STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 2, June 2009

A CONSTRUCTION OF ADMISSIBLE STRATEGIES FOR AMERICAN OPTIONS ASSOCIATED WITH PIECEWISE CONTINUOUS PROCESSES

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Abstract. We provide the construction of some admissible strategies in a "feedback shape" for American Options, and where the contingent claim depends on a nontrivial solution of some possibly degenerate elliptic inequation.

1. **Setting of the problem**

Let $W(t)$ be a standard m-dimensional Wiener process over a complete probability space $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\}$, $\{\lambda(t); t \geq 0\}$ and $\{y(t); t \geq 0\}$ piecewise constant adapted processes of dimension n, respectively d defined on the same probability space. $\lambda(t)$ takes values in some subset S of $\mathbb{R}^n$.

We denote $\mu(t) = (y(t), \lambda(t))$, for $t \geq 0$ and

$$\mu(t, \omega) = \mu_k(\omega) = (y_k(\omega), \lambda_k(\omega)),\ t \in [t_k(\omega), t_{k+1}(\omega)),$$

where the sequence $\{t_k; k \geq 0\}$ is increasing and it's elements are positive random variables with $t_0 = 0$, $t_k \to \infty$, $\mathbb{P}a.s.$, as $k \to \infty$ and (y_k, λ_k) are multidimensional $\mathcal{F}_{t_k}$-measurable random variables. Then we may assume $S = \{\lambda_k; k \geq 1\}$.

We make the assumption that the process $W(t)$ and the sequence $\{(t_k, \mu_k); k \geq 1\}$ are mutually independent.

Received by the editors: 28.03.2008.

2000 *Mathematics Subject Classification.* 35B40, 35J70, 60G44, 91B28.

Key words and phrases. American options, admissible strategies, piecewise continuous process, degenerate elliptic equation.

The research of the authors was supported by the *ANCS* research contract Nr. 2-CEx06-11-18/2006.

Consider a small investor acting in a financial market on which is given a *riskless asset* (for instance a *bond*) whose price evolves in time as

$$dS_0(t) = rS_0(t)dt;\ S_0(0) = 1,\ t \geq 0, \tag{1}$$

implying that $S_0(t) = e^{rt}$ and d *risky assets* (that we call *stocks*), for which the vector $S(t,x)$ collecting the prices of the assets satisfies the SDE

$$\begin{cases} dS(t) & = g_0(S(t);\lambda(t))dt + \displaystyle\sum_{j=1}^{m} g_j(S(t);\lambda(t))dW_j(t),\ t \in [t_k, t_{k+1}), \\ S(t_k) & = S_-(t_k) + y_k,\ \text{for any } k \geq 1, \\ S(0) & = x. \end{cases} \tag{2}$$

where the vector fields

$$g_i(y;\lambda) = a_i(\lambda) + A_i(\lambda)y,\ i = 1,\dots,m,\ \lambda \in S,\ y \in \mathbb{R}^d, \tag{3}$$

are assumed continuous and bounded with respect to λ. We denoted $S_-(t_k) = \lim_{t \uparrow t_k} S(t)$. $x = (x_1, x_2, \dots, x_d)$ and x_i represents the amount of money invested at the initial time $t = 0$ in the stock i, for $i = 1, \dots, d$. x_i may be negative and this happens if the quantity $-x_i$ is borrowed at the interest rate r.

The unique solution of the system (2) is a piecewise continuous and $\{\mathcal{F}_t\}$-adapted process $\{S(t,x); t \geq 0\}$, such that at each jump time t_k, the jump $S(t_k,x) - S_-(t_k,x) = y_k$ occurs. The linear shape of $g_0(y;\lambda)$ is not required and we assume that $g_0(y;\lambda)$ is global Lipschitz continuous with respect to $y \in \mathbb{R}^d$.

A portofolio problem for an American Option with maturity T and its admissible strategies can be described by a value function of the following form

$$V(t,x) = e^{rt}\theta_0(t,x) + \theta(t,x) \cdot S(t,x),\ t \in [0,T],\ x \in \mathbb{R}^d, \tag{4}$$

where $\theta_0(t,x) \in \mathbb{R}, \theta(t,x) \in \mathbb{R}^d$ are some $\mathcal{F}_t^1$-adapted processes, for each fixed $x \in \mathbb{R}^d$ representing the amount of assets form the bond, respectively the quantities of stocks possessed by the investor.

We accept only self-financing portfolios, i.e. portfolios for which the differential of the value function is given by

$$dV(t,x) = \theta_0(t,x)de^{rt} + \theta(t,x)\cdot dS(t,x),\ t\in[0,T],$$

and this formula is understood in the integral sense, i.e.

$$\begin{aligned} V(t,x) &= V(t_k,x) + r\int_{t_k}^{t}\theta_0(s,x)e^{rs}ds + \int_{t_k}^{t}\theta(s,x)\cdot dS(s,x) \\ &= \theta_0(t_k,x)e^{rt_k} + \theta(0,x)\cdot x + r\int_{t_k}^{t}\theta_0(s,x)e^{rs}ds \\ &\quad + \int_{t_k}^{t}\theta(s,x)\cdot g_0(S(s,x);\lambda_k)ds \\ &\quad + \sum_{j=1}^{m}\int_{t_k}^{t}\theta(s,x)\cdot g_j(S(s,x);\lambda_k)dW_j(s),\ t\in[t_k\wedge T, t_{k+1}\wedge T). \end{aligned} \tag{5}$$

Instead of $[t_k\wedge T, t_{k+1}\wedge T)$, we shall simply write $[t_k, t_{k+1})$.

American options, in contrast with European options may be exercised at any moment of time between 0 and T, and thus the value function for an admissible strategy has to satisfy the constraint

$$V(t,x) \geq h_\gamma(t,x),\ 0\leq t\leq T, \tag{6}$$

where $h_\gamma(t,x)$ is a positive $\mathcal{F}_t$-measurable random variable which stands for the value of the option at the moment t, i.e. the amount of money that the investor has to be able to provide at time t.

We consider here only functionals of the form

$$h_\gamma(t,x) := e^{\gamma t}\varphi_\gamma(S(t,x),\lambda(t)), \tag{7}$$

where γ is a negative constant and $\varphi_\gamma(y,\lambda)\in\mathcal{P}_2(y;\lambda)$, the set consisting of second degree polynomials with respect to the variables $(y_1,\ldots,y_d)=y$, whose coefficients are continuous and bounded functions of λ.

$\mathcal{P}_2(y)\subseteq\mathcal{P}_2(y;\lambda)$ stands for the set of constant coefficients polynomials.

We consider functions φ_γ of a particular form, which we shall make precise later on.

In order to find such strategies, we need to emphasize those conditions which allow to get them in a "feedback shape"

$$\theta(t,x) = e^{\gamma t}\nabla_y\varphi_\gamma(S(t,x);\lambda(t)),\ t\in[0,T],\ x\in\mathbb{R}^d \tag{8}$$

and

$$\theta_0(t_k,x) = e^{(\gamma-r)t_k}\varphi_\gamma(0,\lambda_k). \tag{9}$$

Remark 1. *For the sake of simplicity, when computing admissible strategies we shall include the "feedback shape"* (8) *and* (9) *in the definition of such strategies and we look for appropriate* (γ,φ_γ), $\varphi_\gamma\in\mathcal{P}_2(y;\lambda)$, *such that the equations* (5) *and* (6) *are fulfilled. We emphasize that this approach will lead us to an admissible couple* $(\theta_0(t,x),\theta(t,x))\in\mathbb{R}^{d+1}$, *provided*

(a) $\varphi_\gamma\in\mathcal{P}_2(y;\lambda)$ *is a convex function with respect to* $y\in\mathbb{R}^d$*;*

(b) (γ,φ_γ) *is a nontrivial solution of the following elliptic inequality*

$$\gamma\varphi_\gamma(y;\lambda)+\sum_{j=1}^m\frac{1}{2}\left\langle\partial_y^2\varphi_\gamma(y;\lambda)\,g_j(y;\lambda),g_j(y;\lambda)\right\rangle\le 0,\ (y,\lambda)\in\mathbb{R}^d\times S. \tag{10}$$

The "feedback shape" (8) agrees with the constraints (5) and (6), without involving the convexity property (a) and the analysis can be reduced to the elliptic inequality (10).

2. Auxiliary results

Set $L:\mathcal{P}_2(y;\lambda)\to\mathcal{P}_2(y;\lambda)$ the second order linear operator defined as

$$L(\psi)(y;\lambda):=\sum_{j=1}^m\frac{1}{2}\langle\partial_y^2\psi(y;\lambda)g_j(y;\lambda),g_j(y;\lambda)\rangle,\text{for }\psi\in\mathcal{P}_2(y;\lambda), \tag{11}$$

where we denoted $\partial_y^2\psi(y;\lambda)$ the Hessian matrix of ψ with respect to y.

Notice that L is a *possibly degenerate* elliptic operator.

Lemma 1. *Let* $f\in\mathcal{P}_2(y)$ *such that* $f(y)\ge 0,\ \forall y\in\mathbb{R}^d$ *and* γ *a nonzero constant such that the elliptic equation*

$$L(\psi)(y;\lambda)+\gamma\psi(y;\lambda)+f(y)=0,\ \text{for any } y\in\mathbb{R}^d,\ \lambda\in S \tag{12}$$

has a nontrivial solution $\varphi_\gamma\in\mathcal{P}_2(y;\lambda)$.

Then the following estimate holds true

$$h_\gamma(t,x) \le \exp(\gamma t_k)\varphi_\gamma(S(t_k,x);\lambda_k) + \int_{t_k}^{t} \exp(\gamma s)\nabla_y\varphi_\gamma(S(s,x);\lambda_k)\cdot dS(s,x), \quad (13)$$

for any $t \in [t_k, t_{k+1})$.

Proof. Apply the Itô formula for the process $h_\gamma(t,x) = e^{\gamma t}\varphi_\gamma(S(t,x),\lambda(t))$ on the interval $[t_k, t_{k+1})$ and get

$$\begin{aligned}
h_\gamma(t,x) &:= \exp(\gamma t_k)\varphi_\gamma(S(t_k,x);\lambda_k) + \int_{t_k}^{t} \exp(\gamma s)\nabla_y\varphi_\gamma(S(s,x);\lambda_k)\cdot g_0(S(s,x);\lambda_k)ds \\
&\quad + \int_{t_k}^{t} \exp(\gamma s)\left[\gamma\varphi_\gamma + f + L(\varphi_\gamma)(S(s,x);\lambda_k)\right] ds \\
&\quad + \sum_{j=1}^{m}\int_{t_k}^{t} \exp(\gamma s)\nabla_y\varphi_\gamma(S(s,x);\lambda_k)\cdot g_j(S(s,x);\lambda_k)dW_j(s) \\
&\quad - \int_{t_k}^{t} \exp(\gamma s) f(S(s,x))ds = \exp(\gamma t_k)\varphi_\gamma(S(t_k,x);\lambda_k) \\
&\quad + \int_{t_k}^{t} \exp(\gamma s)\nabla_y\varphi_\gamma(S(s,x);\lambda_k)\cdot g_0(S(s,x);\lambda_k)ds \\
&\quad + \sum_{j=1}^{m}\int_{t_k}^{t} \exp(\gamma s)\nabla_y\varphi_\gamma(S(s,x);\lambda_k)\cdot g_j(S(s,x);\lambda_k)dW_j(s) \\
&\quad - \int_{t_k}^{t} \exp(\gamma s) f(S(s,x))ds \\
&= \exp(\gamma t_k)\varphi_\gamma(S(t_k,x);\lambda_k) + \int_{t_k}^{t} \exp(\gamma s)\nabla_y\varphi_\gamma(S(s,x);\lambda_k)\cdot dS(s,x) \\
&\quad - \int_{t_k}^{t} \exp(\gamma s) f(S(s,x))ds,
\end{aligned} \quad (14)$$

for any $t \in [t_k, t_{k+1})$, by virtue of our assumptions.

This leads us to the conclusion of the lemma, since f takes positive values. □

Lemma 2. *Let the assumptions of the Lemma 1 be in force and, in addition, we make the hypothesis that a nontrivial solution* φ_γ *of the elliptic equation* (12) *is a convex function. Define*

$$\theta(t,x) := e^{\gamma t}\nabla_y\varphi_\gamma(S(t,x);\lambda(t)),\ 0 \le t \le T,\ x \in \mathbb{R}^d \quad (15)$$

and let $\{\theta_0(t,x); t \in [0,T]\}$ *be the piecewise continuous process satisfying the integral equation* (5)*, with*

$$\theta_0(t_k, x) := e^{(\gamma-r)t_k}\varphi_\gamma(0;\lambda_k). \tag{16}$$

Moreover, we assume that

$$\theta_0(t,x) \geq 0,\ \forall t \in [0,T],\ x \in \mathbb{R}^n. \tag{17}$$

Then $(\theta_0(t,x), \theta(t,x)) \in \mathbb{R}^{d+1}$ *is an admissible strategy (see the formulas* (5) *and* (6)*) satisfying the "feedback shape"* (8) *and* (9)*.*

Proof. The value function V considered at the time t_k may be written as

$$V(t_k,x) = \theta_0(t_k,x)e^{rt_k} + e^{\gamma t_k}\nabla_y\varphi_\gamma(S(t_k,x);\lambda_k)\cdot S(t_k,x)$$

and we require that

$$V(t_k,x) \geq \exp(\gamma t_k)\varphi_\gamma(S(t_k,x);\lambda_k), \tag{18}$$

where we used the choice (15) for $\theta(t,x)$.

The equation (18) is equivalent with

$$\theta_0(t_k,x)e^{rt_k} + e^{\gamma t_k}\nabla_y\varphi_\gamma(S(t_k,x);\lambda_k)\cdot S(t_k,x) \geq e^{\gamma t_k}\varphi_\gamma(S(t_k,x),\lambda_k). \tag{19}$$

Since φ_γ is a convex function, its gradient $\partial_y\varphi_\gamma(y;\lambda)$ satisfies

$$\langle \nabla_y\varphi_\gamma(y_2;\lambda) - \nabla_y\varphi_\gamma(y_1;\lambda),\ y_2 - y_1\rangle \geq 0, \text{ for any } y_1, y_2 \in \mathbb{R}^d \text{ and } \lambda \in S. \tag{20}$$

and thus, if $\theta_0(t_k,x)$ is defined as in (16), we easily get the estimate (19) fulfilled, via the Lagrange Mean Value Theorem.

$\theta_0(t,x)$ is finally obtained as the unique solution of the integral equation

$$\begin{aligned} V(t,x) &= e^{rt}\theta_0(t,x) + e^{\gamma t}\nabla_y\varphi_\gamma(S(t,x);\lambda(t))\cdot S(t,x) \\ &= V(t_k,x) + r\int_{t_k}^t \theta_0(s,x)e^{rs}ds + \int_{t_k}^t e^{\gamma t}\nabla_y\varphi_\gamma(S(t,x);\lambda(t))\cdot dS(s,x), \end{aligned} \tag{21}$$

for $t \in [t_k, t_{k+1})$.

Let t be arbitrary chosen in some interval $[t_k, t_{k+1})$. Then

$$\begin{aligned}
V(t,x) &= e^{\gamma t_k}\varphi_\gamma(0;\lambda_k) + e^{\gamma t_k}\nabla_y\varphi_\gamma(S(t_k,x);\lambda_k)\cdot S(t_k,x) + r\int_{t_k}^{t}\theta_0(s,x)e^{rs}ds \\
&\quad + \int_{t_k}^{t} e^{\gamma s}\nabla_y\varphi_\gamma(S(s,x);\lambda(s))\cdot dS(s,x) \\
&\geq e^{\gamma t_k}\varphi_\gamma(0;\lambda_k) + e^{\gamma t_k}\nabla_y\varphi_\gamma(S(t_k,x);\lambda_k)\cdot S(t_k,x) \\
&\quad + \int_{t_k}^{t} e^{\gamma s}\nabla_y\varphi_\gamma(S(s,x);\lambda(s))\cdot dS(s,x) \\
&\geq e^{\gamma t_k}\varphi_\gamma(S(t_k,x);\lambda_k) + \int_{t_k}^{t} e^{\gamma s}\nabla_y\varphi_\gamma(S(s,x);\lambda(s))\cdot dS(s,x) \\
&\geq h_\gamma(t,x),
\end{aligned} \tag{22}$$

where we used the self-financing equation (5), the asumption (17), the convexity property of φ_γ with respect to y and the Lemma 1. The conclusion of the lemma is now straightforward. □

Remark 2. *For a fixed $f \in \mathcal{P}_2(y)$, a solution (γ, φ_γ) of the elliptic equation* (12) *is constructed using the following series*

$$\varphi_\gamma(y;\lambda) = \frac{1}{|\gamma|}\left[\sum_{k=0}^{\infty} L_{|\gamma|}^{k}(f)(y;\lambda)\right], \text{ for } \gamma < 0, \tag{23}$$

where $L_{|\gamma|} = \frac{1}{|\gamma|}L$ and $L : \mathcal{P}_2(y;\lambda) \to \mathcal{P}_2(y;\lambda)$ stands for the linear operator defined in the formula (11).

As far as the linear operator $L_{|\gamma|}$ is acting on $\mathcal{P}_2(y;\lambda)$, for the sake of simplicity we shall assume that $f(y) = (\langle q, y\rangle)^2$, where $q \neq 0$ is a common eigen vector of the matrices $A_j(\lambda)$, such that $A_j^*(\lambda)q = \mu_j(\lambda)q$ and $\mu_j : S \to \mathbb{R}$ is continuous and bounded, for any $1 \leq j \leq m$.

Lemma 3. *Let $f \in \mathcal{P}_2(y)$ and $g_j(y;\lambda) = A_j(\lambda)y + a_j(\lambda)$, $j = 1, \dots, m$, be given as above. Let $\gamma < 0$ such that $\frac{\|\mu\|}{|\gamma|} \leq 1$, where $\mu(\lambda) = \sum_{j=1}^{m}\mu_j^2(\lambda)$ and $\|\mu\| =$*

$\sup_{\lambda\in S}\mu(\lambda)$. *Then the function*

$$\varphi_\gamma(y;\lambda) = \frac{1}{|\gamma|}\left[\sum_{k=0}^{\infty} L^k_{|\gamma|}(f)(y;\lambda)\right] \tag{24}$$

$$= \frac{1}{|\gamma|-\mu(\lambda)}\left[f(y) + \frac{b(\lambda)}{|\gamma|}\langle q,y\rangle + \frac{a(\lambda)}{|\gamma|}\right], y\in\mathbb{R}^d, \lambda\in S \tag{25}$$

is a solution of the elliptic equation (12)*, where* $b(\lambda) = 2\sum_{j=1}^m \mu_j(\lambda)\langle q, a_j(\lambda)\rangle$ *and* $a(\lambda) = \sum_{j=1}^m (\langle q, a_j(\lambda)\rangle)^2$.

Proof. By hypothesis, we easily see that

$$L(f)(y;\lambda) = \sum_{j=1}^m \left[A_j(\lambda)y + a_j(\lambda)\right]^* q\, q^* \left[A_j(\lambda)y + a_j(\lambda)\right] \tag{26}$$

$$= \sum_{j=1}^m \left(\langle q, A_j(\lambda)y + a_j(\lambda)\rangle\right)^2 = \mu(\lambda)f(y) + b(\lambda)\langle q,y\rangle + a(\lambda).$$

Hence

$$L_{|\gamma|}(f)(y;\lambda) = \frac{\mu(\lambda)}{|\gamma|}f(y) + \frac{b(\lambda)}{|\gamma|}\langle q,y\rangle + \frac{a(\lambda)}{|\gamma|}. \tag{27}$$

An induction argument leads us to

$$L^k_{|\gamma|}(f)(y;\lambda) = \left(\frac{\mu(\lambda)}{|\gamma|}\right)^k f(y) + \left(\frac{\mu(\lambda)}{|\gamma|}\right)^{k-1}\left[\frac{b(\lambda)}{|\gamma|}\langle q,y\rangle\right] \tag{28}$$

$$+ \left(\frac{\mu(\lambda)}{|\gamma|}\right)^{k-1}\left[\frac{a(\lambda)}{|\gamma|}\right], \text{ for any } k\geq 1.$$

Denote $\rho_\gamma(\lambda) = \frac{\mu(\lambda)}{|\gamma|}$ and

$$T(\lambda) = \sum_{k=0}^{\infty}[\rho_\gamma(\lambda)]^k = \frac{|\gamma|}{|\gamma|-\mu(\lambda)},$$

where $\rho_\gamma(\lambda) < 1$, for any $\lambda\in S$ (see $\frac{\|\mu\|}{|\gamma|}\leq 1$). Inserting the formula (28) in (24), we obtain

$$\varphi_\gamma(y;\lambda) = \frac{1}{|\gamma|}T(\lambda)f(y) + \frac{1}{|\gamma|}T(\lambda)\frac{b(\lambda)}{|\gamma|}\langle q,y\rangle + \frac{1}{|\gamma|}T(\lambda)\frac{a(\lambda)}{|\gamma|}$$

and substituting $T(\lambda)$ we get the conclusion fulfilled. □

Remark 3. *Notice that*

$$\theta_0(t_k,x) = e^{(\gamma-r)t_k}\varphi_\gamma(0;\lambda_k) = e^{(\gamma-r)t_k}\frac{a(\lambda)}{|\gamma|(|\gamma|-\mu(\lambda))}\geq 0.$$

Therefore, the assumption that $\theta_0(t,x)\geq 0$*, for all* $t\in[0,T]$*,* $x\in\mathbb{R}^n$ *is very reasonable.*

Remark 4. *The solution of the function* φ_γ *makes use of a special convex function* $f(y)=(\langle q,y\rangle)^2$*, with* $q\in\mathbb{R}^d$ *as a common eigen vector of the matrices* $A_j(\lambda)$*,* $j=1,\dots,m$*.*

Assuming that there exist several eigen vectors $Q=(q_1,\dots,q_s)$*,* $s\leq d$*, such that*

$$Q^*A_j(\lambda)=\mu_j(\lambda)Q^*,\ \mu_j(\lambda)\in\mathbb{R},\ j=1,\dots,m, \tag{29}$$

then $f(y)=\langle Q^*y,Q^*y\rangle$ *agrees with the conclusion of the Lemma 3 and the computation of the convex function* $\varphi_\gamma\in\mathcal{P}_2(y)$ *follows the same procedure.*

In addition, for an arbitrarily fixed $y_0\in\mathbb{R}^d$*, we may consider a convex function*

$$f(y)=\langle Q^*(y-y_0),Q^*(y-y_0)\rangle, \tag{30}$$

where $\tilde{S}(t,x)=S(t,x)-y_0$*,* $t\geq 0$*, satisfies the following linear system*

$$\begin{cases} dz(t) &= h_0(z(t);\lambda)dt+\sum_{j=1}^{m}h_j(z(t);\lambda)dW_j(t),\ t\geq 0\\ z(0) &= x-y_0. \end{cases} \tag{31}$$

Here $h_i(z;\lambda)=A_i(\lambda)z+d_i(\lambda)$*,* $d_i(\lambda)=a_i(\lambda)+A_i(\lambda)y_0$*,* $i=0,1,\dots,m$ *replaces the original vector fields* $g_i(y;\lambda$ *of the system* (2) *and the function* $f(z)=\langle Q^*z,Q^*z\rangle$ *satisfies* (29)*.*

3. Main results

We conclude the above given analysis by the following

Theorem 1. *Let* $g_j(y;\lambda)=A_j(\lambda)y+a_j(\lambda)$ *be given such that the* $(d\times d)$ *matrix* $A_j(\lambda)$ *and the vector* $a_j(\lambda)\in\mathbb{R}^d$ *are continuous and bounded with respect to* $\lambda\in S$*, for any* $j=1,\dots,m$ *and* $d\leq n$*. Consider a continuous vector field* $g_0(y;\lambda)\in\mathbb{R}^d$ *which is globally Lipschitz continuous with respect to* $y\in\mathbb{R}^d$*, uniformly in* $\lambda\in S$*.*

Define a convex function $f\in\mathcal{P}_2(y)$ *by*

$$f(y)=\langle Q^*(y-y_0),Q^*(y-y_0)\rangle, \tag{32}$$

where $y_0 \in \mathbb{R}^d$ is arbitrarily fixed and $Q = (q_1, \ldots, q_s),\ q_i \in \mathbb{R}^d,\ s \leq d$ stand for some common eigen vectors satisfying

$$Q^* A_j(\lambda) = \mu_j(\lambda) Q^*,\ \mu_j(\lambda) \in \mathbb{R},\ j = 1, \ldots, m. \tag{33}$$

Let $\gamma < 0$ be such that $\frac{\|\tilde{\mu}\|}{|\gamma|} < 1$, where $\mu(\lambda) = \sum_{j=1}^m \mu_j^2(\lambda)$ and $\|\tilde{\mu}\| = \sup_{k \geq 0} \mu(\tilde{\lambda}_k)$.

Then

$$\begin{aligned} \varphi_\gamma(y;\lambda) =& \frac{1}{|\gamma|}\left[\sum_{k=0}^{\infty} L_{|\gamma|}^k(f)(y;\lambda)\right] = \frac{1}{|\gamma| - \mu(\lambda)} \\ &\times \left[f(y) + \langle \frac{b(\lambda)}{|\gamma|}, Q^*(y - y_0)\rangle + \frac{a(\lambda)}{|\gamma|}\right],\ y \in \mathbb{R}^d,\ \lambda \in S, \end{aligned} \tag{34}$$

is a solution of the elliptic equation (12)*, where $b(\lambda) = 2\sum_{j=1}^m \mu_j(\lambda) Q^* d_j(\lambda)$, $a(\lambda) = \sum_{j=1}^m \|Q^* d_j(\lambda)\|^2$, $d_j(\lambda) = a_j(\lambda) + A_j(\lambda) y_0,\ j = 1, \ldots, m$.*

Proof. Using the linear mapping $z = y - y_0$, we rewrite

$$f(y) = \tilde{f}(z) = \langle Q^* z, Q^* z\rangle$$

and the solution $\{S(t,x); t \geq 0\}$ satisfying (2) is shifted into $\tilde{S}(t,x) = S(t,x) - y_0$, which satisfies the system (31). Here $h_j(z;\lambda) = A_j(z;\lambda) z + d_j(\lambda),\ j = 1, \ldots, m$ and $h_0(z;\lambda) = g_0(z + y_0;\lambda)$.

The procedure employed in the proof of the Lemma 3 is applicable here and the convex function $\varphi_\gamma \in \mathcal{P}_2(y;\lambda)$ given in (34) satisfies the equation (12). □

Theorem 2. *Assume that the assumptions of the previous theorem and also the estimate* (17) *stand in force. Define*

$$\theta(t,x) = \nabla_y \varphi_\gamma(\hat{y}(t,x); \hat{\lambda}(t)),\ t \in [0,T],\ x \in \mathbb{R}^d \tag{35}$$

and let $\{\theta_0(t,x); t \geq 0\}$ be the piecewise continuous process satisfying the integral equation (5)*, where*

$$\theta_0(t_k, x) = \exp(\gamma t_k) \nabla_y \varphi_\gamma(y_0; \lambda_k),\ k \geq 0,\ x \in \mathbb{R}^d. \tag{36}$$

Then $(\theta_0(t,x), \theta(t,x)) \in \mathbb{R}^{d+1}$ is an admissible strategy corresponding to the value function

$$V(t,x) = \theta_0(t,x) e^{rt} + \theta(t,x) \cdot (S(t,x) - y_0).$$

Proof. By hypothesis, the nontrivial solution $(f, \gamma, \varphi_\gamma)$ of the equation (12) constructed in the Theorem 1 fulfills the conditions assumed in the Lemma 2. The "feedback shape" recommended by the equations (16) and (15) uses the deterministic values $\theta_0(t_k, x) = \exp(\gamma t_k)\varphi_\gamma(0; \lambda_k)$, for $k \geq 0$, which are not correlated with the special form that we obtain here for the convex functions $f \in \mathcal{P}_2(y)$, $\varphi_\gamma \in \mathcal{P}_2(y; \lambda)$.

According to the expression of φ_γ given in the formula (34), the simplest values are obtained for $y = y_0 \in \mathbb{R}^d$, i.e.

$$\varphi_\gamma(y_0, \lambda_k) = \frac{1}{|\gamma| - \mu(\lambda_k)} \frac{a(\lambda_k)}{|\gamma|}, \ k \geq 0.$$

This is a slight changing in the definition of the "feedback shape" (see the formulas (8) and (9)) and it agrees with the linear mapping $z = y - y_0$ used in the proof of the Theorem 1, for which $z = 0$ corresponds to the special "feedback shape" given in (16) and (15).

As a consequence, $(\theta_0(t, x), \theta(t, x)) \in \mathbb{R}^{d+1}$ defined in (35) and (36) is an admissible strategy corresponding to the value function

$$V(t, x) = \theta_0(t, x)e^{rt} + \theta(t, x) \cdot (S(t, x) - y_0), \ t \geq 0, \ x \in \mathbb{R}^d$$

and $\tilde{S}(t, x) = S(t, x) - y_0, \ t \geq 0$, is the solution of the system (31). □

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