STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 2, June 2009

FUGLEDE-PUTNAM THEOREM FOR log-HYPONORMAL OR CLASS $\mathcal{Y}$ OPERATORS

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Abstract. The equation $AX = XB$ implies $A^*X = XB^*$ when A and B are normal is known as the familiar Fuglede-Putnam's theorem. In this paper we will extend Fuglede-Putnam's theorem to a more general class of operators. We show that if A is log-hyponormal and B^* is a class $\mathcal{Y}$ operator, then A, B satisfy Fuglede-Putnam's theorem. Other related results are also given.

1. **Introduction**

Let $\mathcal{H}, \mathcal{K}$ be complex Hilbert spaces and $B(\mathcal{H}), B(\mathcal{K})$ the algebras of all bounded linear operators on $\mathcal{H}, \mathcal{K}$. The familiar Fuglede-Putnam's theorem is as follows:

Theorem 1.1. *(Fuglede-Putnam) Let $A \in B(\mathcal{H}), B \in B(\mathcal{K})$ be normal operators. If $AX = XB$ for some $X \in B(\mathcal{K}, \mathcal{H})$, then $A^*X = XB^*$.*

Many authors have extented this theorem for several classes of operators, for example (see [7, 10, 11, 22, 24]). We say that A, B satisfy Fuglede-Putnam's theorem if $AX = XB$ implies $A^*X = XB^*$. In [22] A. Uchiyama proved that if A, B^* are class $\mathcal{Y}$ operators, then A, B satisfy Fuglede-Putnam's theorem. In [10] the authors showed that Fuglede-Putnam's theorem holds when A is p-hyponormal and B^* is a class A operator. The aim of this paper is to show that if A is log-hyponormal and B^* is a class $\mathcal{Y}$ operator, then A, B satisfy Fuglede-Putnam's theorem.

Received by the editors: 28.03.2008.

2000 *Mathematics Subject Classification.* Primary 47B47, 47A30, 47B20, Secondary 47B10.

Key words and phrases. Fuglede-Putnam's theorem, log-hyponormal operator, class $\mathcal{Y}$ operator.

For any operator $A \in B(\mathcal{H})$ set, as usual, $|A| = (A^*A)$ and $[A^*, A] = A^*A - AA^* = |A|^2 - |A^*|^2$ (the self commutator of A), and consider the following standard definitions: A is normal if $A^*A = AA^*$, hyponormal if $A^*A - AA^* \geq 0$. A is said to be a class $\mathcal{Y}_\alpha$ operator for $\alpha \geq 1$ (or $A \in \mathcal{Y}_\alpha$) if there exists a positive number k_α such that

$$|AA^* - A^*A|^\alpha \leq k_\alpha^2 (A-\lambda)^*(A-\lambda) \,\text{for all}\, \lambda \in \mathbb{C}.$$

It is known that $\mathcal{Y}_\alpha \subset \mathcal{Y}_\beta$ if $1 \leq \alpha \leq \beta$. Let $\mathcal{Y} = \bigsqcup_{1\leq\alpha} \mathcal{Y}_\alpha$. We remark that a class $\mathcal{Y}_1$ operator A is M-hyponormal, i.e., there exists a positive number M such that

$$(A-\lambda I)(A-\lambda I)^* \leq M^2 (A-\lambda I)^*(A-\lambda I) \,\text{for all}\, \lambda \in \mathbb{C},$$

and M-hyponormal operators are class $\mathcal{Y}_2$ operators (see [22]). A is said to be dominant if for any $\lambda \in \mathbb{C}$ there exists a positive number M_λ such that

$$(A-\lambda I)(A-\lambda I)^* \leq M_\lambda^2 (A-\lambda I)^*(A-\lambda I).$$

It is obvious that dominant operators are M-hyponormal. But it is known that there exists a dominant operator which is not a class $\mathcal{Y}$ operator, and also there exists a class $\mathcal{Y}$ operator which is not dominant. In this paper we will extend Fuglede-Putnam's theorem for log-hyponormal operators and class $\mathcal{Y}$ operators.

A is said to be log-hyponormal if A is invertible and satisfies the following equality

$$\log(A^*A) \geq \log(AA^*).$$

It is known that invertible p-hyponormal operators are log-hyponormal operators but the converse is not true [18]. However it is very interesting that we may regard log-hyponormal operators are 0-hyponormal operators [18, 19]. The idea of log-hyponormal operator is due to Ando [3] and the first paper in which log-hyponormality appeared is [6].

2. **Results**

We will recall some known results which will be used in the sequel.

Lemma 2.1. [16] *Let $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$. Then the following assertions are equivalent*

(i) The pair (A, B) satisfies Fuglede-Putnam's theorem;

(ii) if $AC = CB$ for some $C \in B(\mathcal{K}, \mathcal{H})$, then $\overline{\mathrm{ran}(C)}$ reduces A, $(\ker C)^{\perp}$ reduces B and $A|_{\overline{ran(C)}}$ and $B|_{(\ker C)^{\perp}}$ are normal operators.

Lemma 2.2. *(Stampfli and Wadhwa[15]) Let $A \in B(\mathcal{H})$ be a dominant operator and $\mathcal{M} \subset \mathcal{H}$ invariant under A. If $A|_M$ is normal, then $\mathcal{M}$ reduces A.*

Lemma 2.3. [22] *Let $A \in B(\mathcal{H})$ be a class $\mathcal{Y}$ operator and $\mathcal{M} \subset \mathcal{H}$ invariant under A. If $A|_M$ is normal, then $\mathcal{M}$ reduces A.*

Lemma 2.4. *(Stampfli and Wadhwa[15]) Let $A \in B(\mathcal{H})$ be dominant. Let $\delta \subset \mathbb{C}$ be closed. If there exists a bounded function $f : \mathbb{C} \setminus \delta \mapsto \mathcal{H}$ such that $(A - \lambda) f(\lambda) = x \neq 0$ for some $x \in \mathcal{H}$, then there exists an analytic function $g : \mathbb{C} \setminus \delta \mapsto \mathcal{H}$ such that $(A - \lambda) g(\lambda) = x$.*

Lemma 2.5. [22] *Let $A \in B(\mathcal{H})$ be a class $\mathcal{Y}$ operator and $\mathcal{M} \subset \mathcal{H}$ invariant under A. Then $A|_M$ is a class $\mathcal{Y}$ operator.*

Lemma 2.6. [20] *Let $A \in B(\mathcal{H})$ be* log-*hyponormal and $\mathcal{M} \subset \mathcal{H}$ invariant under A. Then $A|_M$ is* log-*hyponormal.*

Theorem 2.1. *Let $A \in B(\mathcal{H})$ be* log-*hyponormal and $B^* \in B(\mathcal{K})$ be class $\mathcal{Y}$. If $AC = CB$ for some operator $C \in B(\mathcal{K}, \mathcal{H})$, then $A^*C = CB^*$. Moreover the closure $\overline{\mathrm{ran} C}$ of the range of C reduces A, $(\ker C)^{\perp}$ reduces B and $A|_{\overline{\mathrm{ran} C}}, B|_{(\ker C)^{\perp}}$ are unitary equivalent normal operators.*

Proof. Since B^* is class $\mathcal{Y}$, there exist positive numbers α and k_α such that

$$|BB^* - B^*B|^{\alpha} \leq k_\alpha^2 (B - \lambda)(B - \lambda)^*, \text{ for all } \lambda \in \mathbb{C}.$$

Hence for $x \in |BB^* - B^*B|^{\frac{\alpha}{2}} \mathcal{K}$ there exists a bounded function $f : \mathbb{C} \mapsto \mathcal{K}$ such that

$$(B - \lambda) f(\lambda) = x, \text{ for all } \lambda \in \mathbb{C}$$

by [4]. Let $A = U|A|$ be the polar decomposition of A and define its Aluthge transform by $\tilde{A} = |A|^{\frac{1}{2}} U |A|^{\frac{1}{2}}$. Let $\tilde{A} = V|\tilde{A}|$, and define the second Aluthge transform of A by

$\hat{A} = |\tilde{A}|^{\frac{1}{2}} V |\tilde{A}|^{\frac{1}{2}}$. Then $\hat{A}$ is hyponormal [7]. Therefore

$$(\hat{A} - \lambda I) f(\lambda) = |\tilde{A}|^{\frac{1}{2}} (\tilde{A} - \lambda I) C f(\lambda)$$

$$= |\tilde{A}|^{\frac{1}{2}} C (B - \lambda) f(\lambda) = |\tilde{A}|^{\frac{1}{2}} C x, \text{ for all } \lambda \in \mathbb{C}.$$

We claim that $|\tilde{A}|^{\frac{1}{2}} C x = 0$. Because if $|\tilde{A}|^{\frac{1}{2}} C x \neq 0$, there exists a bounded entire analytic function $g : \mathbb{C} \mapsto \mathcal{H}$ such that $(\hat{A} - \lambda) g(\lambda) = |\tilde{A}|^{\frac{1}{2}} C x$ by Lemma 2.4. Since

$$g(\lambda) = (\hat{A} - \lambda)^{-1} |\tilde{A}|^{\frac{1}{2}} C x \to 0 \text{ as } \lambda \to \infty,$$

we have $g(\lambda) = 0$ by Liouville's theorem, and hence $|\tilde{A}|^{\frac{1}{2}} C x = 0$. This is a contradiction. Thus $|\tilde{A}|^{\frac{1}{2}} C |BB^* - B^* B|^{2n-1} \mathcal{K} = \{0\}$ and hence

$$C |BB^* - B^* B|^2 \mathcal{K} = \{0\}. \tag{1}$$

It follows from $AC = CB$ that $\overline{\operatorname{ran} C}$ and $(\ker C)^{\perp}$ are invariant subspaces of A and B^* respectively. Then A and B can be written

$$A = \begin{bmatrix} A_1 & S \\ 0 & A_2 \end{bmatrix} \text{ on } \mathcal{H} = \overline{\operatorname{ran} C} \oplus \overline{\operatorname{ran} C}^{\perp}$$

$$B = \begin{bmatrix} B_1 & 0 \\ S & B_2 \end{bmatrix} \text{ on } \mathcal{K} = (\ker C)^{\perp} \oplus \ker C$$

$$C = \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix} : (\ker C)^{\perp} \oplus \ker C \mapsto \overline{\operatorname{ran} C} \oplus \overline{\operatorname{ran} C}^{\perp}.$$

Hence C_1 is injective, with dense range and

$$A_1 C_1 = C_1 A_1. \tag{2}$$

We have

$$BB^* - B^* B = \begin{bmatrix} B_1 & 0 \\ S & B_2 \end{bmatrix} \begin{bmatrix} B_1^* & S^* \\ 0 & B_2^* \end{bmatrix} - \begin{bmatrix} B_1^* & S^* \\ 0 & B_2^* \end{bmatrix} \begin{bmatrix} B_1 & 0 \\ S & B_2 \end{bmatrix}$$

$$= \begin{bmatrix} B_1 B_1^* - B_1^* B_1 - S^* S & B_1 S^* - S^* B_2 \\ (B_1 S^* - S^* B_2)^* & SS^* + B_2 B_2^* - B_2^* B_2 \end{bmatrix}$$

$$\begin{bmatrix} B_1B_1^* - B_1^*B_1 - S^*S & E_1 \\ E_1^* & F_1 \end{bmatrix}.$$

Thus

$$|BB^* - B^*B|^2 = \begin{bmatrix} (B_1B_1^* - B_1^*B_1 - S^*S)^2 + E_1E_1^* & E_2 \\ E_2^* & F_2 \end{bmatrix}.$$

Since $C|BB^* - B^*B|^2(\ker C)^\perp = \{0\}$ by (1), we have

$$C[B_1B_1^* - B_1^*B_1 - S^*S)^2 + E_1E_1^*] = 0$$

and since C_1 is injective, $(B_1B_1^* - B_1^*B_1 - S^*S)^2 + E_1E_1^* = 0$. Hence $B_1B_1^* - B_1^*B_1 - S^*S = 0$, that is, B_1^* is hyponormal. Multiply the two members of (2) by $|\tilde{A}|^{\frac{1}{2}}$ and since the polar decomposition of $\tilde{A} = V|\tilde{A}|$, we get

$$\hat{A}_1(|\tilde{A}_1|^{\frac{1}{2}}C_1) = (|\tilde{A}_1|^{\frac{1}{2}}C_1)B_1.$$

Since the second Aluthge transform $\hat{A} = |\tilde{A}|^{\frac{1}{2}}V|\tilde{A}|^{\frac{1}{2}}$ is hyponormal and B_1^* is hyponormal, we have $\hat{A}_1, B_1$ satisfy Fuglede-Putnam's theorem. Thus

$$\hat{A_1}^*(|\tilde{A}_1|^{\frac{1}{2}}C_1) = (|\tilde{A}_1|^{\frac{1}{2}}C_1)B_1^*.$$

Hence $\hat{A}_1|_{\overline{\mathrm{ran}(|\tilde{A}|^{\frac{1}{2}})C_1}}$ and $B_1|_{\ker(|\tilde{A}|^{\frac{1}{2}}C_1)^\perp}$ are normal operators by Lemma 2.1. Since $|\tilde{A}_1|^{\frac{1}{2}}$ and C_1 are injective, $|\tilde{A}_1|^{\frac{1}{2}}C_1$ is also injective. Hence

$$[\ker(|\tilde{A}|^{\frac{1}{2}}C_1)]^\perp = 0^\perp = (\ker C_1)^\perp = (\ker C)^\perp.$$

By the same arguments as above, we have

$$\overline{\mathrm{ran}(|\tilde{A}|^{\frac{1}{2}})C_1} = C_1^* \ker(|\tilde{A}|^{\frac{1}{2}})^\perp = 0^\perp = \overline{\mathrm{ran} C_1} = \overline{\mathrm{ran} C}.$$

Hence $\hat{A}_1$ is normal. This implies that A_1 is normal by [20]. Hence $\overline{\mathrm{ran} C}$ reduces A_1 by Lemma 2.5 and $(\ker C_1)^\perp$ reduces B_1^* by [24]. Since A_1 is normal, B_1^* is hyponormal and $A_1C_1 = C_1B_1$, we obtain $A_1^*C_1 = C_1B_1^*$ by the Fuglede-Putnam's theorem, and so $A^*C = CB^*$. The rest follows from Lemma 2.1. □

Corollary 2.1. *Let $A \in B(\mathcal{H})$ be log-hyponormal and $B^* \in B(\mathcal{K})$ be class $\mathcal{Y}$. If $AC = CB$ for some operator $C \in B(\mathcal{K}, \mathcal{H})$, then $A^*C = CB^*$. Moreover the closure $\overline{\mathrm{ran} C}$ of the range of C reduces A, $(\ker C)^\perp$ reduces B and $A|_{\overline{\mathrm{ran} C}}, B|_{(\ker C)^\perp}$ are unitary equivalent normal operators.*

Proof. Since $AC = CB$, we have $B^*C^* = C^*A^*$. Hence $BC^* = B^{**}C^* = C^*A^{**} = C^*A$ by the previous theorem. Hence $A^*C = CB^*$. The rest follows from Lemma 2.1. □

Corollary 2.2. *Let $A \in B(\mathcal{H})$. Then A is normal if and only if A is* log*-hyponormal and A^* is class $\mathcal{Y}$.*

The following version of the Fuglede-Putnam's theorem for log-hyponormal operators is immediate from Theorem 2.1 and [9, Theorem 4].

Corollary 2.3. *Let $A \in B(\mathcal{H})$ be log-hyponormal and $B^* \in B(\mathcal{K})$ be class $\mathcal{Y}$. If $AX_n - X_nB \to 0$ for a bounded sequence $\{X_n\}$, $X_n : \mathcal{K} \mapsto \mathcal{H}$, then $A^*X_n - X_nB^* \to 0$.*

Corollary 2.4. *Let $A \in B(\mathcal{H})$ and $B^* \in B(\mathcal{K})$ be such that $AX = XB$. If either A is pure* log*-hyponormal and B^* is class $\mathcal{Y}$, or A is* log*-hyponormal and B^* is pure class $\mathcal{Y}$, then $X = 0$.*

Proof. The hypotheses imply that $AX = XB$ and $A^*X = XB^*$ similtaneously by Theorem 2.1. Therefore $A|_{\overline{\mathrm{ran} X}}$ and $B|_{(\ker X)^\perp}$ are unitarily equivalent normal operators, which contradicts the hypotheses that A or B^* is pure. Hence we must have $X = 0$. □

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