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# A DIFFERENTIAL SANDWICH THEOREM FOR ANALYTIC FUNCTIONS DEFINED BY THE INTEGRAL OPERATOR

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**Abstract**. Let $q_1$ and $q_2$ be univalent in the unit disk $U$, with $q_1(0) = q_2(0) = 1$. We give an application of first order differential subordination to obtain sufficient condition for normalized analytic functions $f \in \mathcal{A}$ to satisfy

$$q_1(z) \prec \left(\frac{I^n f(z)}{z}\right)^{\delta} \prec q_2(z),$$

where $I^n$ is an integral operator.

## 1. Introduction

Let $\mathcal{H} = \mathcal{H}(U)$ denote the class of functions analytic in

$$U = \{z \in \mathbb{C} : \ |z| < 1\}.$$

For $n$ a positive integer and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H} : \ f(z) = a + a_n z^n + \dots\}.$$

We also consider the class

$$\mathcal{A} = \{f \in \mathcal{H} : \ f(z) = z + a_2 z^2 + \dots\}.$$

We denote by $Q$ the set of functions $f$ that are analytic and injective on $\overline{U} \setminus E(f)$, where

$$E(f) = \left\{\zeta \in \partial U : \ \lim_{z \to \zeta} f(z) = \infty\right\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

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Since we use the terms of subordination and superordination, we review here those definitions.

Let $f, F \in \mathcal{H}$. The function $f$ is said to be subordinate to $F$ or $F$ is said to be superordinate to $f$, if there exists a function $w$ analytic in $U$, with $w(0) = 0$ and $|w(z)| < 1$, and such that $f(z) = F(w(z))$. In such a case we write $f \prec F$ or $f(z) \prec F(z)$. If $F$ is univalent, then $f \prec F$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$.

Since most of the functions considered in this paper and conditions on them are defined uniformly in the unit disk $U$, we shall omit the requirement "$z \in U$".

Let $\psi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$, let $h$ be univalent in $U$ and $q \in Q$. In [3] the authors considered the problem of determining conditions on admissible function $\psi$ such that

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z) \tag{1.1}$$

implies $p(z) \prec q(z)$, for all functions $p \in \mathcal{H}[a, n]$ that satisfy the differential subordination (1.1).

Moreover, they found conditions so that the function $q$ is the "smallest" function with this property, called the best dominant of the subordination (1.1).

Let $\varphi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$, let $h \in \mathcal{H}$ and $q \in \mathcal{H}[a, n]$. Recently, in [4] the authors studied the dual problem and determined conditions on $\varphi$ such that

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z) \tag{1.2}$$

implies $q(z) \prec p(z)$, for all functions $p \in Q$ that satisfy the above differential superordination.

Moreover, they found conditions so that the function $q$ is the "largest" function with this property, called the best subordinant of the superordination (1.2).

For two functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

the Hadamard product of $f$ and $g$ is defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

The integral operator $I^n$ of a function $f$ is defined in [6] by

$$I^0 f(z) = f(z),$$
$$I^1 f(z) = If(z) = \int_0^z f(t)t^{-1}dt,$$
$$I^n f(z) = I(I^{n-1}f(z)), \quad z \in U.$$

In this paper we will determine some properties on admissible functions defined with the integral operator.

## 2. Preliminaries

**Theorem 2.1.** [3] *Let $q$ be univalent in $U$ and let $\theta$ and $\phi$ be analytic in a domain $D$ containing $q(U)$, with $\phi(w) \neq 0$, when $w \in q(U)$. Set*

$$Q(z) = zq'(z) \cdot \phi[q(z)], \quad h(z) = \theta[q(z)] + Q(z)$$

*and suppose that either $h$ is convex or $Q$ is starlike. In addition, assume that*

$$\mathrm{Re}\ \frac{zh'(z)}{Q(z)} > 0.$$

*If $p$ is analytic in $U$, with $p(0) = q(0)$, $p(U) \subset D$ and*

$$\theta[p(z)] + zp'(z) \cdot \phi[p(z)] \prec \theta[q(z)] + zp'(z) \cdot \phi[q(z)] = h(z),$$

*then $p \prec q$, and $q$ is the best dominant.*

By taking $\theta(w) := w$ and $\phi(w) := \gamma$ in Theorem 2.1, we get

**Corollary 2.2.** *Let $q$ be univalent in $U$, $\gamma \in \mathbb{C}^*$ and suppose*

$$\mathrm{Re}\ \left[1 + \frac{zq''(z)}{q'(z)}\right] > \max\left\{0, -\mathrm{Re}\ \frac{1}{\gamma}\right\}.$$

*If $p$ is analytic in $U$, with $p(0) = q(0)$ and*

$$p(z) + \gamma zp'(z) \prec q(z) + \gamma zq'(z),$$

*then $p \prec q$, and $q$ is the best dominant.*

**Theorem 2.3.** ([4]) *Let $\theta$ and $\phi$ be analytic in a domain $D$ and let $q$ be univalent in $U$, with $q(0) = a$, $q(U) \subset D$. Set*

$$Q(z) = zq'(z) \cdot \phi[q(z)], \quad h(z) = \theta[q(z)] + Q(z)$$

*and suppose that*

*(i)* $\operatorname{Re}\left\{\dfrac{\theta'[q(z)]}{\phi[q(z)]}\right\} > 0$ *and*

*(ii) $Q(z)$ is starlike.*

*If $p \in \mathcal{H}[a, 1] \cap Q$, $p(U) \subset D$ and $\theta[p(z)] + zp'(z) \cdot \phi[p(z)]$ is univalent in $U$, then*

$$\theta[q(z)] + zp'(z)\phi[q(z)] \prec \theta[p(z)] + zp'(z)\phi[p(z)] \ \Rightarrow \ q \prec p$$

*and $q$ is the best subordinant.*

By taking $\theta(w) := w$ and $\phi(w) := \gamma$ in Theorem 2.3, we get

**Corollary 2.4.** ([2]) *Let $q$ be convex in $U$, $q(0) = a$ and $\gamma \in \mathbb{C}$, $\operatorname{Re} \gamma > 0$. If $p \in \mathcal{H}[a, 1] \cap Q$ and $p(z) + \gamma zp'(z)$ is univalent in $U$, then*

$$q(z) + \gamma zq'(z) \prec p(z) + \gamma zp'(z) \ \Rightarrow \ q \prec p$$

*and $q$ is the best subordinant.*

## 3. Main results

**Theorem 3.1.** *Let $q$ be univalent in $U$ with $q(0) = 1$, $\alpha \in \mathbb{C}^*$, $\delta > 0$ and suppose*

$$\operatorname{Re}\left[1 + \frac{zq''(z)}{q'(z)}\right] > \max\left\{0, -\operatorname{Re}\frac{\delta}{\alpha}\right\}.$$

*If $f \in \mathcal{A}$ satisfies the subordination*

$$(1-\alpha)\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta} + \alpha\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta} \cdot \frac{I^n f(z)}{I^{n+1}f(z)} \prec q(z) + \frac{\alpha}{\delta} zq'(z), \tag{3.1}$$

*then*

$$\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta} \prec q(z)$$

*and $q$ is the best dominant.*

**Proof.** We define the function

$$p(z) := \left(\frac{I^{n+1}f(z)}{z}\right)^{\delta}.$$

By calculating the logarithmic derivative of $p$, we obtain

$$\frac{zp'(z)}{p(z)} = \delta\left(\frac{z(I^{n+1}f(z))'}{I^{n+1}f(z)} - 1\right). \tag{3.2}$$

Because the integral operator $I^n$ satisfies the identity:

$$z[I^{n+1}f(z)]' = I^n f(z), \tag{3.3}$$

equation (3.2) becomes

$$\frac{zp'(z)}{p(z)} = \delta\left(\frac{I^n f(z)}{I^{n+1}f(z)} - 1\right)$$

and, therefore,

$$\frac{zp'(z)}{\delta} = \left(\frac{I^{n+1}f(z)}{z}\right)^{\delta}\left(\frac{I^n f(z)}{I^{n+1}f(z)} - 1\right).$$

The subordination (3.1) from the hypothesis becomes

$$p(z) + \frac{\alpha}{\delta}zp'(z) \prec q(z) + \frac{\alpha}{\delta}zq'(z).$$

We apply now Corollary 2.4 with $\gamma = \dfrac{\alpha}{\delta}$ to obtain the conclusion of our theorem. □

If we consider $n = 0$ in Theorem 3.1, we obtain the following result.

**Corollary 3.2.** *Let $q$ be univalent in $U$ with $q(0) = 1$, $\alpha \in \mathbb{C}^*$, $\delta > 0$ and suppose*

$$\operatorname{Re}\left[1 + \frac{zq''(z)}{q'(z)}\right] > \max\left\{0, -\operatorname{Re}\frac{\delta}{\alpha}\right\}.$$

*If $f \in \mathcal{A}$ satisfies the subordination*

$$(1-\alpha)\left(\frac{If(z)}{z}\right)^{\delta} + \alpha\left(\frac{If(z)}{z}\right)^{\delta} \cdot \frac{f(z)}{If(z)} \prec q(z) + \frac{\alpha}{\delta}zq'(z) \tag{3.4}$$

*then*

$$\left(\frac{If(z)}{z}\right)^{\delta} \prec q(z)$$

*and $q$ is the best dominant.*

We consider a particular convex function

$$q(z) = \frac{1+Az}{1+Bz}$$

to give the following application to Theorem 3.1.

**Corollary 3.3.** *Let $A, B, \alpha \in \mathbb{C}$, $A \neq B$ be such that $|B| \leq 1$, $\operatorname{Re} \alpha > 0$ and let $\delta > 0$. If $f \in \mathcal{A}$ satisfies the subordination*

$$(1-\alpha)\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta} + \alpha\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta} \cdot \frac{I^n f(z)}{I^{n+1}f(z)}$$

$$\prec \frac{1+Az}{1+Bz} + \frac{\alpha}{\delta} \cdot \frac{(A-B)z}{(1+Bz)^2},$$

*then*

$$\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta} \prec \frac{1+Az}{1+Bz}$$

*and $q(z) = \dfrac{1+Az}{1+Bz}$ is the best dominant.*

**Theorem 3.4.** *Let $q$ be convex in $U$ with $q(0) = 1$, $\alpha \in \mathbb{C}$, $\operatorname{Re} \alpha > 0$, $\delta > 0$. If $f \in \mathcal{A}$ such that*

$$\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta} \in \mathcal{H}[1,1] \cap Q,$$

$$(1-\alpha)\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta} + \alpha\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta} \cdot \frac{I^n f(z)}{I^{n+1}f(z)}$$

*is univalent in $U$ and satisfies the superordination*

$$q(z) + \frac{\alpha}{\delta} z q'(z) \prec (1-\alpha)\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta} + \alpha\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta} \cdot \frac{I^n f(z)}{I^{n+1}f(z)}, \tag{3.5}$$

*then*

$$q(z) \prec \left(\frac{I^{n+1}f(z)}{z}\right)^{\delta}$$

*and $q$ is the best subordinant.*

**Proof.** Let

$$p(z) := \left(\frac{I^{n+1}f(z)}{z}\right)^{\delta}.$$

If we proceed as in the proof of Theorem 3.1, the subordination (3.5) become

$$q(z) + \frac{\alpha}{\delta} z q'(z) \prec p(z) + \frac{\alpha}{\delta} z p'(z).$$

The conclusion of this theorem follows by applying the Corollary 2.4. □

If $n = 0$, then we obtain

**Corollary 3.5.** *Let $q$ be convex in $U$, with $q(0) = 1$, $\alpha \in \mathbb{C}$, with* $\mathrm{Re}\,\alpha > 0$ *and $\delta > 0$. If $f \in \mathcal{A}$ such that*

$$\left(\frac{If(z)}{z}\right)^{\delta} \in \mathcal{H}[1,1] \cap Q$$

$$(1-\alpha)\left(\frac{If(z)}{z}\right)^{\delta} + \alpha\left(\frac{If(z)}{z}\right)^{\delta} \cdot \frac{f(z)}{If(z)}$$

*is univalent in $U$ and satisfies the superordination*

$$q(z) + \frac{\alpha}{\delta} zq'(z) \prec (1-\alpha)\left(\frac{If(z)}{z}\right)^{\delta} + \alpha\left(\frac{If(z)}{z}\right)^{\delta} \cdot \frac{f(z)}{If(z)},$$

*then $q(z) \prec \left(\frac{If(z)}{z}\right)^{\delta}$ and $q$ is the best subordinant.*

**Corollary 3.6.** *Let $q$ be convex in $U$ with $q(0) = 1$, $\alpha \in \mathbb{C}$ with* $\mathrm{Re}\,\alpha > 0$, *$\alpha > 0$. If $f \in \mathcal{A}$ such that*

$$\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta} \in \mathcal{H}[1,1] \cap Q,$$

$$(1-\alpha)\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta} + \alpha\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta} \cdot \frac{I^{n}f(z)}{I^{n+1}f(z)}$$

*is univalent in $U$ and satisfies the superordination*

$$q(z) + \frac{\alpha}{\delta} zq'(z) \prec (1-\alpha)\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta} + \alpha\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta} \cdot \frac{I^{n}f(z)}{I^{n+1}f(z)},$$

*then*

$$q(z) \prec \left(\frac{I^{n+1}f(z)}{z}\right)^{\delta}$$

*and $q$ is the best subordinant.*

Concluding the results of differential subordination and superordination we state the following sandwich result.

**Theorem 3.7.** *Let $q_1, q_2$ be convex in $U$ with $q_1(0) = q_2(0) = 1$, $\alpha \in \mathbb{C}$,* $\mathrm{Re}\,\alpha > 0$, *$\delta > 0$. If $f \in \mathcal{A}$ such that*

$$\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta} \in \mathcal{H}[1,1] \cap Q$$

$$(1-\alpha)\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta}+\alpha\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta}\cdot\frac{I^{n}f(z)}{I^{n+1}f(z)}$$

*is univalent in* $U$ *and satisfies*

$$q_1(z)+\frac{\alpha}{\delta}zq_1'(z)\prec(1-\alpha)\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta}+\alpha\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta}\cdot\frac{I^{n}f(z)}{I^{n+1}f(z)}$$

$$\prec q_2(z)+\frac{\alpha}{\delta}zq_2'(z),$$

*then*

$$q_1(z)\prec\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta}\prec q_2(z)$$

*and* $q_1,q_2$ *are the best subordinant and the best dominant respectively.*

**Corollary 3.8.** *Let* $q_1,q_2$ *be convex in* $U$ *with* $q_1(0)=q_2(0)=1$, $\alpha\in\mathbb{C}$ *with* $\operatorname{Re}\alpha>0$, $\delta>0$. *If* $f\in\mathcal{A}$ *such that*

$$\left(\frac{If(z)}{z}\right)^{\delta}\in\mathcal{H}[1,1]\cap Q,$$

$$(1-\alpha)\left(\frac{If(z)}{z}\right)^{\delta}+\alpha\left(\frac{If(z)}{z}\right)^{\delta}\cdot\frac{f(z)}{If(z)}$$

*is univalent in* $U$ *and satisfies*

$$q_1(z)+\frac{\alpha}{\delta}zq_1'(z)\prec(1-\alpha)\left(\frac{If(z)}{z}\right)^{\delta}+\alpha\left(\frac{If(z)}{z}\right)^{\delta}\cdot\frac{f(z)}{If(z)}$$

$$\prec q_2(z)+\frac{\alpha}{\delta}zq_2'(z),$$

*then*

$$q_1(z)\prec\left(\frac{If(z)}{z}\right)^{\delta}\prec q_2(z)$$

*and* $q_1,q_2$ *are the best subordinant and the best dominant respectively.*

**Corollary 3.9.** *Let* $q_1,q_2$ *be convex in* $U$ *with* $q_1(0)=q_2(0)=1$, $\alpha\in\mathbb{C}$, $\operatorname{Re}\alpha>0$, $\delta>0$. *If* $f\in\mathcal{A}$ *such that*

$$\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta}\in\mathcal{H}[1,1]\cap Q,$$

$$(1-\alpha)\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta}+\alpha\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta}\cdot\frac{I^{n}f(z)}{I^{n+1}f(z)}$$

*is univalent in* $U$ *and satisfies*

$$q_1(z) + \frac{\alpha}{\delta} z q_1'(z) \prec (1-\alpha)\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta} + \alpha\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta} \cdot \frac{I^n f(z)}{I^{n+1}f(z)}$$

$$\prec q_2(z) + \frac{\alpha}{\delta} z q_2'(z),$$

*then*

$$q_1(z) \prec \left(\frac{I^{n+1}f(z)}{z}\right)^{\delta} \prec q_2(z)$$

*and* $q_1, q_2$ *are the best subordinant and the best dominant respectively.*

Similar results was obtained by D. Răducanu and V.O. Nechita in [5] for differential Sălăgean operator defined in [6].

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