STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 2, June 2009

DIFFERENTIAL SUBORDINATIONS AND SUPERORDINATIONS FOR ANALYTIC FUNCTIONS DEFINED BY CONVOLUTION STRUCTURE

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Abstract. In the present investigation, we obtain some subordination and superordination results involving Hadamard product for certain normalized analytic functions in the open unit disk. Relevant connections of the results, which are presented in this paper, with various other known results also pointed out.

1. **Introduction**

Let $\mathcal{H}$ be the class of analytic functions in $\mathcal{U} := \{z : |z| < 1\}$ and $\mathcal{H}(a, n)$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots .$$

Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = z + a_2 z^2 + \dots .$$

Let $p, h \in \mathcal{H}$ and let $\phi(r, s, t; z) : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$.

If p and $\phi(p(z), zp'(z), z^2 p''(z); z)$ are univalent and if p satisfies the second order superordination

$$h(z) \prec \phi(p(z), zp'(z), z^2 p''(z); z), \tag{1.1}$$

Received by the editors: 15.01.2009.

2000 *Mathematics Subject Classification.* Primary 30C45, Secondary 30C80.

Key words and phrases. Univalent functions, starlike functions, convex functions, differential subordination, differential superordination, Hadamard product (convolution), Dziok-Srivastava linear operator.

then p is a solution of the differential superordination (1.1). (If f is subordinate to F, then F is superordinate to f.) An analytic function q is called a *subordinant* if $q \prec p$ for all p satisfying (1.1). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (1.1) is said to be the best subordinant. Recently Miller and Mocanu [12] obtained conditions on h, q and ϕ for which the following implication holds:

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z).$$

For two functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g * f)(z).$$

For $\alpha_j \in \mathbb{C}$ $(j = 1, 2, \ldots, l)$ and $\beta_j \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ $(j = 1, 2, \ldots m)$, the *generalized hypergeometric function* ${}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$ is defined by the infinite series

$${}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n} \frac{z^n}{n!}$$

$$(l \leq m + 1; l, m \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}),$$

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & (n = 0); \\ a(a+1)(a+2)\ldots(a+n-1), & (n \in \mathbb{N} := \{1, 2, 3\ldots\}). \end{cases}$$

Corresponding to the function

$$h(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := z\ {}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z),$$

the Dziok-Srivastava operator [6] (see also [7, 22]) $H_m^l(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)$ is defined by the Hadamard product

$$\begin{aligned} H_m^l(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)f(z) &:= h(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1} \ldots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \ldots (\beta_m)_{n-1}} \frac{a_n z^n}{(n-1)!}. \end{aligned} \tag{1.2}$$

For brevity, we write

$$H_m^l[\alpha_1]f(z) := H_m^l(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m)f(z).$$

It is easy to verify from (1.2) that

$$z(H_m^l[\alpha_1]f(z))' = \alpha_1 H_m^l[\alpha_1+1]f(z) - (\alpha_1 - 1)H_m^l[\alpha_1]f(z). \tag{1.3}$$

Special cases of the Dziok-Srivastava linear operator includes the Hohlov linear operator [8], the Carlson-Shaffer linear operator $L(a,c)$ [5], the Ruscheweyh derivative operator D^n [17], the generalized Bernardi-Libera-Livingston linear integral operator (*cf.* [2], [9], [10]) and the Srivastava-Owa fractional derivative operators (*cf.* [15], [16]).

Using the results of Miller and Mocanu [12], Bulboacă [4] considered certain classes of first order differential superordinations as well as superordination-preserving integral operators (see [3]). Further, using the results of Mocanu [12] and Bulboacă [4] many researchers [1, 18, 19, 20, 21] have obtained sufficient conditions on normalized analytic functions f by means of differential subordinations and superordinations.

Recently, Murugusundaramoorthy and Magesh [13, 14] obtained sufficient conditions for a normalized analytic functions f to satisfy

$$q_1(z) \prec \left(\frac{H_m^l[\alpha_1]f(z)}{(z)}\right)^{\delta} \prec q_2(z), \quad q_1(z) \prec \frac{(f*\Phi)(z)}{f*\Psi)(z)} \prec q_2(z)$$

and

$$q_1(z) \prec \frac{H_m^l[\alpha_1+1](f*\Phi)(z)}{H_m^l[\alpha_1](f*\Psi)(z)} \prec q_2(z)$$

where q_1, q_2 are given univalent functions in $\mathcal{U}$ with $q_1(0) = 1$ and $q_2(0) = 1$.

The main object of the present paper is to find sufficient condition for certain normalized analytic functions $f(z)$ in $\mathcal{U}$ such that $(f*\Psi)(z) \neq 0$ and f to satisfy

$$q_1(z) \prec \frac{H_m^l[\alpha_1](f*\Phi)(z)}{H_m^l[\alpha_1+1](f*\Psi)(z)} \prec q_2(z),$$

where q_1, q_2 are given univalent functions in $\mathcal{U}$ and $\Phi(z) = z + \sum\limits_{n=2}^{\infty} \lambda_n z^n$, $\Psi(z) = z + \sum\limits_{n=2}^{\infty} \mu_n z^n$ are analytic functions in $\mathcal{U}$ with $\lambda_n \geq 0$, $\mu_n \geq 0$ and $\lambda_n \geq \mu_n$. Also, we obtain the number of known results as their special cases.

2. **Subordination and Superordination results**

For our present investigation, we shall need the following:

Definition 2.1. [12] Denote by Q, the set of all functions f that are analytic and injective on $\overline{\mathcal{U}} - E(f)$, where

$$E(f) = \{\zeta \in \partial\mathcal{U} : \lim_{z\to\zeta} f(z) = \infty\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathcal{U} - E(f)$.

Lemma 2.2. [11] *Let q be univalent in the unit disk $\mathcal{U}$ and θ and ϕ be analytic in a domain D containing $q(\mathcal{U})$ with $\phi(w) \neq 0$ when $w \in q(\mathcal{U})$. Set*

$$\psi(z) := zq'(z)\phi(q(z)) \quad \text{and} \quad h(z) := \theta(q(z)) + \psi(z).$$

Suppose that

1. *$\psi(z)$ is starlike univalent in $\mathcal{U}$ and*
2. *$Re\left\{\frac{zh'(z)}{\psi(z)}\right\} > 0$ for $z \in \mathcal{U}$.*

If p is analytic with $p(0) = q(0)$, $p(\mathcal{U}) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{2.1}$$

then

$$p(z) \prec q(z)$$

and q is the best dominant.

Lemma 2.3. [4] *Let q be convex univalent in the unit disk $\mathcal{U}$ and ϑ and φ be analytic in a domain D containing $q(\mathcal{U})$. Suppose that*

1. *$Re\{\vartheta'(q(z))/\varphi(q(z))\} > 0$ for $z \in \mathcal{U}$ and*
2. *$\psi(z) = zq'(z)\varphi(q(z))$ is starlike univalent in $\mathcal{U}$.*

If $p(z) \in \mathcal{H}[q(0), 1] \cap Q$, with $p(\mathcal{U}) \subseteq D$, and $\vartheta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in $\mathcal{U}$ and

$$\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z)), \tag{2.2}$$

then $q(z) \prec p(z)$ and q is the best subordinant.

Using Lemma 2.2, we first prove the following theorem.

Theorem 2.4. *Let $\Phi, \Psi \in \mathcal{A}$, $\gamma_4 \neq 0$, γ_1, γ_2, γ_3 be the complex numbers and q be convex univalent in $\mathcal{U}$ with $q(0) = 1$. Further assume that*

$$Re\left\{\frac{\gamma_3}{\gamma_4} + \frac{2\gamma_2}{\gamma_4}q(z) + \left(1 + \frac{zq''(z)}{q'(z)}\right)\right\} > 0 \quad (z \in \mathcal{U}). \tag{2.3}$$

If $f \in \mathcal{A}$ satisfies

$$\Upsilon(f,\, \Phi, \Psi,\, \gamma_1, \gamma_2, \gamma_3, \gamma_4) \prec \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 z q'(z), \tag{2.4}$$

where

$$\Upsilon(f,\, \Phi, \Psi,\, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$$
$$:= \begin{cases} \gamma_1 + \gamma_2 \left(\frac{H_m^l[\alpha_1](f*\Phi)(z)}{H_m^l[\alpha_1+1](f*\Psi)(z)}\right)^2 + \gamma_3 \frac{H_m^l[\alpha_1](f*\Phi)(z)}{H_m^l[\alpha_1+1](f*\Psi)(z)} \\ +\gamma_4 \left(\alpha_1 \frac{H_m^l[\alpha_1+1](f*\Phi)(z)}{H_m^l[\alpha_1](f*\Phi)(z)} - (\alpha_1+1) \frac{H_m^l[\alpha_1+2](f*\Psi)(z)}{H_m^l[\alpha_1+1](f*\Psi)(z)} + 1\right) \\ \times \left(\frac{H_m^l[\alpha_1](f*\Phi)(z)}{H_m^l[\alpha_1+1](f*\Psi)(z)}\right), \end{cases} \tag{2.5}$$

then

$$\frac{H_m^l[\alpha_1](f*\Phi)(z)}{H_m^l[\alpha_1+1](f*\Psi)(z)} \prec q(z)$$

and q is the best dominant.

Proof. Define the function p by

$$p(z) := \frac{H_m^l[\alpha_1](f*\Phi)(z)}{H_m^l[\alpha_1+1](f*\Psi)(z)} \quad (z \in \mathcal{U}). \tag{2.6}$$

Then the function p is analytic in $\mathcal{U}$ and $p(0) = 1$. Therefore, by making use of (2.6), we obtain

$$\begin{aligned} &\gamma_1 + \gamma_2 \left(\frac{H_m^l[\alpha_1](f*\Phi)(z)}{H_m^l[\alpha_1+1](f*\Psi)(z)}\right)^2 + \gamma_3 \frac{H_m^l[\alpha_1](f*\Phi)(z)}{H_m^l[\alpha_1+1](f*\Psi)(z)} \\ &+\gamma_4 \left(\alpha_1 \frac{H_m^l[\alpha_1+1](f*\Phi)(z)}{H_m^l[\alpha_1](f*\Phi)(z)} - (\alpha_1+1) \frac{H_m^l[\alpha_1+2](f*\Psi)(z)}{H_m^l[\alpha_1+1](f*\Psi)(z)} + 1\right) \\ &\times \left(\frac{H_m^l[\alpha_1](f*\Phi)(z)}{H_m^l[\alpha_1+1](f*\Psi)(z)}\right) \\ &= \gamma_1 + \gamma_2 p^2(z) + \gamma_3 p(z) + \gamma_4 z p'(z). \end{aligned} \tag{2.7}$$

By using (2.7) in (2.4), we have

$$\gamma_1 + \gamma_2 p^2(z) + \gamma_3 p(z) + \gamma_4 z p'(z) \prec \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 z q'(z). \tag{2.8}$$

By setting

$$\theta(w) := \gamma_1 + \gamma_2\omega^2(z) + \gamma_3\omega \quad \text{and} \quad \phi(\omega) := \gamma_4,$$

it can be easily observed that $\theta(w)$ and $\phi(w)$ are analytic in $\mathbb{C}-\{0\}$ and that $\phi(w) \neq 0$. Hence the result now follows by an application of Lemma 2.2. □

When $l = 2$, $m = 1$, $\alpha_1 = a$, $\alpha_2 = 1$ and $\beta_1 = c$ in Theorem 2.4, we state the following corollary.

Corollary 2.5. *Let $\Phi, \Psi \in \mathcal{A}$. Let $\gamma_4 \neq 0$, γ_1, γ_2, γ_3 be the complex numbers and q be convex univalent in $\mathcal{U}$ with $q(0) = 1$ and (2.3) holds true. If $f \in \mathcal{A}$ satisfies*

$$\Upsilon_1(f,\, \Phi, \Psi,\, \gamma_1, \gamma_2, \gamma_3, \gamma_4) \prec \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 z q'(z)$$

where

$$\Upsilon_1(f,\, \Phi, \Psi,\, \gamma_1, \gamma_2, \gamma_3, \gamma_4) := \begin{cases} \gamma_1 + \gamma_2\left(\frac{L(a,c)(f*\Phi)(z)}{L(a+1,c)(f*\Psi)(z)}\right)^2 + \gamma_3\frac{L(a,c)(f*\Phi)(z)}{L(a+1,c)(f*\Psi)(z)} \\ +\gamma_4\left(a\frac{L(a+1,c)(f*\Phi)(z)}{L(a,c)(f*\Phi)(z)} - (a+1)\frac{L(a+2,c)(f*\Psi)(z)}{L(a+1,c)(f*\Psi)(z)} + 1\right) \\ \times\left(\frac{L(a,c)(f*\Phi)(z)}{L(a+1,c)(f*\Psi)(z)}\right), \end{cases} \tag{2.9}$$

then

$$\frac{L(a,c)(f*\Phi)(z)}{L(a+1,c)(f*\Psi)(z)} \prec q(z)$$

and q is the best dominant.

By fixing $\Phi(z) = \frac{z}{1-z}$ and $\Psi(z) = \frac{z}{1-z}$ in Theorem 2.4, we obtain the following corollary.

Corollary 2.6. *Let $\gamma_4 \neq 0$, γ_1, γ_2, γ_3 be the complex numbers and q be convex univalent in $\mathcal{U}$ with $q(0) = 1$ and (2.3) holds true. If $f \in \mathcal{A}$ satisfies*

$$\begin{aligned} &\gamma_1 + \gamma_2\left(\frac{H_m^l[\alpha_1]f(z)}{H_m^l[\alpha_1+1]f(z)}\right)^2 + \gamma_3\frac{H_m^l[\alpha_1]f(z)}{H_m^l[\alpha_1+1]f(z)} \\ &+\gamma_4\left(\alpha_1\frac{H_m^l[\alpha_1+1]f(z)}{H_m^l[\alpha_1]f(z)} - (\alpha_1+1)\frac{H_m^l[\alpha_1+2]f(z)}{H_m^l[\alpha_1+1]f(z)} + 1\right)\left(\frac{H_m^l[\alpha_1]f(z)}{H_m^l[\alpha_1+1]f(z)}\right) \\ &\quad \prec \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 z q'(z), \end{aligned}$$

then

$$\frac{H_m^l[\alpha_1]f(z)}{H_m^l[\alpha_1+1]f(z)} \prec q(z)$$

and q is the best dominant.

By taking $l = 2$, $m = 1$, $\alpha_1 = 1$, $\alpha_2 = 1$ and $\beta_1 = 1$ in Theorem 2.4, we state the following corollary.

Corollary 2.7. *Let $\Phi, \Psi \in \mathcal{A}$. Let $\gamma_4 \neq 0$, γ_1, γ_2, γ_3 be the complex numbers and q be convex univalent in $\mathcal{U}$ with $q(0) = 1$ and (2.3) holds true. If $f \in \mathcal{A}$ satisfies*

$$\gamma_1+\gamma_2\left(\frac{(f*\Phi)(z)}{z(f*\Psi)'(z)}\right)^2+\frac{(f*\Phi)(z)}{z(f*\Psi)'(z)}\left[(\gamma_3-\gamma_4)+\gamma_4\frac{z(f*\Phi)'(z)}{(f*\Phi)(z)}-\gamma_4\frac{z(f*\Psi)''(z)}{(f*\Psi)'(z)}\right]$$
$$\prec \gamma_1+\gamma_2 q^2(z)+\gamma_3 q(z)+\gamma_4 zq'(z),$$

then

$$\frac{(f*\Phi)(z)}{z(f*\Psi)'(z)} \prec q(z)$$

and q is the best dominant.

By fixing $\Phi(z) = \Psi(z)$ in Corollary 2.7, we obtain the following corollary.

Corollary 2.8. *Let $\Phi \in \mathcal{A}$. Let $\gamma_4 \neq 0$, γ_1, γ_2, γ_3 be the complex numbers and q be convex univalent in $\mathcal{U}$ with $q(0) = 1$ and (2.3) holds true. If $f \in \mathcal{A}$ satisfies*

$$\gamma_1+\gamma_2\left(\frac{(f*\Phi)(z)}{z(f*\Phi)'(z)}\right)^2+\frac{(f*\Phi)(z)}{z(f*\Phi)'(z)}\left[(\gamma_3-\gamma_4)+\gamma_4\frac{z(f*\Phi)'(z)}{(f*\Phi)(z)}-\gamma_4\frac{z(f*\Phi)''(z)}{(f*\Phi)'(z)}\right]$$
$$\prec \gamma_1+\gamma_2 q^2(z)+\gamma_3 q(z)+\gamma_4 zq'(z),$$

then

$$\frac{(f*\Phi)(z)}{z(f*\Phi)'(z)} \prec q(z)$$

and q is the best dominant.

By fixing $\Phi(z) = \frac{z}{1-z}$ in Corollary 2.8, we obtain the following corollary.

Corollary 2.9. *Let $\gamma_4 \neq 0$, γ_1, γ_2, γ_3 be the complex numbers and q be convex univalent in $\mathcal{U}$ with $q(0) = 1$ and (2.3) holds true. If $f \in \mathcal{A}$ satisfies*

$$\gamma_1+\gamma_2\left(\frac{f(z)}{zf'(z)}\right)^2+\frac{f(z)}{zf'(z)}\left[(\gamma_3-\gamma_4)+\gamma_4\frac{zf'(z)}{f(z)}-\gamma_4\frac{zf''(z)}{f'(z)}\right]$$
$$\prec \gamma_1+\gamma_2 q^2(z)+\gamma_3 q(z)+\gamma_4 zq'(z),$$

then

$$\frac{f(z)}{zf'(z)} \prec q(z)$$

and q is the best dominant.

Remark 2.10. For the choices of $\gamma_1 = \gamma_2 = 0$ and $\gamma_3 = 1$ in Corollary 2.9, we get the result obtained by Shanmugam et.al [19].

By taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ in Theorem 2.4, we have the following corollary.

Corollary 2.11. *Assume that (2.3) holds. If $f \in \mathcal{A}$ and*

$$\Upsilon(f,\, \Phi, \Psi,\, \gamma_1, \gamma_2, \gamma_3, \gamma_4) \prec \gamma_1 + \gamma_2 (\frac{1+Az}{1+Bz})^2 + \gamma_3 \frac{1+Az}{1+Bz} + \gamma_4 \frac{(A-B)z}{(1+Bz)^2},$$

then

$$\frac{H_m^l[\alpha_1](f*\Phi)(z)}{H_m^l[\alpha_1+1](f*\Psi)(z)} \prec \frac{1+Az}{1+Bz}$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Now, by applying Lemma 2.3, we prove the following theorem.

Theorem 2.12. *Let $\Phi, \Psi \in \mathcal{A}$. Let γ_1, γ_2, γ_3 and $\gamma_4 \neq 0$ be the complex numbers. Let q be convex univalent in $\mathcal{U}$ with $q(0) = 1$. Assume that*

$$Re \left\{ \frac{\gamma_3}{\gamma_4} + \frac{2\gamma_2}{\gamma_4} q(z) \right\} \geq 0. \tag{2.10}$$

Let $f \in \mathcal{A}$, $\frac{H_m^l[\alpha_1](f*\Phi)(z)}{H_m^l[\alpha_1+1](f*\Psi)(z)} \in H[q(0),1] \cap Q$. Let $\Upsilon(f,\, \Phi, \Psi,\, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ be univalent in $\mathcal{U}$ and

$$\gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 z q'(z) \prec \Upsilon(f,\, \Phi, \Psi,\, \gamma_1, \gamma_2, \gamma_3, \gamma_4), \tag{2.11}$$

where $\Upsilon(f,\, \Phi, \Psi,\, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ is given by (2.5), then

$$q(z) \prec \frac{H_m^l[\alpha_1](f*\Phi)(z)}{H_m^l[\alpha_1+1](f*\Psi)(z)}$$

and q is the best subordinant.

Proof. Define the function p by

$$p(z) := \frac{H_m^l[\alpha_1](f*\Phi)(z)}{H_m^l[\alpha_1+1](f*\Psi)(z)}. \tag{2.12}$$

Simple computation from (2.12), we get,

$$\Upsilon(f,\ \Phi, \Psi,\ \gamma_1, \gamma_2, \gamma_3, \gamma_4) = \gamma_1 + \gamma_2 p^2(z) + \gamma_3 p(z) + \gamma_4 z p'(z),$$

then

$$\gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 z q'(z) \prec \gamma_1 + \gamma_2 p^2(z) + \gamma_3 p(z) + \gamma_4 z p'(z).$$

By setting $\vartheta(w) = \gamma_1 + \gamma_2 w^2 + \gamma_3 w$ and $\phi(w) = \gamma_4$, it is easily observed that $\vartheta(w)$ is analytic in $\mathbb{C}$. Also, $\phi(w)$ is analytic in $\mathbb{C} - \{0\}$ and that $\phi(w) \neq 0$.

Since $q(z)$ is convex univalent function, it follows that

$$\text{Re}\left\{\frac{\vartheta'(q(z))}{\phi(q(z))}\right\} = \Re\left\{\frac{\gamma_3}{\gamma_4} + \frac{2\gamma_2}{\gamma_4} q(z)\right\} > 0, \quad z \in \mathcal{U}.$$

Now Theorem 2.12 follows by applying Lemma 2.3. □

When $l = 2$, $m = 1$, $\alpha_1 = a$, $\alpha_2 = 1$ and $\beta_1 = c$ in Theorem 2.12, we state the following corollary.

Corollary 2.13. *Let $\Phi, \Psi \in \mathcal{A}$. Let γ_1, γ_2, γ_3 and $\gamma_4 \neq 0$ be the complex numbers. Let q be convex univalent in $\mathcal{U}$ with $q(0) = 1$ and (2.10) holds true. If $f \in \mathcal{A}$ $\frac{L(a,c)(f*\Phi)(z)}{L(a+1,c)(f*\Psi)(z)} \in H[q(0), 1] \cap Q$. Let $\Upsilon_1(f,\ \Phi, \Psi,\ \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ be univalent in $\mathcal{U}$ and*

$$\gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 z q'(z) \prec \Upsilon_1(f,\ \Phi, \Psi,\ \gamma_1, \gamma_2, \gamma_3, \gamma_4),$$

where $\Upsilon_1(f,\ \Phi, \Psi,\ \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ is given by (2.9), then

$$q(z) \prec \frac{L(a,c)(f*\Phi)(z)}{L(a+1,c)(f*\Psi)(z)}$$

and q is the best subordinant.

When $l = 2$, $m = 1$, $\alpha_1 = 1$, $\alpha_2 = 1$ and $\beta_1 = 1$ in Theorem 2.12, we derive the following corollary.

Corollary 2.14. *Let $\Phi, \Psi \in \mathcal{A}$. Let γ_1, γ_2, γ_3 and $\gamma_4 \neq 0$ be the complex numbers. Let q be convex univalent in $\mathcal{U}$ with $q(0) = 1$ and (2.10) holds true. If $f \in \mathcal{A}$, $\frac{(f*\Phi)(z)}{z(f*\Psi)'(z)} \in H[q(0), 1] \cap Q$. Let*

$$\gamma_1 + \gamma_2\left(\frac{(f*\Phi)(z)}{z(f*\Psi)'(z)}\right)^2 + \frac{(f*\Phi)(z)}{z(f*\Psi)'(z)}\left[(\gamma_3 - \gamma_4) + \gamma_4\frac{z(f*\Phi)'(z)}{(f*\Phi)(z)} - \gamma_4\frac{z(f*\Psi)''(z)}{(f*\Psi)'(z)}\right]$$

be univalent in $\mathcal{U}$ *and*

$$\gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 z q'(z)$$

$$\prec \gamma_1 + \gamma_2 \left(\frac{(f*\Phi)(z)}{z(f*\Psi)'(z)}\right)^2 + \frac{(f*\Phi)(z)}{z(f*\Psi)'(z)} \left[(\gamma_3 - \gamma_4) + \gamma_4 \frac{z(f*\Phi)'(z)}{(f*\Phi)(z)} - \gamma_4 \frac{z(f*\Psi)''(z)}{(f*\Psi)'(z)}\right],$$

then

$$q(z) \prec \frac{(f*\Phi)(z)}{z(f*\Psi)'(z)}$$

and q *is the best subordinant.*

By fixing $\Phi(z) = \Psi(z)$ in Corollary 2.14, we obtain the following corollary.

Corollary 2.15. *Let* $\Phi \in \mathcal{A}$. *Let* γ_1, γ_2, γ_3 *and* $\gamma_4 \neq 0$ *be the complex numbers. Let* q *be convex univalent in* $\mathcal{U}$ *with* $q(0) = 1$ *and (2.10) holds true. If* $f \in \mathcal{A}$, $\frac{(f*\Phi)(z)}{z(f*\Phi)'(z)} \in H[q(0), 1] \cap Q$. *Let*

$$\gamma_1 + \gamma_2 \left(\frac{(f*\Phi)(z)}{z(f*\Phi)'(z)}\right)^2 + \frac{(f*\Phi)(z)}{z(f*\Phi)'(z)} \left[(\gamma_3 - \gamma_4) + \gamma_4 \frac{z(f*\Phi)'(z)}{(f*\Phi)(z)} - \gamma_4 \frac{z(f*\Phi)''(z)}{(f*\Phi)'(z)}\right]$$

be univalent in $\mathcal{U}$ *and*

$$\gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 z q'(z)$$

$$\prec \gamma_1 + \gamma_2 \left(\frac{(f*\Phi)(z)}{z(f*\Phi)'(z)}\right)^2 + \frac{(f*\Phi)(z)}{z(f*\Phi)'(z)} \left[(\gamma_3 - \gamma_4) + \gamma_4 \frac{z(f*\Phi)'(z)}{(f*\Phi)(z)} - \gamma_4 \frac{z(f*\Phi)''(z)}{(f*\Phi)'(z)}\right],$$

then

$$q(z) \prec \frac{(f*\Phi)(z)}{z(f*\Phi)'(z)}$$

and q *is the best subordinant.*

By fixing $\Phi(z) = \frac{z}{1-z}$ in Corollary 2.15, we obtain the following corollary.

Corollary 2.16. *Let* γ_1, γ_2, γ_3 *and* $\gamma_4 \neq 0$ *be the complex numbers. Let* q *be convex univalent in* $\mathcal{U}$ *with* $q(0) = 1$ *and (2.10) holds true. If* $f \in \mathcal{A}$, $\frac{f(z)}{zf'(z)} \in H[q(0), 1] \cap Q$. *Let* $\gamma_1 + \gamma_2 \left(\frac{f(z)}{zf'(z)}\right)^2 + \frac{f(z)}{zf'(z)} \left[(\gamma_3 - \gamma_4) + \gamma_4 \frac{zf'(z)}{f(z)} - \gamma_4 \frac{zf''(z)}{f'(z)}\right]$ *be univalent in* $\mathcal{U}$ *and*

$$\gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 z q'(z) \prec$$
$$\gamma_1 + \gamma_2 \left(\frac{f(z)}{zf'(z)}\right)^2 + \frac{f(z)}{zf'(z)} \left[(\gamma_3 - \gamma_4) + \gamma_4 \frac{zf'(z)}{f(z)} - \gamma_4 \frac{zf''(z)}{f'(z)}\right],$$

then

$$q(z) \prec \frac{f(z)}{zf'(z)}$$

and q is the best subordinant.

By taking $q(z) = (1 + Az)/(1 + Bz)$ $(-1 \leq B < A \leq 1)$ in Theorem 2.12, we obtain the following corollary.

Corollary 2.17. *Assume that (2.10) holds true. If $f \in \mathcal{A}$, $\frac{H_m^l[\alpha_1](f*\Phi)(z)}{H_m^l[\alpha_1+1](f*\Psi)(z)} \in H[q(0), 1] \cap Q$. Let $\Upsilon(f, \Phi, \Psi, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ be univalent in $\mathcal{U}$ and*

$$\gamma_1 + \gamma_2(\frac{1+Az}{1+Bz})^2 + \gamma_3\frac{1+Az}{1+Bz} + \gamma_4\frac{(A-B)z}{(1+Bz)^2} \prec \Upsilon(f, \Phi, \Psi, \gamma_1, \gamma_2, \gamma_3, \gamma_4),$$

then

$$\frac{1+Az}{1+Bz} \prec \frac{H_m^l[\alpha_1](f*\Phi)(z)}{H_m^l[\alpha_1+1](f*\Psi)(z)}$$

and $\frac{1+Az}{1+Bz}$ is the best subordinant.

3. **Sandwich results**

We conclude this paper by stating the following sandwich results.

Theorem 3.1. *Let q_1 and q_2 be convex univalent in $\mathcal{U}$, γ_1, γ_2, γ_3 and $\gamma_4 \neq 0$ be the complex numbers. Suppose q_2 satisfies (2.3) and q_1 satisfies (2.10). Let $\Phi, \Psi \in \mathcal{A}$. Moreover suppose $\frac{H_m^l[\alpha_1](f*\Phi)(z)}{H_m^l[\alpha_1+1](f*\Psi)(z)} \in \mathcal{H}[1, 1] \cap Q$ and $\Upsilon(f, \Phi, \Psi, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ is univalent in $\mathcal{U}$. If $f \in \mathcal{A}$ satisfies*

$$\gamma_1 + \gamma_2 q_1^2(z) + \gamma_3 q_1(z) + \gamma_4 z q_1'(z) \prec \Upsilon(f, \Phi, \Psi, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$$

$$\prec \gamma_1 + \gamma_2 q_2^2(z) + \gamma_3 q_2(z) + \gamma_4 z q_2'(z),$$

where $\Upsilon(f, \Phi, \Psi, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ is given by (2.5), then

$$q_1(z) \prec \frac{H_m^l[\alpha_1](f*\Phi)(z)}{H_m^l[\alpha_1+1](f*\Psi)(z)} \prec q_2(z)$$

and q_1, q_2 are respectively the best subordinant and best dominant.

By taking

$$q_1(z) = \frac{1 + A_1 z}{1 + B_1 z} \ (-1 \leq B_1 < A_1 \leq 1)$$

and

$$q_2(z) = \frac{1 + A_2 z}{1 + B_2 z} \ (-1 \leq B_2 < A_2 \leq 1)$$

in Theorem 3.1 we obtain the following result.

Corollary 3.2. *Let* $\Phi, \Psi \in \mathcal{A}$. *If* $f \in \mathcal{A}$,

$$\frac{(f * \Phi)(z)}{z(f * \Psi)'(z)} \in \mathcal{H}[1, 1] \cap Q$$

and $\Upsilon(f,\, \Phi, \Psi,\, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ *is univalent in U. Further*

$$\gamma_1 + \gamma_2 (\frac{1 + A_1 z}{1 + B_1 z})^2 + \gamma_3 \frac{1 + A_1 z}{1 + B_1 z} + \gamma_4 \frac{(A_1 - B_1)z}{(1 + B_1 z)^2}$$

$$\prec \Upsilon(f,\, \Phi, \Psi,\, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$$

$$\prec \gamma_1 + \gamma_2 (\frac{1 + A_2 z}{1 + B_2 z})^2 + \gamma_3 \frac{1 + A_2 z}{1 + B_2 z} + \gamma_4 \frac{(A_2 - B_2)z}{(1 + B_2 z)^2}$$

where $\Upsilon(f,\, \Phi, \Psi,\, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ *is given by (2.5), then*

$$\frac{1 + A_1 z}{1 + B_1 z} \prec \frac{(f * \Phi)(z)}{z(f * \Psi)'(z)} \prec \frac{1 + A_2 z}{1 + B_2 z}$$

and $\frac{1+A_1 z}{1+B_1 z}$, $\frac{1+A_2 z}{1+B_2 z}$ *are respectively the best subordinant and best dominant.*

We remark that Theorem 3.1 can easily restated, for the different choices of $\Phi(z)$, $\Psi(z)$, l, m, $\alpha_1, \alpha_2, \ldots \alpha_l$, $\beta_1, \beta_2, \ldots \beta_m$ and for $\gamma_1, \gamma_2, \gamma_3, \gamma_4$.

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