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ON SUBCLASSES OF PRESTARLIKE FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract. The present paper is aim at defining new subclasses of prestarlike functions with negative coefficients in unit disc U and study there basic properties such as coefficient estimates, closure properties. Further distortion theorem involving generalized fractional calculus operator for functions $f(z)$ belonging to these subclasses are also established.

1. **Introduction**

Let A denote the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the unit disc $U = \{z : |z| < 1\}$ and let S denote the subclass of A, consisting functions of the type (1.1) which are normalized and univalent in U. A function $f \in S$, is said to be starlike of order $\mu (0 \leq \mu < 1)$ in U if and only if

$$Re\left(\frac{zf'(z)}{f(z)}\right) \geq \mu. \tag{1.2}$$

We denote by $S^*(\mu)$, the class of all functions in S, which are starlike of order μ in U.

It is well-known that

$$S^*(\mu) \subseteq S^*(0) \equiv S^*.$$

The class $S^*(\mu)$ was first introduced by Robertson [7] and further it was rather extensively studied by Schild [8], MacGregor [2].

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Also

$$S_\mu(z) = \frac{z}{(1-z)^{2(1-\mu)}} \tag{1.3}$$

is the familiar extremal function for class $S^*(\mu)$. Setting

$$C(\mu,n) = \frac{\prod_{k=2}^{n}(k-2\mu)}{(n-1)!}, n \in \mathbb{N}\backslash\{1\}, \mathbb{N} = \{1,2,3,...\}. \tag{1.4}$$

The function $S_\mu(z)$ can be written in the form

$$S_\mu(z) = z + \sum_{n=2}^{\infty} C(\mu,n)\ z^n. \tag{1.5}$$

We note that $C(\mu,n)$ is decreasing function in μ and that

$$\lim_{n\to\infty} C(\mu,n) = \begin{cases} \infty, & \mu< 1/2 \\ 1, & \mu= 1 \\ 0, & \mu> 1. \end{cases} \tag{1.6}$$

We say that $f \in S$, is in the class $S^*(\alpha,\beta,\gamma)$ if and only if it satisfies the following condition

$$\left| \frac{\dfrac{zf'(z)}{f(z)} - 1}{\gamma\dfrac{zf'(z)}{f(z)} + 1 - (1+\gamma)\alpha} \right| < \beta, \tag{1.7}$$

where $0 \le \alpha < 1, 0 < \beta \le 1, 0 \le \gamma \le 1$.

Furthermore, a function f is said to be in the class $K(\alpha,\beta,\gamma)$ if and only if

$$zf'(z) \in S^*(\alpha,\beta,\gamma).$$

Let $f(z)$ be given by (1.1) and $g(z)$ be given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \tag{1.8}$$

then the Hadamard product(or convolution) of (1.1) and (1.8) is given by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{1.9}$$

Let $R_\mu(\alpha,\beta,\gamma)$ be the subclass of A consisting functions $f(z)$ such that

$$\left|\frac{\dfrac{zh'(z)}{h(z)}-1}{\gamma\dfrac{zh'(z)}{h(z)}+1-(1+\gamma)\alpha}\right|<\beta \tag{1.10}$$

where,

$$h(z)=(f*S_\mu(z)),0\leq\mu<1. \tag{1.11}$$

Also, let $C_\mu(\alpha,\beta,\gamma)$ be the subclass of A consisting functions $f(z)$, which satisfy the condition

$$zf'(z)\in R_\mu(\alpha,\beta,\gamma).$$

We note that $R_\mu(\alpha,1,1)=R_\mu(\alpha)$ is the class functions introduced by Sheil-Small *et al* [9]and such type of classes were studied by Ahuja and Silverman[1].

Finally, let T denote the subclass of S consisting of functions of the form

$$f(z)=z-\sum_{n=2}^{\infty}a_nz^n,a_n\geq 0. \tag{1.12}$$

We denote by $T^*(\alpha,\beta,\gamma)$, $C^*(\alpha,\beta,\gamma)$, $R_\mu[\alpha,\beta,\gamma]$ and $\mathrm{C}_\mu[\alpha,\beta,\gamma]$ the classes obtained by taking the intersection of the classes $S^*(\alpha,\beta,\gamma)$, $K(\alpha,\beta,\gamma)$, $R_\mu(\alpha,\beta,\gamma)$ and $\mathrm{C}_\mu(\alpha,\beta,\gamma)$ with the class T. In the present paper we aim at finding various interesting properties and characterization of aforementioned general classes $R_\mu[\alpha,\beta,\gamma]$ and $\mathrm{C}_\mu[\alpha,\beta,\gamma]$. Further we note that such classes were studied by Owa and Uralegaddi [6], Silverman and Silvia [10] and Owa and Ahuja [4].

2. **Basic Characterization**

Theorem 1. *A function $f(z)$ defined by* (1.12) *is in the class $R_\mu[\alpha,\beta,\gamma]$ if and only if*

$$\sum_{n=2}^{\infty}C(\mu\ ,n)\left\{(n-1)+\beta[\gamma n+1-(1+\gamma)\alpha]\right\}a_n\leq\beta(1+\gamma)(1-\alpha). \tag{2.1}$$

The result (2.1) *is sharp and is given by*

$$f(z) = z - \frac{\beta(1+\gamma)(1-\alpha)}{C(\mu,n)\left\{(n-1)+\beta[\gamma n+1-(1+\gamma)\alpha]\right\}} z^n, n \in \mathbb{N}\backslash\{1\}. \tag{2.2}$$

Proof. The proof of Theorem 1 is straightforward and hence details are omitted. □

Theorem 2. *Let $f(z) \in T$, then $f(z)$ is in the class $C_\mu[\alpha,\beta,\gamma]$ if and only if*

$$\sum_{n=2}^{\infty} C(\mu\ , n)n\left\{(n-1)+\beta[\gamma n+1-(1+\gamma)\alpha]\right\} a_n \leq \beta(1+\gamma)(1-\alpha). \tag{2.3}$$

The result (2.3) *is sharp for the function $f(z)$ given by*

$$f(z) = z - \frac{\beta(1+\gamma)(1-\alpha)}{C(\mu,n)n\left\{(n-1)+\beta[\gamma n+1-(1+\gamma)\alpha]\right\}} z^n, n \in \mathbb{N}\backslash\{1\}. \tag{2.4}$$

Proof. Since $f(z) \in C_\mu[\alpha,\beta,\gamma]$ if and only if $zf'(z) \in R_\mu[\alpha,\beta,\gamma]$, we have Theorem 2, by replacing a_n by na_n in Theorem 1. □

Corollary 1. *Let $f(z) \in T$, be in the class $R_\mu[\alpha,\beta,\gamma]$ then*

$$a_n \leq \frac{\beta(1+\gamma)(1-\alpha)}{C(\mu,n)\left\{(n-1)+\beta[\gamma n+1-(1+\gamma)\alpha]\right\}}, n \in \mathbb{N}\backslash\{1\}. \tag{2.5}$$

Equality holds true for the function $f(z)$ given by (2.2).

Corollary 2. *Let $f(z) \in T$, be in the class $C_\mu[\alpha,\beta,\gamma]$ then*

$$a_n \leq \frac{\beta(1+\gamma)(1-\alpha)}{C(\mu,n)n\left\{(n-1)+\beta[\gamma n+1-(1+\gamma)\alpha]\right\}}, n \in \mathbb{N}\backslash\{1\}. \tag{2.6}$$

Equality in (2.6) *holds true for the function $f(z)$ given by* (2.4).

3. **Closure Properties**

Theorem 3. *The class $R_\mu[\alpha,\beta,\gamma]$ is closed under convex linear combination.*

Proof. Let, each of the functions $f_1(z)$ and $f_2(z)$ be given by

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n, a_{n,j} \geq 0, j = 1,2 \tag{3.1}$$

be in the class $R_\mu[\alpha,\beta,\gamma]$ .It is sufficient to show that the function $h(z)$ defined by

$$h(z) = \lambda f_1(z) + (1-\lambda) f_2(z), 0 \leq \lambda \leq 1 \tag{3.2}$$

is also in the class $R_\mu[\alpha,\beta,\gamma]$. Since, for $0 \leq \lambda \leq 1$,

$$h(z) = z - \sum_{n=2}^{\infty}[\lambda a_{n,1} + (1-\lambda)a_{n,2}]z^n \tag{3.3}$$

by using Theorem 1,we have

$$\sum_{n=2}^{\infty} C(\mu\ , n)\left\{(n-1) + \beta[\gamma n + 1 - (1+\gamma)\alpha]\right\}[\lambda a_{n,1} + (1-\lambda)a_{n,2}] \leq \beta(1+\gamma)(1-\alpha) \tag{3.4}$$

which proves that $h(z) \in R_\mu[\alpha,\beta,\gamma]$.

Similarly we have □

Theorem 4. *The class $C_\mu[\alpha,\beta,\gamma]$ is closed under convex linear combination.*

Theorem 5. *Let,*

$$f_1(z) = z \tag{3.5}$$

and,

$$f_n(z) = z - \frac{\beta(1-\alpha)(1+\gamma)}{C(\mu,n)\left\{(n-1) + \beta[\gamma n + 1 - (1+\gamma)\alpha]\right\}}z^n. \tag{3.6}$$

Then $f(z)$ is in the class $R_\mu[\alpha,\beta,\gamma]$ if and only if it can be expressed as

$$f(z) = \sum_{n=1}^{\infty}\lambda_n f_n(z) \tag{3.7}$$

where, $\lambda_n \geq 0$ and $\sum_{n=1}^{\infty}\lambda_n = 1$.

Proof. Let,

$$\begin{aligned} f(z) &= \sum_{n=2}^{\infty}\lambda_n f_n(z) \\ &= z - \sum_{n=2}^{\infty}\frac{\beta(1-\alpha)(1+\gamma)}{C(\mu,n)\left\{(n-1) + \beta[\gamma n + 1 - (1+\gamma)\alpha]\right\}}\lambda_n z^n. \end{aligned} \tag{3.8}$$

Then it follows that

$$\sum_{n=2}^{\infty}\frac{\beta(1-\alpha)(1+\gamma)}{C(\mu,n)\left\{(n-1) + \beta[\gamma n + 1 - (1+\gamma)\alpha]\right\}}\lambda_n\frac{C(\mu,n)\left\{(n-1) + \beta[\gamma n + 1 - (1+\gamma)\alpha]\right\}}{\beta(1-\alpha)(1+\gamma)}$$

$$= \sum_{n=2}^{\infty}\lambda_n = 1 - \lambda_1 < 1. \tag{3.9}$$

Therefore by Theorem 1, $f(z) \in R_\mu[\alpha,\beta,\gamma]$.

Conversely, assume that the function $f(z)$ defined by (1.12) belongs to the class $R_\mu[\alpha,\beta,\gamma]$, and then we have

$$a_n \leq \frac{\beta(1+\gamma)(1-\alpha)}{C(\mu,n)\left\{(n-1)+\beta[\gamma n+1-(1+\gamma)\alpha]\right\}}, n \in \mathbb{N}\backslash\{1\}. \tag{3.10}$$

Setting

$$\lambda_n = a_n \frac{C(\mu,n)\left\{(n-1)+\beta[\gamma n+1-(1+\gamma)\alpha\right\}}{\beta(1-\alpha)(1+\gamma)}, n \in \mathbb{N}\backslash\{1\}, \tag{3.11}$$

and

$$\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n, \tag{3.12}$$

we see that $f(z)$ can be expressed in the form(3.7).This completes the proof of Theorem 5.

In the same manner we can prove, □

Theorem 6. *Let,*

$$f_1(z) = z \tag{3.13}$$

and

$$f_n(z) = z - \frac{\beta(1-\alpha)(1+\gamma)}{C(\mu,n)n\left\{(n-1)+\beta[\gamma n+1-(1+\gamma)\alpha]\right\}} z^n, n \in \mathbb{N}\backslash\left\{1\right\}. \tag{3.14}$$

Then $f(z)$ is in the class $C_\mu[\alpha,\beta,\gamma]$ if and only it can be expressed as

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z) \tag{3.15}$$

where, $\lambda_n \geq 0$ and $\sum_{n=1}^{\infty} \lambda_n = 1$.

4. **Generalized Fractional Integral Operator**

Various operators of fractional calculus, that is fractional derivative operator, fractional integral operator have been studied in the literature rather extensively for *e.g.* [3, 5, 11, 12]. In the present section we shall make use of generalized fractional integral operator $I_{0,z}^{\lambda,\delta,\eta}$ given by Srivastava *et al* [13].

Definition. For real numbers $\lambda > 0, \delta$ and η the generalized fractional integral operator $I_{0,z}^{\lambda,\delta,\eta}$ is defined as

$$I_{0,z}^{\lambda,\delta,\eta} f(z) = \frac{z^{-\lambda-\delta}}{\Gamma(\lambda)} \int_0^z (z-t)^{\lambda-1} {}_2F_1(\lambda+\delta, -\eta, 1 - {}^t\!/_z) f(t)dt \tag{4.1}$$

where $f(z)$ is an analytic function in a simply connected region of the z-plane containing origin with order

$$f(z) = 0(|z|)^{\varepsilon}, (z \to 0, \varepsilon > \max[0,\delta{-}\eta] - 1) \tag{4.2}$$

$${}_2F_1(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n} \frac{z^n}{n} \tag{4.3}$$

and $(\nu)_n$ is the Pochhammer symbol defined by

$$(\nu)_n = \frac{\Gamma(\nu+n)}{\Gamma(\nu)} = \begin{cases} 1 \\ \nu(\nu+1)...(\nu+n+1), \nu \in \mathbb{N} \end{cases} \tag{4.4}$$

an the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ to be real when $(z-t) > 0$.

In order to prove the results for generalized fractional integral operator $I_{0,z}^{\lambda,\delta,\eta}$, we recall here the following lemma due to Srivastava *et al* [13].

Lemma 1 (Srivastava *et al* [13])**.** *If* $\lambda > 0$ and $k > \delta{-}\eta{-}1$ *then*

$$I_{0,z}^{\lambda,\delta,\eta} z^k = \frac{\Gamma(k+1)\Gamma(k-\delta+\eta+1)}{\Gamma(k-\delta+1)\Gamma(k+\lambda+\eta+1)} z^{k-\delta}. \tag{4.5}$$

Theorem 7. *Let* $\lambda > 0, \delta < 2, \lambda + \eta > -2, \delta - \eta < 2$ and $\delta(\lambda+\eta) \le 3\lambda$. *If* $f(z) \in T$ *is in the class* $R_\mu[\alpha,\beta,\gamma]$ *with* $0 \le \mu \le {}^1\!/_2$, $0 < \beta \le 1, 0 \le \alpha < 1$ *and* $0 \le \gamma \le 1$ *then*

$$\frac{\Gamma(2-\delta+\eta)\,|z|^{1-\delta}}{\Gamma(2-\delta)\Gamma(2+\lambda+\eta)} \left\{1 - \frac{(2-\delta+\eta)\beta(1-\alpha)(1+\gamma)}{1+\beta\{\gamma(2-\alpha)+1-\alpha\}(1-\mu)(2-\delta)(2+\lambda+\eta)}\,|z|\right\}$$
$$\le \left|I_{0,z}^{\lambda,\delta,\eta} f(z)\right| \le$$
$$\frac{\Gamma(2-\delta+\eta)\,|z|^{1-\delta}}{\Gamma(2-\delta)\Gamma(2+\lambda+\eta)} \left\{1 + \frac{(2-\delta+\eta)\beta(1-\alpha)(1+\gamma)}{1+\beta\{\gamma(2-\alpha)+1-\alpha\}(1-\mu)(2-\delta)(2+\lambda+\eta)}\,|z|\right\}, \tag{4.6}$$

when

$$U_0 = \begin{cases} U, \delta \le 1 \\ U\backslash\{1\}, \delta > 1. \end{cases} \tag{4.7}$$

Equality in (4.6) *is attended for the function given by*

$$f(z) = z - \frac{\beta(1-\alpha)(1+\gamma)}{2\left\{1+\beta[\gamma(2-\alpha)+1-\alpha]\right\}} z^2. \tag{4.8}$$

Proof. By making use of Lemma 1, we have

$$I_{0,z}^{\lambda,\delta,\eta} f(z) = \frac{\Gamma(2-\delta+\eta)}{\Gamma(2-\delta)\Gamma(2+\lambda+\eta)} z^{1-\delta} - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(n-\delta+\eta+1)}{\Gamma(n-\delta+1)\Gamma(n+\lambda+\eta+1)} a_n z^{n-\delta}. \tag{4.9}$$

Letting,

$$\begin{aligned} H(z) &= \frac{\Gamma(2-\delta)\Gamma(2+\lambda+\eta)}{\Gamma(2-\delta+\eta)} z^{\delta} I_{0,z}^{\lambda,\delta,\eta} \\ &= z - \sum_{n=2}^{\infty} \psi(n) a_n z^n \end{aligned} \tag{4.10}$$

where,

$$\psi(n) = \frac{(2-\delta+\eta)\,(1)_n}{(2-\delta)_{n-1}(2+\lambda+\eta)}, n \in \mathbb{N}\backslash\{1\}. \tag{4.11}$$

We can see that $\psi(n)$ is non -increasing for integers $n, n \in \mathbb{N}\backslash\{1\}$, and we have

$$0 < \psi(n) \leq \psi(2) = \frac{2(2-\delta+\eta)}{(2-\delta)(2+\lambda+\eta)}, n \in \mathbb{N}\backslash\{1\}. \tag{4.12}$$

Now in view of Theorem 1 and (4.12), we have

$$|H(z)| \geq |z| - \psi(2)\,|z|^2 \sum_{n=2}^{\infty} a_n$$

$$\geq |z| - \frac{(2-\delta+\eta)\beta(1-\alpha)(1+\gamma)}{1+\beta[\gamma(2-\alpha)+1-\alpha](1-\mu)(2-\delta)(2+\lambda+\eta)} |z|^2 \tag{4.13}$$

and

$$\begin{aligned} |H(z)| &\leq |z| + \psi(2)|z|^2 \sum_{n=2}^{\infty} a_n \\ &\geq |z| + \frac{(2-\delta+\eta)\beta(1-\alpha)(1+\gamma)}{1+\beta[\gamma(2-\alpha)+1-\alpha](1-\mu)(2-\delta)(2+\lambda+\eta)} |z|^2. \end{aligned} \tag{4.14}$$

This completes the proof of Theorem 7.

Now, by applying Theorem 2 to the functions $f(z)$ belonging to the class $C_\mu[\alpha,\beta,\gamma]$, we can derive □

Theorem 8. *Let* $\lambda > 0, \delta < 2, \lambda + \eta > -2, \delta - \eta < 2$ and $\delta(\lambda + \eta) \leq 3\lambda$ *.If* $f(z) \in T$ *is in the class* $C_\mu[\alpha, \beta, \gamma]$ *with* $0 \leq \mu \leq 1/2$, $0 < \beta \leq 1, 0 \leq \alpha < 1$ *and* $0 \leq \gamma \leq 1$ *then*

$$\frac{\Gamma(2-\delta+\eta)\,|z|^{1-\delta}}{\Gamma(2-\delta)\Gamma(2+\lambda+\eta)}\left\{1 - \frac{(2-\delta+\eta)\beta(1-\alpha)(1+\gamma)}{2[1+\beta\{\gamma(2-\alpha)+1-\alpha\}](1-\mu)(2-\delta)(2+\lambda+\eta)}\,|z|\right\} \tag{4.15}$$

$$\leq \left|I_{0,z}^{\lambda,\delta,\eta} f(z)\right| \leq$$

$$\frac{\Gamma(2-\delta+\eta)\,|z|^{1-\delta}}{\Gamma(2-\delta)\Gamma(2+\lambda+\eta)}\left\{1 + \frac{(2-\delta+\eta)\beta(1-\alpha)(1+\gamma)}{2[1+\beta\{\gamma(2-\alpha)+1-\alpha\}](1-\mu)(2-\delta)(2+\lambda+\eta)}\,|z|\right\} \tag{4.16}$$

where U_0 *is defined by* (4.7)*. Equality in* (4.6) *is attended for the function given by*

$$f(z) = z - \frac{\beta(1-\alpha)(1+\gamma)}{2\{1+\beta[\gamma(2-\alpha)+1-\alpha]\}}z^2.$$

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