\mathbf{f}$ and q given by (3.29) and the estimate (4.29) imply that $\mathbf{u}^* \in W^{1,\infty}(0,T;V)$ and (4.30) implies that $\varphi^* \in W^{1,\infty}(0,T;W)$. We conclude by (4.7), (4.8), (4.17) and (4.18) that $(\mathbf{u}^*, \varphi^*, \alpha^*)$ is a solution of problem PV and it satisfies (4.1)-(4.3).

Uniqueness. The uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operator Λ defined by (4.20).

Let $(\mathbf{u}, \varphi, \alpha)$ be a solution of problem PV which satisfies (4.1)-(4.3). Using arguments in remark 3.1 we deduce that $\alpha \in \mathcal{Z}$, moreover, it follows from (3.30)-(3.31) that $(\mathbf{u}, \varphi)$ is a solution to problem PV_α and since by lemma 4.2 this problem has a unique solution denoted $(\mathbf{u}_\alpha, \varphi_\alpha)$, we obtain

$$\mathbf{u} = \mathbf{u}_\alpha \text{ and } \varphi = \varphi_\alpha. \tag{4.31}$$

We replace $\mathbf{u}$ by $\mathbf{u}_\alpha$ in (3.32) and use the initial condition (3.33) to see that α is a solution to problem PV_θ. Since by Lemma 4.3 this last problem has a unique solution denoted θ_α, we find

$$\alpha = \theta_\alpha. \tag{4.32}$$

We use now (4.20) and (4.32) to see that $\Lambda\alpha = \alpha$, i.e. α is a fixed point of the operator Λ. It follows now from lemma 4.4 that

$$\alpha = \alpha^*. \tag{4.33}$$

The uniqueness part of the theorem is now a consequence of equalities (4.31), (4.32) and (4.33). □

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